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ON THE MEMORY OF ATOMIC OPERATORS

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ABSTRACT. Properties of the *memory* of operator are considered. Roughly speaking, the *memory* is an information about the preimages the operator remembers given some information about images. We show that some properties of the *memory* allow to single out two classes of nonlinear operators generalizing the notion of local operator between ideal function spaces. The first class, named *atomic*, contains in particular all the linear shifts (inner superpositions), while the second one, called *coatomic*, contains all the adjoints to the atomic operators, and, in particular, the conditional expectations. Both classes include local (in particular, Nemytskiĭ) operators and are closed with respect to compositions of operators. Results about representation of operators of both classes are provided.

1. INTRODUCTION

The question whether it is possible to represent a σ-homomorphism between two measure spaces by an inverse of a measurable point map has by now rather long history. The first and very remarkable result on this topic can be traced back to the paper by J. von Neumann [20] and to his subsequent paper with P.R. Halmos [9]. In these papers it was proven, in modern terms, that (up to some insignificant details) every σ-automorphism of standard measure space is representable by a Borel measurable bijection. Further generalizations as well as some references on representation of automorphisms of measure spaces can be found in [2]. Another principal result in this direction was proven by R. Sikorski in [19]: he showed that a σ-homomorphism of a standard measure space into an arbitrary measure space is representable by a measurable point transformation, which in this case is, generally speaking, not invertible.

The mentioned principles provide a powerful tool in representation theory of linear operators. For instance, one of the most famous results obtained with the essential use of this tool, was the theorem on representation of isometries of Lebesgue spaces as *weighted shift* operators proven by J. Lamperti in [11] (see also chapter 15.5 of [17] and chapter 10 of [16], where the dependence of this representation result on one of the above principles is made even clearer). Another consequence of the same tool is the class of statements on representation of linear *disjointness preserving* operators (also called *D-operators* [21] or Riesz homomorphisms [22]), i.e. the operators mapping functions with disjoint supports into functions with disjoint supports, in various spaces of measurable functions as weighted shifts. Such a representation theorem can be easily proven for all linear continuous in measure D-operators in the space of classes of measurable functions with the usual topology of convergence in measure. This result in fact gave rise to many profound generalizations and is widely used in many modern mathematical theories, especially in the

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theory of Banach lattices. For example, a general representation theorem for linear order continuous D-operators as weighted shifts in Banach lattices of measurable functions can be found in [22]. However, most of the generalizations have been obtained in the framework of the *linear* operator theory, while only few particular results are known as far as nonlinear operators are concerned.

We are inclined to think that it is unreasonable to expect as beautiful result on representation of general nonlinear D-operators like as in the linear case. A step in this direction was undertaken in [3]. A new class of operators (called *atomic*), providing such a reasonable generalization of inear D-operators to the nonlinear case and inheriting many nice properties from the linear theory, was introduced there. It was shown that the operators of this class arize naturally in various applications like e.g. functional-differential equations or description of stochastic periodic processes. All the constructions were heavily based on the notion of the *memory* of an operator, which was introduced in the paper. Roughly speaking, it is a piece of information about the preimages the operator is able to remember given a piece of information about the images. One easily understands this definition while considering the classical notion of *local* (or locally defined) operators [18], which in particular includes *Nemytskiĭ* operators (also called superposition operators) [1], stochastic integrals [13] in Lebesgue spaces, and differential operators in the spaces of smooth functions [21, 10]. In fact, one easily observes that the well-known definition of a local operator by I.V. Shragin [18] involves only the "structure of the memory". The definition of atomic operators follows the same idea and, moreover, includes that of the local operators. Moreower, the considerations based on the notion of the memory enable the authors of [3] to define another interesting class of operators (called *coatomic*), which in a sense is dual to the class of atomic operators. It includes, for instance, conditional expectations as well as all adjoint operators to weighted shifts. The local operators may be viewed then as a particular case of both atomic and coatomic operators.

In the present paper we will show that some properties of the operators' memory allow to define σ-homomorphisms, generating *atomic* and *coatomic* operators and to describe the structure of these operators.

Thus, unlike in [3], our definitions of *atomic* and *coatomic* operators already are not based on an apriori assumption on existence of a corresponding σ-homomorphism.

