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ITERATIVE APPROXIMATION OF FIXED POINTS OF A CLASS OF MAPPINGS IN A HILBERT SPACE

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Abstract. In the present paper, we show the strong convergence of a Mann type iterative scheme for a class of mappings to fixed points

1. Introduction

In the present paper, we consider an iterative scheme for a class of mappings in a Hilbert space H and establish the strong convergence of the iteration to fixed points of the mappings. Throughout this paper, we denote by H a real Hilbert space. The inner product and the norm of H is denoted by $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$, respectively. A mapping $T : D(T) \longrightarrow H$ is said to be strictly pseudocontractive if there exists a number $t > 1$ such that the inequality

$$\|x - y\| \leq \|(1 + r)(x - y) - rt(Tx - Ty)\|$$

holds for all $x, y \in D(T)$ and $r > 0$, where $D(T)$ denotes the domain of T. A mapping A is said to be strongly accretive if there exists a positive number k such that

$$\langle Ax - Ay, x - y \rangle \geq k \|x - y\|^2 \qquad \text{for all } x, y \in D(A).$$

It is known that a mapping $T : D(A) \longrightarrow H$ is strictly pseudocontractive if and only if $I - T$ is strongly accretive(cf. Chidume[2]). Though we restricted ourselves to a Hilbert space, one can see that the definition of strictly pseudocontractive mapping is valid in any Banach space. For pseudocontractive mappings, the strong convergence of Mann type iteration defined by

$$\begin{cases} x_1 \in H, \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T x_n \qquad n \geq 1 \end{cases}$$

has been studied by many authors(cf. [2], [3], [4], and [5]).

In the present paper, we consider a class of mappings defined below. Let E_1, E_2 be subspaces of H such that $E_1 \perp E_2$ and $H = E_1 \oplus E_2$. We impose the following conditions on $T : D(T) \longrightarrow H$

(T1) For each $y \in D(T) \cap E_2$, $T(y + \cdot) : D(T) \cap E_1 \longrightarrow H$ is a Lipschitz mapping with Lipschitz constant l_1;
(T2) For each $x \in D(T) \cap E_1$, $T(x + \cdot) : D(T) \cap E_2 \longrightarrow H$ is a Lipschitz mapping with Lipschitz constant l_2;

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(T3) For each $y \in D(T) \cap E_2$, $T(y + \cdot) : D(T) \cap E_1 \longrightarrow H$ is strictly pseudo contractive. i.e., there exists $k_1 > 0$ such that

$$\langle (I-T)(x_1+y) - (I-T)(x_2+y), x_1 - x_2 \rangle \geq k_1 \|x_1 - x_2\|^2$$

for all $x_1, x_2 \in D(T) \cap E_1$;

(T4) For each $x \in D(T) \cap E_1$, $T(x + \cdot) : D(T) \cap E_2 \longrightarrow H$ is strictly pseudo contractive, i.e., there exists $k_2 > 0$ such that

$$\langle (I-T)(x+y_1) - (I-T)(x+y_2), y_1 - y_2 \rangle \geq k_2 \|y_1 - y_2\|^2$$

for all $y_1, y_2 \in D(T) \cap E_2$;

A broad class of mappings satisfies conditions (T1)-(T4). Here we give examples of mappings which satisfy (T1)-(T4). First we consider a variational case. Let $f(x,y) = x^4 + y^4 - 5xy$ for $x, y \in \mathbb{R}$. One can easily verify that $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is not a convex function. But we have that for each $x \in \mathbb{R}$, $y \longrightarrow f(x,y)$ is a convex function. We also have that for each $y \in \mathbb{R}$, $x \longrightarrow f(x,y)$ is a convex function. Then we can see that on any bounded subset D of $\mathbb{R}^2$, the mapping $T : (x,y) \longrightarrow (x + \partial f(x,y)/\partial x, y + \partial f(x,y)/\partial y)$ satisfies (T1)-(T4). A fixed point $(x,y) \in \mathbb{R}^2$ of T corresponds to the point (x,y) satisfying $\partial f(x,y)/\partial x = \partial f(x,y)/\partial y = 0$. Similarly, we can consider a concave-convex function. Let $g(x,y) = x^4 - y^4 - 5xy$ for $x, y \in \mathbb{R}$. Then as above, we can see that for each $x \in \mathbb{R}$, $y \longrightarrow f(x,y)$ is concave function, and for each $y \in \mathbb{R}$, $x \longrightarrow f(x,y)$ is a convex function. Then if we put $T(x,y) = (x + \partial g(x,y)/\partial x, y - \partial g(x,y)/\partial y)$ for $x, y \in \mathbb{R}^2$. Then T satisfies (T1)-(T4) on any bounded set $D \subset \mathbb{R}^2$. Let $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be a mapping defined by

