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WELL-POSED SADDLE POINT PROBLEMS

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Abstract. We provide a new well-posedness concept for saddle-point problems. We characterize it by means of the behavior of the sublevel sets of an associated function. We then study the concave-convex case in Euclidean spaces. Applying these results in the setting of Convex Programming, we get a result on the convergence of the pair solution-Lagrange multiplier of approximating problems to the pair solution-Lagrange multiplier of the limit problem.

1. Introduction

Let $(\mathcal{F}, d)$ be some complete metric space of functions $f : X \times Y \longrightarrow \mathbb{R}$, where X, Y are complete metric spaces.

We are interested in the saddle point problem engendered by f, denoted by $S(f)$, i.e. in finding $(\bar{x}, \bar{y})$ such that:

$$f(x, \bar{y}) \leq f(\bar{x}, \bar{y}) \leq f(\bar{x}, y),$$

$\forall x \in X$, $\forall y \in Y$. As it is well-known, and immediate to prove, for any function f, the following inequality holds:

$$\sup_{x\in X} \inf_{y\in Y} f(x, y) \leq \inf_{y\in Y} \sup_{x\in X} f(x, y).$$

Let us define

$$\beta_f(y) = \sup_{x\in X} f(x, y),$$
$$\alpha_f(x) = \inf_{y\in Y} f(x, y)$$

and

$$\omega_f(x, y) = \beta_f(y) - \alpha_f(x)$$

(We shall omit the subscript f when it is not needed, and we shall use sometimes the subscript n when we deal with a sequence of functions $\{f_n\}$).
Then, from the inequality above we have, for each $(x, y) \in X \times Y$,

$$\omega(x, y) \geq 0.$$

The following proposition is known and easy to prove.

Proposition 1.1. *The following are equivalent:*

- *There exists* $(\bar{x}, \bar{y})$ *such that:*
  (1) $\sup_{x\in X} \alpha_f(x) = \alpha_f(\bar{x})$;
  (2) $\inf_{y\in Y} \beta_f(y) = \beta_f(\bar{y})$;

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(3) $\sup_{x\in X}\inf_{y\in Y} f(x,y) = \inf_{y\in Y}\sup_{x\in X} f(x,y)$;
- $(\bar{x},\bar{y})$ *is a saddle point for* f.

We say that the saddle point problem engendered by f *has value*, provided Condition 3. holds. Thus, for every point $\bar{x}$ maximizing α, for every point $\bar{y}$ minimizing β, if the problem has value, then the pair $(\bar{x},\bar{y})$ is a saddle point for f. Conversely, the existence of a saddle point $(\bar{x},\bar{y})$ guarantees that the problem has value, that $\bar{x}$ maximizes α and that $\bar{y}$ minimizes β. I. e. the following proposition holds.

Proposition 1.2. *Solving the saddle point problem* $S(f)$ *is equivalent to solving the following (minimum) problem:*
find $(\bar{x},\bar{y}) \in X \times Y$ *such that* $\omega(\bar{x},\bar{y}) = 0$.

Let us then define the set of all saddle points of the function f:

$$S = \{(\bar{x},\bar{y}) \in X \times Y \text{ such that } \omega(\bar{x},\bar{y}) = 0\}.$$

The following well-posedness concept for a saddle point problem is due to Cavazzuti and Morgan ([CM]).

Definition 1.1. The saddle point problem $S(f)$ is Tykhonov well-posed if:

(1) there exists only one saddle point $(\bar{x},\bar{y})$;
(2) $\forall (x_n,y_n) \in X \times Y$ such that $\omega(x_n,y_n) \longrightarrow 0$, it is $(x_n,y_n) \longrightarrow (\bar{x},\bar{y})$.

This means that the minimum problem engendered by ω_f has value zero, and it is Tykhonov well-posed.

Sometimes uniqueness of the saddle point is a too stringent requirement. Thus, as in the case of Tykhonov well-posedness, the previous definition can be adapted by requiring only existence (and not uniqueness) in the first item, and convergence to a saddle point, up to a subsequence, in the second one.