2. Notation and preliminaries

Let $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ be two measure spaces, and $\Sigma_1^0 \subset \Sigma_1$, $\Sigma_2^0 \subset \Sigma_2$ be the σ-ideals of μ_1- and μ_2-nullsets respectively. We denote by $\tilde{\Sigma}_i := \Sigma_i/\Sigma_i^0$, $i = 1, 2$ the respective *measure algebrae* (see § 42 of [19]). For the elements of $\tilde{\Sigma}_i$ (i.e. the equivalence classes of sets) will be denoted $\tilde{e}_i$ or $[e_i]$, $i = 1, 2$. Further on we will however frequently abuse the notation and identify the elements of the measure algebrae $\tilde{\Sigma}_i$ with the elements of the respective original σ-algebrae of sets Σ_i. A map F: $\tilde{\Sigma}_1 \to \tilde{\Sigma}_2$ is called a *σ-homomorphism*, if $F(\Omega_1) = \Omega_2$, $F(\Omega_1 \backslash e) = \Omega_2 \backslash F(e)$

whenever $e \in \tilde{\Sigma}_1$ and

$$F\left(\bigsqcup_{i=1}^{\infty} e_i\right) = \bigsqcup_{i=1}^{\infty} F(e_i),$$

for any pairwise disjoint collection of $\{e_i\}_{i=1}^{\infty} \subset \tilde{\Sigma}_1$, where $\bigsqcup$ stands for the disjoint union. Every (Σ_2, Σ_1)-measurable map $g\colon \Omega_2 \to \Omega_1$ satisfying

$$\mu_2(g^{-1}(e_1)) = 0 \text{ when } \mu_1(e_1) = 0 \tag{2.1}$$

generates a σ-homomorphism according to the formula $F(\tilde{e}_1) := [g^{-1}(e_1)]$. The latter σ-homomorphism is said to be induced by a point map g, and in this case we will write $F = g^{-1}$.

A measure space (Ω, Σ, μ) is called standard, if Ω is a Polish space, Σ is either the Borel σ-algebra or its completion with respect to finite Borel measure μ.

All the measure spaces we will be dealing with in the sequel are assumed to be complete, and, for the sake of simplicity, the measures will be supposed to be finite. Further, the notation $L^p(\Omega, \Sigma, \mu; \mathcal{X})$, where $\mathcal{X}$ is a separable Banach space, will stand, as usual, for the classical Lebesgue space of $\mathcal{X}$-valued functions measurable with respect to Σ and μ-summable with power p (if $p \in [1, +\infty)$) or μ-essentially bounded (if $p = +\infty$). These spaces are silently assumed to be equipped with their strong topologies. If $\mathcal{X}$ is a separable metric space, then $L^0(\Omega, \Sigma, \mu; \mathcal{X})$ stands for the metric space of $\mathcal{X}$-valued functions measurable with respect to Σ equipped with the topology of convergence in measure.

Whenever there is no possibility for confusion, the references to $\mathcal{X}$, Ω, Σ and/or μ will be omitted. We will also omit in sequel sign $(\tilde{\ })$, assuming that all considerations are done modulo equivalence classes of sets.

3. Memory and comemory of an operator

Let $X_i := X(\Omega_i, \Sigma_i, \mu_i; \mathcal{X}_i)$, $i = 1, 2$. Consider an operator $T\colon X_1 \to X_2$. Following [3], we introduce now the concept of *memory* and the related concept of *comemory* which are basic for our study.

Definition 3.1. We call the **memory** of an operator $T\colon X_1 \to X_2$ on a set $e_2 \in \Sigma_2$ the family of all possible $e_1 \in \Sigma_1$ such that for any $x, y \in X_1$ satisfying $x\,|_{e_1} = y\,|_{e_1}$ it follows that $T(x)\,|_{e_2} = T(y)\,|_{e_2}$. In other words,

$$\mathrm{Mem}_T(e_2) := \{e_1 \in \Sigma_1 \,:\, x\,|_{e_1} = y\,|_{e_1} \;\Rightarrow\; T(x)\,|_{e_2} = T(y)\,|_{e_2}\},$$

Similarly, the **comemory** of operator T on a set $e_1 \in \Sigma_1$ is the family

$$\mathrm{Comem}_T(e_1) := \{e_2 \in \Sigma_2 \,:\, x\,|_{e_1} = y\,|_{e_1} \;\Rightarrow\; T(x)\,|_{e_2} = T(y)\,|_{e_2}\}.$$

Recall that according to our convention all the equalities in the above definition should be understood in almost everywhere sense.

It is clear from the definitions that

$$e_1 \in \mathrm{Mem}_T(e_2) \Longleftrightarrow e_2 \in \mathrm{Comem}_T(e_1).$$

Example 3.1. Let $\mathcal{X} = \mathcal{X}_1 = \mathcal{X}_2$. Define a *shift* operator $T_g\colon L^0(\Omega_1, \Sigma_1, \mu_1; \mathcal{X}) \to L^0(\Omega_2, \Sigma_2, \mu_2; \mathcal{X})$ (sometimes also called inner superposition) by the formula

$$(T_g x)(\omega_2) := x(g(\omega_2)) \tag{3.1}$$

where $g\colon \Omega_2 \to \Omega_1$ is a given (Σ_2, Σ_1)-measurable function. For this operator to be well-defined on the classes of measurable functions we require

$$e_1 \in \Sigma_1,\ \mu_1(e_1) = 0 \Rightarrow \mu_2(g^{-1}(e_1)) = 0. \tag{3.2}$$