$$T : \begin{pmatrix} x \\ y \end{pmatrix} \longrightarrow \begin{pmatrix} x^3 - 4y \\ y^3 - 5x - 1 \end{pmatrix}, \qquad (x,y) \in \mathbb{R}^2$$

satisfies (T1)-(T4) on each bounded subset of $\mathbb{R}^2$. It is easy to see that the mapping T defined above is not variational problem. Iterative shcemes for mappings satisfying conditions (T1)-(T4) are important from the view point of practical applications. For example, the Bathe free energy function which appears in the coding theory satisfies (T1)-(T4)(cf. [6]).

We now state our main result.

Theorem 1. *Let $T : H \longrightarrow H$ satisfy $(T1) - (T4)$. Suppose that $F(T) \neq \phi$ and*

$$c = k_1 k_2 - \max\{(1+l_1)l_1, (1+l_2)l_2\} > 0.$$

Let $\{z_n\}$ be a sequence defined by

$$\text{(P)} \qquad \begin{cases} z_1 = x_1 + y_1, & x_1 \in E_1, y_1 \in E_2, \\ z_n' = x_{n+1} + y_n & x_{n+1} = (1-\alpha_n)x_n + \alpha_n P_1 T(z_n) \\ z_{n+1} = x_{n+1} + y_{n+1} & y_{n+1} = (1-\beta_n)y_n + \beta_n P_2 T(z_n') \end{cases}$$

for $n \geq 1$ with $\{\alpha_n\}, \{\beta_n\} \subset (0,1]$ satisfying

$$(1) \qquad \begin{cases} \lim_{n\longrightarrow\infty} \alpha_n = \lim_{n\longrightarrow\infty} \beta_n = 0, & \sum_{n=1}^{\infty} \gamma_n = \infty, \\ \sum_{m=1}^{\infty} \alpha_m \Pi_{j=1}^{m}(1-\rho\gamma_j) < \infty, \text{and} & \sum_{m=1}^{\infty} \beta_m \Pi_{j=1}^{m}(1-\rho\gamma_j) < \infty \end{cases}$$

for some $0 < \rho < \min\{c/k_1, c/k_2\}$, where $\gamma_n = \min\{\alpha_n, \beta_n\}$ for $n \geq 1$. Then $\{z_n\}$ converges strongly to a fixed point of T.

Remark. The condition 1 is satisfied if $\alpha_n = \beta_n = 1/\rho n$ for $n \geq 2$ and $\alpha_1 = \beta_1 = 0$.

Throughout the rest of this paper, we assume that the assumption of Theorem is satisfied. For $i = 1, 2$, we denote by P_i the projections from E_i onto H.

Lemma 1. (1) *There exists a Lipschitz mapping* $\varphi_2 : E_1 \longrightarrow E_2$ *with Lipschiz constant* l_1/k_2 *such that*

$$P_2 T(x + \varphi_2(x)) = \varphi_2(x) \qquad \textit{for } x \in E_1$$

(2) *There exists a Lipschitz mapping* $\varphi_1 : E_2 \longrightarrow E_1$ *with Lipschiz constant* l_2/k_1 *such that*

$$P_1 T(z + \varphi_1(z)) = \varphi_1(z) \qquad \textit{for } z \in E_2$$

Proof. We prove (1). The assertion (2) follows by the same argument. Let $x \in E_1$. For simplicity, we put $T_2(y) = P_2 T(x + y)$ for $y \in E_2$. Then since $T_2 : E_2 \longrightarrow E_2$ is strictly pseudocontractive, there exists a unique fixed point $y \in E_2$ of T_2(cf. [1]). Therefore the mapping $\varphi_2 : E_1 \longrightarrow E_2$ can be defined as $\varphi_2(x) = y$. We will see that φ_2 is a Lipshitz mapping. Let $x_1, x_2 \in E_1$. Then