Definition 1.2. The saddle point problem $S(f)$ is Tykhonov well-posed in the generalized sense if:

(1) there exists at least one saddle point of f;
(2) $\forall (x_n,y_n) \in X \times Y$ such that $\omega(x_n,y_n) \longrightarrow 0$, there exists a subsequence (x_{n_k},y_{n_k}) such that $(x_{n_k},y_{n_k}) \longrightarrow (\bar{x},\bar{y})$, where $(\bar{x},\bar{y})$ is a saddle point of f.

We want now to provide a new well-posedness concept for the saddle point problem, in the spirit of an analogous one given for minimum problems in [Zo1, Zo2] (see also [LZ]). This definition looks interesting since, differently from Tykhonov well-posedness, it takes into account possible small perturbations in the target function, that frequently occur, due to the introduction of approximations in the model, or to round off errors and so on. Since we are supposing $\mathcal{F}$ to be endowed with some metric, we can consider sequences $\{f_n\} \subset \mathcal{F}$ converging to f and associated $\{\beta_n\}$, $\{\alpha_n\}$ and $\{\omega_n\}$.

Definition 1.3. The saddle point problem $S(f)$ is well-posed (with respect to the metric d on $\mathcal{F}$) if:

(1) there exists only one saddle point $(\bar{x},\bar{y})$;

(2) $\forall f_n \longrightarrow f$,

$$\begin{cases} \alpha_n(x) > -\infty & \text{for at least one x,} \\ \beta_n(y) < +\infty & \text{for at least one y} \end{cases}$$

eventually, and $\forall (x_n, y_n) \in X \times Y$ such that $\omega_n(x_n, y_n) - \inf \omega_n(x, y) \longrightarrow 0$, it is $(x_n, y_n) \longrightarrow (\bar{x}, \bar{y})$.

Definition 1.4. The saddle point problem $S(f)$ is well-posed (with respect to the metric d on F) in the generalized sense if:

(1) there exists at least one saddle point of f;

(2) $\forall f_n \longrightarrow f$,

$$\begin{cases} \alpha_n(x) > -\infty & \text{for at least one x,} \\ \beta_n(y) < +\infty & \text{for at least one y} \end{cases}$$

eventually, and $\forall (x_n, y_n) \in X \times Y$ such that $\omega_n(x_n, y_n) - \inf \omega_n(x, y) \longrightarrow 0$, there exists a subsequence (x_{n_k}, y_{n_k}) such that $(x_{n_k}, y_{n_k}) \longrightarrow (\bar{x}, \bar{y})$, where $(\bar{x}, \bar{y})$ is a saddle point of f.

A slightly weaker notion of well-posedness can be taken into account, often called in the literature Hadamard well-posedness. This means that in the definition above one considers only sequences $\{(x_n, y_n)\}$ such that $\omega_n(x_n, y_n) = \inf \omega_n$, i.e *exact* and not only approximate solutions of the approximating problems.

2. Characterization of well-posedness via level sets

In this section we want to characterize a well-posed saddle point problem engendered by a function f, by using its associated function ω and its level sets.
Given a lower semicontinuous, lower bounded function g on a complete metric space Z, we set

$$g^a := \{z \in Z : g(z) \leq \inf_{y \in Z} g(y) + a\}$$

and

$$\text{diam } g^a = \sup_{y,z \in g^a} d(y, z).$$

Characterizing well-posedness of the saddle point problem is naturally related to the good behavior of the minimum problem engendered by the function ω. It is well-known that, if Z is a complete metric space and g an extended real valued, lower semicontinuous function on Z, Tykhonov well-posedness of g is equivalent to saying that $\lim_{a \to 0^+} \text{diam } g^a = 0$. Thus, first of all, we need to make assumptions in order to guarantee lower semicontinuity of ω or, equivalently, lower semicontinuity of β and upper semicontinuity of α. To do this, observe that, setting

$$\text{epi } \beta := \{(y, r) : \beta(y) \leq r\},$$

and, dually,

$$\text{hypo } \alpha := \{(x, r) : \alpha(x) \geq r\},$$

we have that

$$\text{epi } \beta = \bigcap_{x \in X} \text{epi } f(x, \cdot), \quad \text{hypo } \alpha = \bigcap_{y \in Y} \text{hypo } f(\cdot, y).$$