Then

$$\mathrm{Mem}_{T_g}(e_2) = \{e_1 \in \Sigma_1 \,:\, e_1 \supset g(e_2)\}.$$

Example 3.2. Let $\Omega \subset \mathbb{R}$ be a compact set supplied with the Lebesgue measure μ, Σ being the respective σ-algebra of measurable subsets of Ω, $\mathcal{X} = \mathbb{R}$. We define operator $K : L^1(\Omega) \to L^1(\Omega)$ by the formula

$$(Kx)(t) := \int_\Omega K(t,s)x(s)\,ds, \quad t \in \Omega,$$

where the function $K(\cdot,\cdot)$ is measurable, positive and essencially bounded on $\Omega \times \Omega$. Then

$$\mathrm{Mem}_K(\mathcal{E}) = \begin{cases} \{\Omega\}, & \mu(\mathcal{E}) \neq 0, \\ \Sigma, & \mu(\mathcal{E}) = 0, \end{cases}$$

Let us present several properties of *memory* and *comemory* (they will appear together with the proofs in the forthcoming paper [4]).

We have the following obvious properties of the *comemory.*

Proposition 3.1. *For every operator $T\colon X_1 \to X_2$ and for all $e_1 \in \Sigma_1$ the following holds:*

(i) *if $e_2 \in Comem_T(e_1)$ and $e_2' \subset e_2$, $e_2' \in \Sigma_2$, then $e_2' \in Comem_T(e_1)$. In particular, $\emptyset \in Comem_T(e_1)$;*
(ii) *$Comem_T(\Omega_1) = \Sigma_2$;*
(iii) *$Comem_T(e_1)$ is closed under at most countable unions of its elements;*
(iv) *$e_1 \subset e_1'$ implies $Comem_T(e_1) \subset Comem_T(e_1')$;*
(v) *the family $Comem_T(e_1)$ contains maximum element (called "the maximum comemory") with respect to the inclusion.*

Remark 3.1. In other terms (see § 3 of [19]), the conditions (i) and (ii) mean that for all $e_1 \in \Sigma_1$ the family $\mathrm{Comem}_T(e_1)$ is a *σ-ideal.*

Below we list some similar properties of *memory.* The omitted proof is straightforward.

Proposition 3.2. *For every operator $T\colon X_1 \to X_2$ and for all $e_2 \in \Sigma_2$ the following holds:*

(i) *if $e_1 \in Mem_T(e_2)$ and $e_1 \subset e_1'$, $e_1' \in \Sigma_1$, then $e_1' \in Mem_T(e_2)$. In particular, $\Omega_1 \in Mem_T(e_2)$;*
(ii) *$Mem_T(e_2)$ is closed under finite intersections of its elements.*

Remark 3.2. Similarly to the case of comemory, the above statement asserts (see § 3 of [19]), that the for all $e_2 \in \Sigma_2$ the family $\mathrm{Mem}_T(e_2)$ is a *filter.*

Note that $\mathrm{Mem}_T(e_2)$ does not need to be closed under countable intersection of its elements and to contain a minimum element with respect to the inclusion (i.e. it is, generally speaking, not a *σ-filter*), as the example below shows.

Example 3.3. Let $\Omega \subset \mathbb{R}^n$ be a compact set supplied with the n-dimensional Lebesgue measure μ, Σ being the respective σ-algebra of measurable subsets of Ω. We define operator $T\colon L^\infty(\Omega) \to L^\infty(\Omega)$ by the formula

$$(Tx)(\omega) := \limsup_{r\to 0^+} \frac{1}{\mu(B_r(x_0))} \int_{B_r(x_0)} x(s)\, ds \cdot 1(\omega),$$

where $B_r(x_0) \subset \mathbb{R}^n$ stands for the ball of radius $r > 0$ centered at $x_0 \in \operatorname{int}\Omega$. This operator is nonlinear, bounded, but discontinuous. One has, obviously, $B_r(x_0) \in \mathrm{Mem}_T(\Omega)$ for all $r > 0$ small enough, but

$$[\{x_0\}] = \emptyset \notin \mathrm{Mem}_T(\Omega).$$

Note that the above example was only possible because the operator was taken to be discontinuous in measure. On the other hand, the following statement is valid.