$$\begin{aligned}
0 &= \langle \varphi_2(x_1) - \varphi_2(x_2), (I - T)(x_1 + \varphi_2(x_1)) - (I - T)(x_2 + \varphi_2(x_2)) \rangle \\
&= \langle \varphi_2(x_1) - \varphi_2(x_2), (I - T)(x_1 + \varphi_2(x_1)) - (I - T)(x_1 + \varphi_2(x_2)) \rangle \\
&\qquad + \langle \varphi_2(x_1) - \varphi_2(x_2), (I - T)(x_1 + \varphi_2(x_2)) - (I - T)(x_2 + \varphi_2(x_2)) \rangle \\
&\geq k_2 \|\varphi_2(x_1) - \varphi_2(x_2)\|^2 \\
&\qquad - \langle \varphi_2(x_1) - \varphi_2(x_2), T(x_1 + \varphi_2(x_2)) - T(x_2 + \varphi_2(x_2)) \rangle .
\end{aligned}$$

Then

$$\begin{aligned}
&k_2 \|\varphi_2(x_1) - \varphi_2(x_2)\|^2 \\
&\qquad \leq \langle T(x_1 + \varphi_2(x_2)) - T(x_2 + \varphi_2(x_2), \varphi_2(x_1) - \varphi_2(x_2) \rangle \\
&\qquad \leq l_1 \|x_1 - x_2\| \|\varphi_2(x_1) - \varphi_2(x_2)\| .
\end{aligned}$$

That is

$$\|\varphi_2(x_1) - \varphi_2(x_2)\| \leq (l_1/k_2) \|x_1 - x_2\| .$$

This completes the proof. □

Proof of Theorem. Let $\{z_n\}$ be the sequence defined by (P). Let $n \geq 1$. Then by the same arguement as in the proof of Liu[3], we have that

$$\begin{aligned}
x_n &= x_{n+1} + \alpha_n x_n - \alpha_n P_1 T z_n \\
&= (1 + \alpha_n) x_{n+1} + \alpha_n P_1 (I - T - k_1 I) z_n' - (2 - k_1) \alpha_n x_{n+1} \\
&\qquad\qquad + \alpha_n x_n + \alpha_n P_1 (T z_n' - T z_n) \\
&= (1 + \alpha_n) x_{n+1} + \alpha_n P_1 (I - T - k_1 I) z_n' \\
&\qquad - (2 - k_1) \alpha_n \left[(1 - \alpha_n) x_n + \alpha_n P_1 T z_n \right] \\
&\qquad\qquad + \alpha_n x_n + \alpha_n P_1 (T z_n' - T z_n)
\end{aligned}$$

$$= (1+\alpha_n)x_{n+1} + \alpha_n P_1(I - T - k_1 I)z_n' - (1-k_1)\alpha_n x_n$$
$$+ (2-k_1)\alpha_n^2 P_1(z_n - Tz_n) + \alpha_n P_1(Tz_n' - Tz_n).$$

Recalling that $P_1T(y_n + \varphi_1(y_n)) = \varphi_1(y_n)$, we have that

$$x_n - \varphi_1(y_n) = (1+\alpha_n)(x_{n+1} - \varphi_1(y_n)) + \alpha_n P_1(I - T - k_1 I)(z_n' - \varphi_1(y_n))$$
$$- (1-k_1)\alpha_n(x_n - \varphi_1(y_n)) + (2-k_1)\alpha_n^2 P_1(z_n - Tz_n)$$
$$+ \alpha_n P_1(Tz_n' - Tz_n).$$

Then since $P_1T(y_n + \cdot)$ is strictly pseudocontractive, we have

$$\|x_n - \varphi_1(y_n)\| \geq (1+\alpha_n)\|x_{n+1} - \varphi_1(y_n)\| - (1-k_1)\alpha_n \|x_n - \varphi_1(y_n)\| \tag{2}$$
$$- (2-k)\alpha_n^2 \|P_1(z_n - Tz_n)\| - \alpha_n \left\|P_1(Tz_n' - Tz_n)\right\|.$$

Noting that $\|P_1Tz_n - \varphi_1(y_n)\| \leq l_1 \|x_n - \varphi_1(y_n)\|$, we have

$$\|P_1(z_n - Tz_n)\| \leq \|P_1(z_n - \varphi_1(y_n))\| + \|P_1(Tz_n - \varphi_1(y_n))\|$$
$$= (1+l_1)\|x_n - \varphi_1(y_n)\|.$$