It follows that, if $x \mapsto f(x,y)$ is an upper semicontinuous function for all $y \in Y$, then hypo $f(\cdot,y)$ is a closed set, and thus hypo α is a closed set and so α is upper semicontinuous. The same line of reasoning can be applied to β. We summarize these simple remarks in the following proposition.

Proposition 2.1. *Let $f : X \times Y \longrightarrow R$ be a given function. Suppose:*

- *$x \mapsto f(x,y)$ is upper semicontinuous for all $y \in Y$;*
- *$y \mapsto f(x,y)$ is lower semicontinuous for all $x \in X$.*

Then β is lower semicontinuous, α is upper semicontinuous and ω is lower semicontinuous.

Define now

$$T_f^a := \cup \left\{ \omega_g^a : g \in \mathcal{F},\ d(g,f) \leq a \right\}.$$

Remark 2.1. It is intended that $\omega_g^a = X \times Y$ if $\inf \omega_g(x,y) = +\infty$. This case must be taken into account since ω itself is defined in terms of suprema, and it is easy to produce examples of functions f for which $\omega_f(x,y) = \infty$ for all pairs (x,y).

We are now able to prove the first result.

Theorem 2.1. *If the saddle point problem $S(f)$ is well-posed, then*

$$\text{diam } (T_f^a) \xrightarrow{a \downarrow 0} 0.$$

Proof. By contradiction suppose there exists $\varepsilon > 0$ such that $\forall a > 0$ $\text{diam}(T_f^a) > \varepsilon$. Take $a = \frac{1}{n}$. Then there exist

$$f_n \longrightarrow f, \qquad g_n \longrightarrow g, \qquad (x_n, y_n) \qquad \text{and} \qquad (u_n, v_n)$$

such that

$$0 \leq \omega_{f_n}(x_n,y_n) \leq \inf \omega_{f_n}(x,y) + \frac{1}{n}, \qquad 0 \leq \omega_{g_n}(u_n,v_n) \leq \inf \omega_{g_n}(x,y) + \frac{1}{n}$$

and

$$\delta((x_n,y_n),(u_n,v_n)) \geq \varepsilon.$$

(Here δ denotes a compatible distance on the space $X \times Y$). This is impossible because the problem is well-posed, and thus the sequences $\{(x_n,y_n)\}$ and $\{(u_n,v_n)\}$ must converge to the same element, the unique saddle point. □

In the other opposite, the following result holds.

Theorem 2.2. *Let $f : X \times Y \longrightarrow R$ be a given function. Suppose:*

- *$x \mapsto f(x,y)$ is upper semicontinuous for all $y \in Y$;*
- *$y \mapsto f(x,y)$ is lower semicontinuous for all $x \in X$.*

Suppose moreover $\inf \omega(x,y) = 0$ *and*

$$\text{diam } (T_f^a) \xrightarrow{a \downarrow 0} 0. \tag{2}$$

Then the saddle point problem $S(f)$ is well-posed.

Proof. First of all there exists one and only one saddle point $(\bar{x}, \bar{y})$ because $\inf \omega(x, y) = 0$, diam $(\omega_f^a) \xrightarrow{a \downarrow 0} 0$, and ω is lower semicontinuous. Moreover, observe that condition (1) in Definition 1.3 holds thanks to (2). Now, take $f_n \longrightarrow f$ and (x_n, y_n) such that

$$\omega_n(x_n, y_n) - \inf \omega_n(x, y) \longrightarrow 0.$$

Define

$$\varepsilon_n := \max \{d(f_n, f), \omega_n(x_n, y_n) - \inf \omega_n(x, y)\}.$$