Proposition 3.3. *For every continuous operator*

$$T : L^0(\Omega_1, \Sigma_1, \mu_1; \mathcal{X}_1) \to L^0(\Omega_2, \Sigma_2, \mu_2; \mathcal{X}_2)$$

and for all $e_2 \in \Sigma_2$ the following holds:

(i) *$Mem_T(e_2)$ is closed under at most countable intersections of its elements (and therefore, is a σ-filter);*
(ii) *$Mem_T(e_2)$ contains minimum element (called "the minimum memory") with respect to the inclusion.*

Example 3.4. The minimum memory, i.e. the minimum element of the set

$$\mathrm{Mem}_{T_g}(e_2) = \{e_1 \in \Sigma_1 \,:\, e_1 \supset g(e_2)\}$$

in the Example 3.1 is given by the unique element $E \in \Sigma_2$ such that $E \supset g(e_2)$ and $\mu_2(E)$ is equal to the outer measure of the (not necessarily measurable) set $g(e_2)$.

4. Local, atomic and coatomic operators

Assume fixed a measure space (Ω, Σ, μ) and define $X_i = L^0(\Omega, \Sigma, \mu; \mathcal{X}_i)$, $i = 1, 2$.

In this section we recall the notions of **local**, **atomic** and **coatomic** operators introduced in [3] together with some examples and propositions helping deaper understanding of the nature of the operators under considereation.

Definition 4.1. An operator $T\colon X_1 \to X_2$ is called **local**, if

$$e \in \mathrm{Mem}_T(e)$$

for all $e \in \Sigma$, that is, if $x\,|_e = y\,|_e$ for $x, y \in X_1$ implies $T(x)\,|_e = T(y)\,|_e$.

The following example is classical.

Example 4.1. Let $\mathcal{X}_1$ and $\mathcal{X}_2$ be separable metric spaces, $f\colon \Omega \times \mathcal{X}_1 \to \mathcal{X}_2$ be a sup-measurable function (i.e. $f(\cdot, x(\cdot))$ is μ-measurable whenever $x(\cdot)$ is μ-measurable). Then the Nemytskiĭ operator $N\colon L^0(\Omega, \Sigma, \mu; \mathcal{X}_1) \to L^0(\Omega, \Sigma, \mu; \mathcal{X}_2)$ (commonly known also under the name *superposition operator* [1]), defined by

$$(Nx)(\omega) := f(\omega, x(\omega))$$

is local. If $f\colon \Omega \times \mathcal{X}_1 \to \mathcal{X}_2$ is a Carathéodory function (i.e. $f(\omega, \cdot)$ is continuous for μ-almost every $\omega \in \Omega$ and $f(\cdot, x)$ is μ-measurable for all $x \in \mathbb{R}$), then the Nemytskiĭ operator N becomes continuous in measure (i.e. as an operator in L^0).

Obviously, the class of local operators is closed under compositions and finite sums (when the latter are defined, e.g. when $\mathcal{X}_2$ is an additive group).

The above general definition of local operators is due to I.V. Shragin [18] who called them *locally defined*. We just reformulated his definition using the introduced notion of memory. The theory of local operators constitutes an important new chapter of functional analysis with applications in stochastic analysis and differential equations (see e.g. [12, 13, 14, 21]). The reader is referred to these works for the classification theory of local operators, their properties, further examples, etc.

Now we recall another definition generalizing the notion of local operator.

Here$X_i = X_i(\Omega_i, \Sigma_i, \mu_i; \mathcal{X}_i), \quad i = 1, 2.$

Definition 4.2. An operator $T\colon\ X_1 \to X_2$ is called **atomic**, if there is a σ-homomorphism $F\colon\ \Sigma_1 \to \Sigma_2$, satisfying

$$[F(e_1)] \in \mathrm{Comem}_T(e_1) \tag{4.1}$$

for all $e_1 \in \Sigma_1$, that is, if from $x\,|_{e_1} = y\,|_{e_1}$ for $x, y \in X_1$ follows $T(x)\,\big|_{F(e_1)} = T(y)\,\big|_{F(e_1)}$.

Obviously, every local operator is atomic. However, the class of atomic operators is richer, as one can conclude from the following example.

Example 4.2. Every shift operator (see example 3.1)

$$T_g\ :\ L^0(\Omega_1, \Sigma_1, \mu_1; \mathcal{X}) \to L^0(\Omega_2, \Sigma_2, \mu_2; \mathcal{X})$$

is atomic. To show this, it is enough to set $F(e_1) := g^{-1}(e_1)$.

The class of atomic operators is obviously closed under compositions.

Example 4.3. Let Σ_2 be a σ-algebra of Borel subsets (or its μ_2-completion) of a metric space Ω_2, $\mathcal{X}_1$ and $\mathcal{X}_2$ be separable metric spaces, $f\colon\ \Omega_2 \times \mathcal{X}_1 \to \mathcal{X}_2$ be a Carathéodory function, and $g\colon\ \Omega_2 \to \Omega_1$ be a (Σ_2, Σ_1)-measurable function satisfying (3.2). Then the operator $T\colon\ L^0(\Omega_1, \Sigma_1, \mu_1; \mathcal{X}_1) \to L^0(\Omega_2, \Sigma_2, \mu_2; \mathcal{X}_2)$ defined by the relationship

$$(Tx)(\omega_2) := f(\omega_2, x(g(\omega_2)))$$

is atomic as a composition of a Nemytskiĭ operator

$$N\ :\ y(\cdot) \mapsto f(\cdot, y(\cdot)),$$

which is local, and a shift T_g.