We also have

$$\left\|P_1(Tz_n' - Tz_n)\right\| \leq l_1 \|x_{n+1} - x_n\|$$
$$\leq l_1(l_1+1)\alpha_n \|x_n - \varphi_1(y_n)\|.$$

Therefore we find

$$\|x_n - \varphi_1(y_n)\| \geq (1+\alpha_n)\|x_{n+1} - \varphi_1(y_n)\| - (1-k_1)\alpha_n\|x_n - \varphi_1(y_n)\|$$
$$- [(2-k_1)\alpha_n^2(1+l_1) + l_1(1+l_1)\alpha_n^2]\|x_n - \varphi_1(y_n)\|.$$

Here we put

$$C(k_1, l_1, \alpha_n) = [1 + (1-k_1)\alpha_n + (1+l_1)\alpha_n^2(2 - k_1 + l_1)](1+\alpha_n)^{-1}.$$

Then we find

$$\|x_{n+1} - \varphi_1(y_n)\| \leq C(k_1, l_1, \alpha_n)\|x_n - \varphi_1(y_n)\|.$$

It then follows that

$$\|x_{n+1} - \varphi_1(y_{n+1})\|$$
$$\leq C(k_1, l_1, \alpha_n)(\|x_n - \varphi_1(y_n)\| + \|\varphi_1(y_{n+1}) - \varphi_1(y_n)\|)$$
$$\leq C(k_1, l_1, \alpha_n)(\|x_n - \varphi_1(y_n)\| + (l_2/k_1)\|y_{n+1} - y_n\|)$$
$$\leq C(k_1, l_1, \alpha_n)(\|x_n - \varphi_1(y_n)\| + (l_2/k_1)\beta_n \left\|y_n - P_2Tz_n'\right\|).$$

By the same argument as above, we find that

$$\left\|P_2(y_n - Tz_n')\right\|$$
$$\leq \left\|P_2(z_n' - \varphi_2(x_{n+1}))\right\| + \left\|P_2(Tz_n' - \varphi_2(x_{n+1}))\right\|$$
$$= (1+l_2)\|y_n - \varphi_2(x_{n+1})\|.$$

Therefore we have

$$\|y_{n+1} - y_n\| \leq (1+l_2)\beta_n \|y_n - \varphi_2(x_{n+1})\|$$

We also have by (T2) that

$$\begin{aligned}\|y_n-\varphi_2(x_{n+1})\| &\le \|y_n-\varphi_2(x_n)\|+\|\varphi_2(x_n)-\varphi_2(x_{n+1})\|\\ &\le \|y_n-\varphi_2(x_n)\|+(l_1/k_2)\,\|x_n-x_{n+1}\|\\ &\le \|y_n-\varphi_2(x_n)\|+(l_1/k_2)(l_1+1)\alpha_n\,\|x_n-\varphi_1(y_n)\|\,.\end{aligned}$$

Combining inequalities above, we find

$$\begin{aligned}&\|x_{n+1}-\varphi_1(y_{n+1})\|\\ &\quad\le C(k_1,l_1,\alpha_n)(1+(l_1l_2/k_1k_2)(1+l_2)(1+l_1)\alpha_n\beta_n)\,\|x_n-\varphi_1(y_n)\|\\ &\qquad\qquad+C(k_1,l_1,\alpha_n)(1+l_2)(l_2/k_1)\beta_n\,\|y_n-\varphi_2(x_n)\|\,.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}y_n &= y_{n+1}+\beta_n y_n-\beta_n P_2Tz_n'\\ &=(1+\beta_n)y_{n+1}+\beta_nP_2(I-T-k_1I)z_{n+1}-(2-k_1)\alpha_ny_{n+1}\\ &\qquad+\beta_ny_n+\beta_nP_1(Tz_{n+1}-Tz_n')\\ &=(1+\beta_n)y_{n+1}+\beta_nP_2(I-T-kI)z_{n+1}\\ &\qquad-(2-k_2)\beta_n\left[\beta_ny_n+(1-\beta_n)P_2Tz_n'\right]\\ &\qquad+\beta_ny_n+\beta_nP_2(Tz_{n+1}-Tz_n')\\ &=(1+\beta_n)y_{n+1}+\beta_nP_2(I-T-k_2I)z_{n+1}-(1-k_2)\beta_ny_n\\ &\qquad+(2-k_2)\beta_n^2P_2(z_n'-Tz_n')+\beta_nP_2(Tz_{n+1}-Tz_n').\end{aligned}$$