It is $\varepsilon_n \downarrow 0$ and for each fixed $\varepsilon > 0$ there exists N such that $\forall n \geq N$

$$\text{diam } (T_f^{\varepsilon_n}) < \varepsilon.$$

This means that $\{(x_n, y_n)\}$ is a Cauchy sequence and so it converges. Let (x_0, y_0) be its limit. Then $(x_0, y_0) \in \cap_{a>0}\overline{(T_f^a)}$. This set however must be a singleton, and it contains $(\bar{x}, \bar{y})$ too. Thus $(x_0, y_0) = (\bar{x}, \bar{y})$ and the proof is complete. □

3. Well-posedness and existence theorems

The Weierstrass theorem is considered the most general and elegant existence theorem for minimum problems. Actually, it does not merely state existence but, as it is easy to see from its proof, it claims Tykhonov well-posedness in generalized sense. In this section we want to analyze the relation between well-posedness and general existence theorems for saddle points.

Theorem 3.1. *Let X and Y be compact sets, let $f : X \times Y \to R$ be such that:*

- *$x \mapsto f(x, y)$ is upper semicontinuous for all $y \in Y$;*
- *$y \mapsto f(x, y)$ is lower semicontinuous for all $x \in X$.*

Suppose moreover that the saddle problem has a solution. Then $S(f)$ is Tykhonov well-posed in the generalized sense.

Proof. Take $(x_n, y_n) \in X \times Y$ such that $\omega(x_n, y_n) \longrightarrow 0$. By compactness of $X \times Y$ there exists a subsequence (x_{n_k}, y_{n_k}) such that $(x_{n_k}, y_{n_k}) \longrightarrow (x^*, y^*)$. Being ω lower semicontinuous it is

$$0 = \liminf_{n \longrightarrow +\infty} \omega(x_{n_k}, y_{n_k}) \geq \omega(x^*, y^*) \geq 0,$$

that is $(x^*, y^*) \in S$. □

The next theorem extends the above result, in a straightforward way, to generalized well-posedness. The distance considered in the space $\mathcal{F}$ is a compatible one with uniform convergence of functions.

Theorem 3.2. *In the assumptions of Theorem 3.1, the problem $S(f)$ is well-posed in the generalized sense, with respect to uniform convergence.*

Proof. Take $f_n \longrightarrow f$ uniformly. Then, as it is easy to see,

$$\omega_n(x_n, y_n) - \omega(x_n, y_n) \longrightarrow 0$$

and

$$\inf \omega_n(x, y) \longrightarrow \inf \omega(x, y).$$

Being the problem $S(f)$ Tykhonov well-posed in the generalized sense, it follows that

$$\omega(x_n, y_n) \longrightarrow \inf_{x,y} \omega(x, y) = 0,$$

and we conclude by means of Theorem 3.1. □

Thus, under the topological properties for X, Y and f of Theorem 3.1, we see that well-posedness is a consequence of the existence of the value for the saddle point problem. This follows from proposition 1.1, as α_f is upper semicontinuous, β_f lower semicontinuous, and so they assume maximum (resp. minimum) on the compact set X (resp. Y). We recall now (in a simplified form) the most celebrated theorem guaranteeing that a saddle point problem has a value ([Si]).

Theorem 3.3. *(Sion) Let X and Y be compact and convex sets. Suppose:*

- *$x \mapsto f(x, y)$ is upper semicontinuous and quasi concave for all $y \in Y$;*
- *$y \mapsto f(x, y)$ is lower semicontinuous and quasi convex for all $x \in X$.*

Then

$$\sup_{x \in X} \inf_{y \in Y} f(x, y) = \inf_{y \in Y} \sup_{x \in X} f(x, y).$$

Thus we can conclude:

Corollary 3.1. *Under the assumptions of the Sion theorem, the saddle point problem is well-posed in the generalized sense, with respect to the uniform convergence on the space of the target functions.*