Another concept which seems to be interesting to study, is given by the following definition. Again, here $X_i := X_i(\Omega_i, \Sigma_i, \mu_i; \mathcal{X}_i)$, $i = 1, 2$.

Definition 4.3. An operator $T\colon\ X_1 \to X_2$ is called **coatomic**, if there is a σ-homomorphism $\Phi\colon\ \Sigma_2 \to \Sigma_1$, satisfying

$$[\Phi(e_2)] \in \mathrm{Mem}_T(e_2) \tag{4.2}$$

for all $e_2 \in \Sigma_2$, that is, if from $x\,\big|_{\Phi(e_2)} = y\,\big|_{\Phi(e_2)}$ for $x, y \in X_1$ follows $T(x)\,|_{e_2} = T(y)\,|_{e_2}$.

It is not difficult to observe that a notion of a *coatomic* operator is in certain sense dual to the notion of an *atomic* operator. However, both classes contain local operators. Certainly, the class of coatomic operators is wider than that of local operators.

Example 4.4. Assume that $g\colon \Omega_2 \to \Omega_1$ is a bijection satisfying (3.2) and the inverse function $g^{-1}\colon \Omega_1 \to \Omega_2$ has the same property:

$$e_2 \in \Sigma_2,\ \mu_2(e_2) = 0 \Rightarrow \mu_1(g(e_2)) = 0. \tag{4.3}$$

Then the corresponding shift operator

$$T_g\,:\,L^0(\Omega_1,\Sigma_1) \to L^0(\Omega_2,\Sigma_2)$$

is coatomic, for one can take $\Phi(e) := g(e)$.

In relation to the latter example we observe that, unlike the class of *atomic* operators, the class of *coatomic* operators contains only rather particular shifts, namely, only those described in the example 4.4.

In [3] the following statement is proved.

Proposition 4.1. *Let $(\Omega_2,\Sigma_2,\mu_2)$ be a standard measure space and a (Σ_1,Σ_2)-measurable function $g\colon \Omega_2 \to \Omega_1$ satisfy (3.2). Then the shift operator*

$$T_g\,:\,L^0(\Omega_1,\Sigma_1,\mu_1) \to L^0(\Omega_2,\Sigma_2,\mu_2)$$

is coatomic, if and only if g is μ_2-equivalent to a bijection and satisfies (4.3).

Thus we see that the class of *coatomic* operators does not coincide with that of *atomic* operators. Moreover, it is not contained in the latter one. To prove this, we need the following statement which seems to be of independent interest.

Proposition 4.2. *Let $\mathcal{X}_1$ and $\mathcal{X}_2$ be reflexive Banach spaces. A linear bounded operator*

$$T\,:\,L^p(\Omega_1,\Sigma_1,\mu_1;\mathcal{X}_1) \to L^q(\Omega_2,\Sigma_2,\mu_2;\mathcal{X}_2),$$

$1 \le p,q < +\infty$*, is coatomic (resp. atomic), if and only if its adjoint*

$$T'\,:\,L^{q'}(\Omega_2,\Sigma_2,\mu_2;\mathcal{X}_2') \to L^{p'}(\Omega_1,\Sigma_1,\mu_1;\mathcal{X}_1')$$

is atomic (resp. coatomic).

The proof can also be found in [3].

5. Short memory and full comemory operators

The properties of atomic and coatomic operators are studied in details in [3]. It is shown that both classes inherit from Nemytskiĭ operator the properties of noncompactness in measure and weak degeneracy, while having different relationships of acting, continuity and boundedness, as well as different convergence properties.

Let us point out, that the definition of atomic (coatomic) operator includes the assumption of the existence of some homomorphism satisfying (4.1) ((4.2) respectively). However, one can define some properties of the operators' memory which will garantee the existence of a corresponding homomorphism.

Here again $X_i := X_i(\Omega_i,\Sigma_1,\mu_i;\mathcal{X}_i), \quad i = 1,2.$

Definition 5.1. Operator $T : X_1 \to X_2$ is called **short memory**, if for any $\varepsilon > 0$ there exists $\delta > 0$, such that for any $e_2 \in \Sigma_2$ the condition

$$\mu_2(e_2) < \delta$$

implies the existence of $e_1 \in \mathrm{Mem}_T(e_2)$, satisfying

$$\mu_1(e_1) < \varepsilon.$$

Definition 5.2. Operator $T : X_1 \to X_2$ is called **full comemory**, if for any collection $\{e_{1i} \in \Sigma_1\}$, satisfying the following condition

$$\Omega_1 = \cup_i e_{1i}, \tag{5.1}$$

there exists a collection $\{e_{2i} \in \Sigma_2\}, \quad e_{2i} \in \mathrm{Comem}_T(e_{1i})$, such that the following equality

$$\Omega_2 = \cup_i e_{2i}, \tag{5.2}$$

holds.