Recalling that $P_2T(x_{n+1}+\varphi_2(x_{n+1}))=\varphi_2(x_{n+1})$, we have that

$$\begin{aligned}&y_n-\varphi_2(x_{n+1})\\ &\quad=(1+\beta_n)(y_{n+1}-\varphi_2(x_{n+1}))+\beta_nP_2(I-T-k_2I)(z_{n+1}-\varphi_2(x_{n+1}))\\ &\qquad-(1-k_2)\beta_n(y_n-\varphi_2(x_{n+1}))+(2-k_2)\beta_n^2P_2(z_n'-Tz_n')\\ &\qquad+\beta_nP_2(Tz_n'-Tz_{n+1}).\end{aligned}$$

Then since $P_2T(x_{n+1}+\cdot)$ is strictly pseudocontractive, we have

$$\begin{aligned}&\|y_n-\varphi_2(x_{n+1})\|\\ &\quad\ge(1+\beta_n)\,\|y_{n+1}-\varphi_2(x_{n+1})\|-(1-k_2)\beta_n\,\|y_n-\varphi_2(x_{n+1})\|\\ &\qquad-(2-k_2)\beta_n^2\,\|P_2(z_n'-Tz_n')\|-\beta_n\,\|P_2(Tz_n'-Tz_{n+1})\|\,.\end{aligned}\tag{3}$$

Noting that $\|Tz_n'-\varphi_2(x_{n+1})\|\le l_2\,\|y_n-\varphi_2(x_{n+1})\|$, we have

$$\begin{aligned}\|P_2(z_n'-Tz_n')\| &\le \|P_2(y_n-\varphi_2(x_{n+1}))\|+\|P_2(Tz_n'-\varphi_2(x_{n+1}))\|\\ &=(1+l_2)\,\|y_n-\varphi_2(x_{n+1})\|\,.\end{aligned}$$

We also have

$$\|P_2(Tz_n'-Tz_{n+1})\|\le l_2(l_2+1)\beta_n\,\|y_n-\varphi_2(x_{n+1})\|\,.$$

Therefore we find

$$\begin{aligned}&\|y_n - \varphi_2(x_{n+1})\| \\ &\quad \geq (1+\beta_n)\|y_{n+1} - \varphi_2(x_{n+1})\| - (1-k_2)\beta_n \|y_n - \varphi_2(x_{n+1})\| \\ &\qquad - [(2-k_2)\beta_n^2(1+l_2) + l_2(1+l_2)\beta_n^2]\|y_n - \varphi_2(x_{n+1})\|.\end{aligned}$$

Consequently, we have

$$\|y_{n+1} - \varphi_2(x_{n+1})\| \leq C(k_2, l_2, \beta_n)\|y_n - \varphi_2(x_{n+1})\|,$$

where

$$C(k_2, l_2, \beta_n) = [1 + (1-k_2)\beta_n + (1+l_2)\beta_n^2(2-k_2+l_2)](1+\beta_n)^{-1}.$$

We can see from the definition of $\{x_n\}$ that

$$\|x_{n+1} - x_n\| \leq (1+l_1)\alpha_n \|x_n - \varphi_1(y_n)\|.$$

Then we have

$$\begin{aligned}&\|y_{n+1} - \varphi_2(x_{n+1})\| \\ &\quad \leq C(k_2, l_2, \beta_n)(\|y_n - \varphi_2(x_n)\| + \|\varphi_2(x_{n+1}) - \varphi_2(x_n)\|) \\ &\quad \leq C(k_2, l_2, \beta_n)(\|y_n - \varphi_2(x_n)\| + (l_1/k_2)\|x_{n+1} - x_n\|) \\ &\quad \leq C(k_2, l_2, \beta_n)(\|y_n - \varphi_2(x_n)\| + (l_1/k_2)\alpha_n(1+l_1)\|x_n - \varphi_1(y_n)\|.\end{aligned}$$