4. Concave/Convex case in Euclidean spaces

Let $\mathcal{F}$ be the family of the convex lower semicontinuous functions defined on an Euclidean space. It is known that for $f \in \mathcal{F}$ uniqueness of the minimum point does actually imply its Tykhonov well-posedness. It is not difficult to prove (see [Lu] and also later) that this actually implies well-posedness with respect to Kuratowski convergence on $\mathcal{F}$. We now want to analyze the same issue for the saddle point problem. So we shall assume that $X \times Y$ is an Euclidean space, that $f(\cdot, y)$ is concave and upper semicontinuous $\forall y \in Y$ and that $f(x, \cdot)$ is convex and lower semicontinuous $\forall x \in X$ (for short, we shall say in the following that f is concave/convex). We start with a preliminary result concerning the minimization of convex functions.

Theorem 4.1. *Suppose f concave/convex and that there is only one saddle point $(\bar{x}, \bar{y}) \in X \times Y$. Then the problem $S(f)$ is Tykhonov well-posed.*

Proof. From Proposition 2.1 we know that ω_f is lower semicontinuous. Exactly with the same argument we can conclude that ω is convex too. From the assumption, we conclude that ω has a unique minimum point, that is $(\bar{x}, \bar{y})$, and that $\omega(\bar{x}, \bar{y}) = 0$. Thus the minimum problem engendered by ω is Tykhonov well-posed with value zero and this is equivalent to say that the saddle point problem is Tykhonov well-posed. □

Thus the situation in the saddle point problem in Euclidean spaces, is similar to that one for the minimum problems, where uniqueness of the solution implies Tykhonov well-posedness. Actually, in the setting of the minimum problems, a further step can be done, since uniqueness is actually equivalent to well-posedness, with respect to uniform convergence on bounded sets (for instance). Is it the same for the saddle point problem? The answer is negative, as shown in the following example.

Example 4.1. *Consider, on* $R \times R$

$$f(x, y) = xy$$

and let

$$f_n(x, y) = xy + \frac{1}{n}x(y - 1).$$

The concave/convex problem engendered by f *is not well-posed, with respect to uniform convergence on bounded sets, because condition* (2) *in Definition 1.3 does not hold since*

$$\omega_n(x, y) = +\infty \qquad \forall (x, y).$$

On the other hand, it is clear that when we perturb a function in order to solve an easier problem, the involved perturbations should not be totally arbitrary. Thus, we can ask whether more suitable perturbations allow claiming for well-posedness. This is the content of our next results. To start with, we prove now a result concerning minimum problems. More precisely, we see that uniqueness of the minimum point implies well-posedness of the problem, when we consider Kuratowski convergence on the set of objective functions. We consider in this case Kuratowski convergence, rather than, for instance, uniform convergence on bounded sets, as we are interested in applying the result also to extended real valued functions. For this (larger) class of functions, Kuratowski convergence is more appropriate. We recall that a sequence $\{f_n\}$ of functions converges in Kuratowski sense to a function f if the following conditions are verified:

1. $\forall x_n \longrightarrow x$

$$\liminf_{n \to +\infty} f_n(x_n) \geq f(x);$$

2. $\forall x \qquad \exists x_n \longrightarrow x$ s.t.

$$\limsup_{n \to +\infty} f_n(x_n) \leq f(x).$$

Lemma 4.1. *Let* $f : X \longrightarrow (-\infty, +\infty]$ *be convex, lower semicontinuous and with unique minimum point* $\bar{x}$. *Then* f *is well-posed with respect to Kuratowski convergence of functions, i.e., if* f_n *is a convex and lower semicontinuous function for all* n, *if* $f_n \xrightarrow{K} f$, *then* $\inf f_n > -\infty$ *eventually, and if* $\{x_n\}$ *is such that* $f_n(x_n) - \inf f_n \longrightarrow 0$, *then* $x_n \longrightarrow \bar{x}$.