Theorem 5.1. *Let operator $T : X_1 \to X_2$ be full comemory and let the following condition be fulfilled:*

$$\mathit{Comem}_T(\emptyset_{\mu_1}) = \emptyset_{\mu_2}. \tag{5.3}$$

Then T is atomic.

Proof. Let us define map $F : \Sigma_1 \to \Sigma_2$ as follows

$$F(e_1) = \max \mathrm{Comem}_T(e_1), \quad e_1 \in \Sigma_1. \tag{5.4}$$

Let us show, that the map defined above is a homomorphism.

Indeed, the conditions of the theorem imply that

$$F(\Omega_1) = \Omega_2.$$

Furthermore, let $e_1 \in \Sigma_1$. Then, on one hand,

$$F(\Omega_1 \setminus e_1 \cup e_1) = \Omega_2,$$

and on the other hand (from the definition of **full comemory**)

$$F(\Omega_1 \setminus e_1 \cup e_1) \supset F(\Omega_1 \setminus e_1) \cup F(e_1).$$

Since

$$(\Omega_1 \setminus e_1) \cap e_1 = \emptyset_{\mu_1},$$

then taking into account (5.3)

$$F(\Omega_1 \setminus e_1) \cap F(e_1) = \emptyset_{\mu_2}.$$

Thus,

$$F(\Omega_1 \setminus e_1) = \Omega_2 \setminus F(e_1).$$

the correctness of the equality

$$F(\bigsqcup_{i=1}^{\infty} e_i) = \bigsqcup_{i=1}^{\infty} F(e_i)$$

for each pairwise disjoint collection of μ_1 measurable sets $\{e_i\}_{i=1}^{\infty}$ can be proved by induction. The reference to the definition of *atomic* operator completes the proof. □

Let us discuss the essence of condition (5.3). Define for the operator $T : X_1(\Omega_1, \Sigma_1, \mu_1; \mathcal{X}_1) \to X_2(\Omega_2, \Sigma_2, \mu_2; \mathcal{X}_2)$ a set $\Lambda \in \Sigma_2$ by the following equality

$$\Lambda = \max \mathrm{Comem}_T(\emptyset_{\mu_1}). \tag{5.5}$$

If $\mu_2(\Lambda) > 0$, then we may put into correspondence to the operator $T : X_1 \to X_2$ a function $\varphi_T : \Lambda \to \mathcal{X}_2$ such that for any function $x \in X_1$ the equality

$$(Tx)(t) = \varphi_T(t), \quad t \in \Lambda \tag{5.6}$$

holds. Let us point out that if $T : X_1 \to X_2$ is a linear operator, then

$$\varphi_T(t) = 0, \quad t \in \Lambda. \tag{5.7}$$

Further, let us define an operator $\tilde{T} : X_1(\Omega_1, \Sigma_1, \mu_1; \mathcal{X}_1) \to X_2(\Omega_2 \setminus \Lambda, \Sigma_2(\Omega_2 \setminus \Lambda), \mu_2; \mathcal{X}_2)$ as follows:

$$(\forall x \in X_1) \quad (\tilde{T}x)(t) = (Tx)(t), \quad t \in \Omega_2 \setminus \Lambda. \tag{5.8}$$

Here $\Sigma_2(\Omega_2 \setminus \Lambda)$ is a restriction of Σ_2 on $\Omega_2 \setminus \Lambda$.

In these notations the operator $T : X_1 \to X_2$ can be represented in the form:

$$(\forall x \in X_1) \quad (Tx)(t) = \begin{cases} (\tilde{T}x)(t), & t \in \Omega_2 \setminus \Lambda, \\ \varphi_T(t), & t \in \Lambda. \end{cases} \tag{5.9}$$

Moreover, for $\tilde{T}$ (5.3) holds, i.e.

$$\mathrm{Comem}_{\tilde{T}}(\emptyset_{\mu_1}) = \emptyset_{\mu_2}. \tag{5.10}$$

Corollary 5.1. *Let $T : X_1(\Omega_1, \Sigma_1, \mu_1; \mathcal{X}_1) \to X_2(\Omega_2, \Sigma_2, \mu_2; \mathcal{X}_2)$ be* **full comemory**. *Then $\tilde{T} : X_1(\Omega_1, \Sigma_1, \mu_1; \mathcal{X}_1) \to X_2(\Omega_2 \setminus \Lambda, \Sigma_2(\Omega_2 \setminus \Lambda), \mu_2; \mathcal{X}_2)$, defined by (5.8) is* **atomic**.