It then follows that

$$\begin{aligned}&\|x_{n+1} - \varphi_1(y_{n+1})\| + \|y_{n+1} - \varphi_2(x_{n+1})\| \\ &\quad \leq C(k_1, l_1, \alpha_n)(1 + (l_1 l_2/k_1 k_2)(1+l_2)(1+l_1)\alpha_n\beta_n)\|x_n - \varphi_1(y_n)\|) \\ &\qquad + C(k_1, l_1, \alpha_n)(1+l_2)(l_2/k_1)\beta_n\|y_n - \varphi_2(x_n)\| \\ &\qquad + C(k_2, l_2, \beta_n)(\|y_n - \varphi_2(x_n)\| + (l_1/k_2)\alpha_n(1+l_1)\|x_n - \varphi_1(y_n)\|.\end{aligned}$$

That is

$$\begin{aligned}&\|x_{n+1} - \varphi_1(y_{n+1})\| + \|y_{n+1} - \varphi_2(x_{n+1})\| \\ &\qquad \leq C_n\|x_n - \varphi_1(y_n)\| + D_n\|y_n - \varphi_2(x_n)\|,\end{aligned}$$

where

$$\begin{aligned}C_n = C(k_1, l_1, \alpha_n)&(1 + (l_1 l_2/k_1 k_2)(1+l_2)(1+l_1)\alpha_n\beta_n) \\ &+ C(k_2, l_2, \beta_n)(l_1/k_2)\alpha_n(1+l_1)\end{aligned}$$

and

$$D_n = C(k_2, l_2, \beta_n) + C(k_1, l_1, \alpha_n)(1+l_2)(l_2/k_1)\beta_n.$$

Let $\varepsilon > 0$ such that $\rho + \varepsilon < \min\{c/k_1, c/k_2\}$. Then recalling that $\lim_{n\longrightarrow\infty}\alpha_n = \lim_{n\longrightarrow\infty}\beta_n = 0$, we have, from the definition of $C(k_1, l_1, \alpha_n)$ and $C(k_2, l_2, \beta_n)$, that for n sufficiently large

$$C_n \leq 1 - k_1\alpha_n + (1+l_1)(l_1/k_2)\alpha_n + \varepsilon\alpha_n < 1 - \rho\alpha_n$$

and

$$D_n < 1 - \rho\beta_n.$$

Here we put

$$c_n = \|x_n - \varphi_1(y_n)\| + \|y_n - \varphi_2(x_n)\| \qquad \text{for } n \geq 1.$$

Then from the inequality above, we find

$$c_{n+1} \leq (1-\rho\gamma_n)c_n \qquad \text{for } n \text{ sufficiently large.}$$

That is, there exists $n_0 \geq 1$ such that for each $n \geq n_0$,

$$c_{n+1} \leq \Pi_{k=n_0}^{n}(1-\rho\gamma_k)c_{n_0}. \tag{4}$$

Since $\sum \gamma_n = \infty$, we have that $\lim_{n\longrightarrow\infty} \Pi_{k=n_0}^{n}(1-\rho\gamma_k) = 0$. This implies that $\lim_{n\longrightarrow\infty} c_n = 0$. That is

$$\lim_{n\longrightarrow\infty}(\|x_n - \varphi_1(y_n)\| + \|y_n - \varphi_2(x_n)\|) = 0. \tag{5}$$

Here we recall that

$$\|x_{n+1} - x_n\| \leq (l_1+1)\alpha_n \|x_n - \varphi_1(y_n)\| \qquad \text{for all } n \geq 1.$$

Then we have by (4) that for each $n > n_0$ and $m \geq 1$,

$$\begin{aligned}\|x_{n+m} - x_n\| &\leq \sum_{k=1}^{m} \|x_{n+k} - x_{n+k-1}\| \\ &\leq (l_1+1)\sum_{k=1}^{m} \alpha_{n+k-1} \|x_{n+k-1} - \varphi_1(y_{n+k-1})\| \\ &\leq (l_1+1)\sum_{k=1}^{m} \alpha_{n+k-1}\Pi_{j=n_0}^{n+k-1}(1-\rho\gamma_j).\end{aligned}$$

Then by the assumption, we have that

$$\lim_{n\longrightarrow\infty} \|x_{n+m} - x_n\| = 0 \qquad \text{uniformly for } m \geq 1.$$

Thus we find that $x_n \longrightarrow x \in H$ strongly. Similarly, we have that $y_n \longrightarrow y \in H$ strongly. It then follows from (5) that

$$x = \varphi_2(y) \qquad \text{and } y = \varphi_1(x).$$

That is $T(x+y) = x+y$. This completes the proof. □

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