Proof. First of all, let us see that $\inf f_n > -\infty$ eventually. By contradiction, suppose there are a subsequence $\{n_k\}$ and a sequence $\{x_k\}$ such that $f_{n_k}(x_k) \to -\infty$. If $\{x_k\}$ has a limit point x, then it is $f(x) = -\infty$, and this is a contradiction. Thus it must be $\|x_k\| \to \infty$. Consider $y_k = \frac{x_k}{\|x_k\|}$ and let d be any limit point of y_k. Take $z_k \to \bar{x}$ such that $f_{n_k}(z_k) \to f(\bar{x})$. Consider

$$w_k = (1 - \frac{1}{\|x_k\|})z_k + \frac{1}{\|x_k\|}x_k \to \bar{x} + d.$$

Since
$$f_{n_k}(w_k) \le (1 - \frac{1}{\|x_k\|}) f_{n_k}(z_k)$$
eventually, then
$$f(\bar{x} + d) \le \liminf f_{n_k}(w_k) \le f(\bar{x}).$$
This provides the needed contradiction. The second part of the proof exploits a similar argument. Take $\{x_n\}$ such that $f_n(x_n) - \inf f_n \longrightarrow 0$. If $\{x_n\}$ has a bounded subsequence, then there are a subsequence n_k and x such that $x_{n_k} \longrightarrow x$. Thus
$$f(x) \le \liminf_{k \longrightarrow +\infty} f_{n_k}(x_{n_k}) = \liminf_{k \longrightarrow +\infty} \inf f_{n_k} \le \limsup_{k \longrightarrow +\infty} \inf f_{n_k} \le \inf f.$$
Then $x = \bar{x}$ and this shows that every limit point of x_n is $\bar{x}$. Suppose now $\|x_n\| \to \infty$ and take $z_n \longrightarrow \bar{x}$ such that
$$\limsup f_n(z_n) \le f(\bar{x}).$$
Fix $a > \|\bar{x}\|$ and consider
$$y_n = \lambda_n x_n + (1 - \lambda_n) z_n,$$
with λ_n such that $\|y_n\| = a$ (implying $\lambda_n \longrightarrow 0$). Now, there is a subsequence $\{y_{n_k}\}$ converging to some $\bar{y}$, with $\|\bar{y}\| = a$. It is
$$f(\bar{y}) \le \liminf_{k \longrightarrow +\infty} f_{n_k}(y_{n_k}) \le \liminf_{k \longrightarrow +\infty} (\lambda_{n_k} f_{n_k}(x_{n_k}) + (1 - \lambda_{n_k}) f_{n_k}(z_{n_k})) \le$$
$$\le \limsup_{k \longrightarrow +\infty} \lambda_{n_k} f_{n_k}(x_{n_k}) + \limsup_{k \longrightarrow +\infty} (1 - \lambda_{n_k}) f_{n_k}(z_{n_k}) \le f(\bar{x}).$$
This contradicts uniqueness of the minimum point and the proof is complete. □

Now consider
$$f_n(x, y) = f(x, y) + g_n(y) - h_n(x),$$
where g_n and h_n are convex, nonnegative and such that $g_n, h_n \xrightarrow{UB} 0$, where $\xrightarrow{UB}$ indicates uniform convergence on bounded sets.

Observe that the functions f_n are concave/convex and that $f_n \xrightarrow{UB} f$.

Theorem 4.2. *$\beta_n \xrightarrow{K} \beta$, $-\alpha_n \xrightarrow{K} -\alpha$, and thus $\omega_n \xrightarrow{K} \omega$.*

Proof. We shall prove that $\beta_n \xrightarrow{K} \beta$. The proof for the functions $-\alpha_n$ is the same. We have to prove that:

1. $\forall y_n \longrightarrow y$
$$\liminf_{n \to +\infty} \beta_n(y_n) \ge \beta(y);$$
2. $\forall y \qquad \exists y_n \longrightarrow y$ s.t.
$$\limsup_{n \to +\infty} \beta_n(y_n) \le \beta(y).$$

1. We deal with the case $\beta(y) \in \mathbb{R}$. The case $\beta(y) = \infty$ is analogous and it is left to the reader. Let ε be fixed. There exists $\hat{x}$ s.t.
$$f(\hat{x}, y) \ge \beta(y) - \varepsilon.$$
As
$$f_n(\hat{x}, y_n) \longrightarrow f(\hat{x}, y)$$

and

$$\beta_n(y_n) \geq f_n(\hat{x}, y_n),$$

we have

$$\liminf_{n\to+\infty} \beta_n(y_n) \geq \liminf_{n\to+\infty} f_n(\hat{x}, y_n) \geq f(\hat{x}, y) \geq \beta(y) - \varepsilon.$$

We now conclude as $\varepsilon > 0$ is arbitrary.