Indeed, if T is **full comemory** then $\tilde{T}$ is also **full comemory**. Moreover, $\tilde{T}$ satisfies (5.10) in virtue of the definition of the set Λ. Then it follows from Theorem 5.1 that $\tilde{T}$ is **atomic** with respect to the homeomorphism $\tilde{F} : \Sigma_1 \to \Sigma_2(\Omega_2 \setminus \Lambda)$ defined by the equality:

$$\tilde{F}(e) = \max \mathrm{Comem}_{\tilde{T}}(e_1), \quad e_1 \in \Sigma_1.$$

Theorem 5.2. *If operator $T : X_1 \to X_2$ is* atomic *then it is* **full comemory**.

Proof. Let collection $\{e_{1i}\}$ be such that

$$\Omega_1 = \cup_i e_{1i}.$$

Let us now construct a collection $\{e'_{1i}\}$ according to the following rule:

$$e'_{11} = e_{11}, \quad e'_{12} = e_{12} \setminus e_{11}, ..., e'_{1i} = e_{1i} \setminus \cup_{j=1}^{i-1} e_{1j}, ...$$

Clearly,

$$\Omega_1 = \cup_i e'_{1i}.$$

Moreover, $\{e'_{1i}\}$ consists of mutually disjoint sets.

Let $F : \Sigma_1 \to \Sigma_2$ be a homomorphism, involved in the definition of *atomic* operator $T : X_1 \to X_2$. Then

$$\Omega_2 = \cup F(e'_{1i}).$$

Since

$$F(e'_{1i}) \subset F(e_{1i}),$$

then

$$\Omega_2 = \cup F(e_{1i}).$$

This means that $T : X_1 \to X_2$ is a **full comemory** operator. $\square$

Theorems 5.1 and 5.2 allow to derive (compare with [3]) a representation of full comemory operator and its' properties.

Let us point out, that the sum of **full comemory** operators is, in general, not **full comemory** as the following example shows.

Example 5.1. Let us consider operator $T : L_[0,1] \to L_[0,1]$,

$$(Tx)(t) = x(t) + x(\frac{1}{2}t), \quad t \in [0,1].$$

It could be checked easily that

$$\mathrm{Comem}_T((\frac{1}{2},1)) = \emptyset,$$

$$\max \mathrm{Comem}_T((0,\frac{1}{2})) = (0,\frac{1}{2}).$$

Thus, T is not **full comemory**.

Let $E_j \in \Sigma_j, \quad j = 1,2$. Let us define by $\Sigma_j(E_j)$ a restriction of Σ_j on E_j, $j = 1,2$. For any $e_1 \in \Sigma_1$ define a family $\mathrm{Comem}(E_2)_T(e_1)$ as follows:

$$\mathrm{Comem}(E_2)_T(e_1) = \{e_2 \in \Sigma_2(E_2) : x|_{e_1} = y_{e_1} \Rightarrow Tx|_{e_2} = Ty|_{e_2}\}. \tag{5.11}$$

If $E_2 = \Omega_2$, then E_2 in (5.11) will be omitted, in order to adjust (5.11) with Definition 3.1.

Definition 5.3. We say that operator $T : X_1 \to X_2$ satisfies **I**-condition ($T \in$ **I**-condition) if there exists a collection of disjoint sets $E_{2i}, \quad E_{2i} \in \Sigma_2, \quad i = 1,2,...,$ such that

$$\Omega_2 = \bigsqcup_i E_{2i},$$

and for any set $e_2 \in \Sigma_2$ the following equalities hold:

$$\max \mathrm{Comem}(E_{2i})_T[inf \mathrm{Mem}_T(e_2 \cap E_{2i})] = e_2 \cap E_{2i}, \quad i = 1,2,.... \tag{5.12}$$

Let

$$E_{1i} := [\inf \mathrm{Mem}_T(E_{2i})], \quad i = 1,2,.... \tag{5.13}$$

Theorem 5.3. *Let $T \in$ **I**-condition and*

$$Mem_T(\emptyset_{\mu_2}) = \Sigma_1. \tag{5.14}$$

Then a map

$$\Phi_i : \Sigma_2(E_{2i}) \to \Sigma_1(E_{1i}), \quad i = 1,2,...,$$

defined by

$$(\forall e_{2i} \in \Sigma_2(E_{2i})) \quad \Phi_i(e_{2i}) = [\inf Mem_T(e_{2i})], \quad i = 1,2,..., \tag{5.15}$$

is a homomorphism. Moreover, the operator

$$T_i : X_1(E_{1i}, \Sigma_1(E_{1i}), \mu_1; \mathcal{X}_1) \to X_2(E_{2i}, \Sigma_2(E_{2i}), \mu_2; \mathcal{X}_2), \quad i = 1,2,...,$$

defined as the corresponding restriction of the operator

$$T : X_1(\Omega_1, \Sigma_1, \mu_1; \mathcal{X}_1) \to X_2(\Omega_2, \Sigma_2, \mu_2; \mathcal{X}_2),$$

is coatomic with respect to this homomorphism.