2. Let y be fixed. Take $y_n = y \ \forall n$. As

$$f_n(x, y) = f(x, y) + g_n(y) - h_n(x)$$

the required inequality easily follows from the fact that $g_n(y) \to 0$ and $h_n(x) \geq 0$ $\forall x \in X$. Finally, since ω_n is, for all n, the sum of two functions, one depending only on the variable x, the other one depending only on y, it is clear that $\omega_n \xrightarrow{K} \omega$: the proof is complete. □

Now we can state the following theorem.

Theorem 4.3. *Let f be concave/convex and suppose there exists only one saddle point $(\bar{x}, \bar{y})$. Let $\mathcal{F}$ be the set of functions of the form*

$$f_n(x, y) = f(x, y) + g_n(y) - h_n(x),$$

with g_n and h_n convex, lower semicontinuous, nonnegative and such that $g_n, h_n \xrightarrow{UB} 0$. Then the saddle point problem $S(f)$ is well-posed.

Proof. First of all observe that $\omega(x, y)$ and $\omega_n(x, y)$ are convex and that ω admits only one minimum point $(\bar{x}, \bar{y})$ such that $\omega(\bar{x}, \bar{y}) = 0$. By Theorem 4.2 we have that $\omega_n \xrightarrow{K} \omega$ and by Lemma 4.1 the problem

$$\min \omega(x, y)$$

is well-posed with respect to Kuratowski convergence, i.e. for each (x_n, y_n) such that $\omega_n(x_n, y_n) - \inf \omega_n \longrightarrow 0$, it is $(x_n, y_n) \longrightarrow (\bar{x}, \bar{y})$. □

Remark 4.1. A particular case that can be considered is

$$f_n(x, y) = f(x, y) + \varepsilon_n g(y) - \sigma_n h(x),$$

with $f(x, y)$ concave/convex, $\varepsilon_n, \sigma_n \longrightarrow 0^+$, g and h convex and lower bounded. Observe also that the assumption g_n, h_n non negative can be relaxed to uniformly coercive.

We provide now our last result. It deals with the following convex programming $P(p, a, b)$:

$$\mathbf{P}(p, a, b) \qquad \begin{cases} \textit{minimize} & f(x) - \langle p, x \rangle, \\ \textit{s.t.} \ \ g(x) \leq a, \quad Lx = b, & \end{cases}$$

where $f : R^n \to R$ is a convex function, $g : R^n \to R^l$ is convex (coordinatewise), $L : R^n \to R^k$ is a linear, onto operator, while p, a, b are vectors in the appropriate spaces and serve as parameters. Let us call $V(p, a, b)$ the value function of this problem:

$$V(p, a, b) = \inf\{f(x) - \langle p, x \rangle : \ \textit{s.t.} \ \ g(x) \leq a, \quad Lx = b\}.$$

Even if it is a redundant assumption, we suppose that f is coercive (i.e. $\lim_{\|x\|\to\infty}\frac{f(x)}{\|x\|}=\infty$), in order to have a big set of parameters for which the problem has finite value. It is well-known and easy to prove that $V(\cdot,a,b)$ is a concave function for all (a,b) and $V(p,\cdot,\cdot)$ is convex for all p. Thus V is a concave/convex function.

Let us recall that for a concave/convex function $F(\cdot,\cdot)$ the subdifferential at a point (x,y), denoted by $\partial F(x,y)$, consists of all pairs (x^*,y^*) such that:

$$-F(u,y)\geq -F(x,y)+\langle x^*,u-x\rangle \quad \forall u\in X,$$
$$F(x,v)\geq F(x,y)+\langle y^*,v-y\rangle \quad \forall v\in Y.$$

In other words,

$$x^*\in\partial(-F(\cdot,y))(x),\;\; y^*\in\partial(F(x,\cdot)(y),$$

where the subdifferentials in the line above are intended in the sense of the convex analysis.

As it is well-known, $(\bar{x},\bar{\lambda})$ is a pair (solution-Lagrange multiplier) for the convex programming problem $\mathbf{P}(p,a,b)$ above if and only if

$$(\bar{x},\bar{\lambda})\in\partial V(p,a,b).$$

So, let us suppose, as above, that for a given triple of parameters (p,a,b) there is existence and uniqueness of the pair (solution-Lagrange multiplier). Can we say anything about well-posedness of the problem? Our next result shows that we have in this setting Hadamard well-posedness.

Our proof relies on a result by T. Rockafellar ([Ro], Theorem 35.8) asserting that a concave/convex function whose subdifferential at a given point reduces to a singleton, is actually (Fréchet) differentiable at that point, and on the following result, that holds in general Banach spaces X,Y (with duals X^*,Y^*).

Proposition 4.1. *Suppose the concave/convex function is Fréchet differentiable at a point (x,y). Then the subdifferential multifunction:*

$$\partial F: X\times Y\to X^*\times Y^*,$$

is norm-norm upper semicontinuous at (x,y).

Proof. We can suppose, without loss of generality, that $(x,y)=(0,0)$, $\partial F(0,0)=\{0,0\}$, $F(0,0)=0$. By contradiction, suppose there is $\{(x_n,y_n)\}$ such that $(x_n,y_n)\to(0,0)$, and (x_n^*,y_n^*) such that, for all n, $(x_n^*,y_n^*)\in\partial F(x_n,y_n)$, and $\|(x_n^*,y_n^*)\|>5\varepsilon$, for some $\varepsilon>0$. We assume $\|y_n^*\|>5\varepsilon$ (along a subsequence), the other case being completely analogous. By definition of Fréchet differentiability, there is $\delta>0$ such that:

$$F(u,v)\leq\varepsilon(\|u\|+\|v\|),$$

provided $\|u\|\leq\delta$, $\|v\|\leq\delta$. There is d_n such that $\|d_n\|=1$ and

$$\langle y_n^*,d_n\rangle>5\varepsilon,$$

for all n. We have that

$$F(x_n,v)\geq F(x_n,y_n)+\langle y_n^*,v-y_n\rangle \quad \forall v\in Y,$$

and thus

$$\langle y_n^*,v\rangle\leq F(x_n,v)-F(x_n,y_n)+\langle y_n^*,y_n\rangle.$$

Set $v = \delta d_n$ in the above formula, with n so large that $\|x_n\| < \delta$, $|F(x_n, y_n)| < \varepsilon\delta$, $|\langle y_n^*, y_n\rangle| < \varepsilon\delta$. Thus

$$5\varepsilon\delta \leq |F(x_n, \delta d_n)| + |F(x_n, y_n)| + |\langle y_n^*, y_n\rangle| \leq 2\varepsilon\delta + \varepsilon\delta + \varepsilon\delta,$$

a contradiction. □

Theorem 4.4. *Consider the convex programming $P(p, a, b)$ above. Suppose $P(\bar{p}, \bar{a}, \bar{b})$ has one and only one pair (solution-Lagrange multiplier) (x, λ). Let $a_n \to \bar{a}$, $b_n \to \bar{b}$, $p_n \to \bar{p}$. If (x_n, λ_n) is a pair (solution-Lagrange multiplier) for the problem $P(a_n, b_n, p_n)$, then $x_n \to x, \lambda_n \to \lambda$.*

Proof. As already remarked, a pair (solution-Lagrange multiplier) corresponds to a pair in the subdfferential of the value function. Thus it is enough to apply Theorem 35.8 in [Ro] and Proposition 4.1. □

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