Proof. Let us show that under conditions of the theorem, the map $\Phi_i, \quad i = i, 2, ...,$ defined by (5.14), is a homomorphism.

STEP 1. In virtue of the definition,

$$\Phi_i(E_{2i}) = E_{1i}. \tag{5.16}$$

STEP 2. Let us assume that the sets $e'_{2i}, e''_{2i} \in \Sigma_2(E_{2i})$ are disjoint:

$$e'_{2i} \cap e''_{2i} = \emptyset_{\mu_2}. \tag{5.17}$$

Then the intersection of their images under map Φ_i is also empty:

$$\Phi_i(e'_{2i}) \cap \Phi_i(e''_{2i}) = \emptyset_{\mu_1}. \tag{5.18}$$

Indeed, if we assume that

$$e_{1i} = [\Phi_i(e'_{2i}) \cap \Phi_i(e''_{2i})] \neq \emptyset_{\mu_1},$$

then the set

$$\max \mathrm{Comem}(E_{2i})_T(e_{1i})$$

belongs to e'_{2i} and e''_{2i} simultaneously. Moreover, in virtue of (5.14), this set is not empty. The last statement contradicts our assumption (5.17) and thus proves the correctness of (5.18).

STEP 3. Let $e_{2i} \in \Sigma_2(E_{2i})$. Then

$$\Phi_i([E_{2i} \backslash e_{2i}] \cup e_{2i}) = \Phi_i([E_{2i} \backslash e_{2i}]) \cup \Phi_i(e_{2i}). \tag{5.19}$$

Indeed, the inclusion

$$E_{1i} \supset \Phi_i[E_{2i} \backslash e_{2i}] \cup \Phi_i(e_{2i}) \tag{5.20}$$

is evident. Let us prove the reverse inclusion:

$$E_{1i} \subset \Phi_i[E_{2i} \backslash e_{2i}] \cup \Phi_i(e_{2i}). \tag{5.21}$$

Suppose that

$$E_{1i} \backslash \{\Phi_i[E_{2i} \backslash e_{2i}] \cup \Phi_i(e_{2i})\} = e_{1i}, \tag{5.22}$$

and moreover,

$$e_{1i} \neq \emptyset_{\mu_1}.$$

Then, in virtue of (5.14),

$$\max \mathrm{Comem}(E_{2i})_T(e_{1i}) \neq \emptyset_{\mu_2}.$$

On the other hand,

$$\max \mathrm{Comem}(E_{2i})_T(e_{1i}) \notin E_{2i} \backslash e_{2i},$$
$$\max \mathrm{Comem}(E_{2i})_T(e_{1i}) \notin e_{2i}.$$

Thus, (5.19) holds.

Since in virtue of step 2,

$$\Phi_i([E_{2i} \backslash e_{2i}]) \cap \Phi_i(e_{2i}) = \emptyset_{\mu_1},$$

then (5.19) implies

$$\Phi_i([E_{2i}\backslash e_{2i}]) = E_{2i}\backslash\Phi_i(e_{2i}). \tag{5.23}$$

STEP 4. For any pairwise disjoint collection of $\{e_{2i}^k\}_{k=1}^{\infty} \subset \Sigma_2(E_{2i})$, the equation

$$\Phi_i(\bigsqcup_{k=1}^{\infty} e_{2i}^k) = \bigsqcup_{k=1}^{\infty} \Phi_i(e_{2i}^k). \tag{5.24}$$

can be proved by induction.

Thus, (5.16), (5.23) and (5.24) imply that the map $\Phi_i : \Sigma_2(E_{2i}) \to \Sigma_1(E_{1i}), \quad i = 1, 2, ...,$ is a homomorphism.

STEP 5. Definition 5.3 implies that for all $e_{2i} \in \Sigma_2(E_{2i}), \quad i = 1, 2, ...,$ for $x, y \in X_1(E_{1i}\Sigma_1(E_{1i}), \mu_1; \mathcal{X}_1)$,

$$x|_{\Phi_i(e_{2i})} = y|_{\Phi_i(e_{2i})}$$

implies

$$T_i(x)|_{e_{2i}} = T_i(y)|_{e_{2i}}.$$

This means, in virtue of Definition 4.3, that operator

$$T_i : X_1(E_{1i}, \Sigma_1(E_{1i}), \mu_1; \mathcal{X}_1) \to X_2(E_{2i}, \Sigma_2(E_{2i}), \mu_2; \mathcal{X}_2), \quad i = 1, 2, ...,$$

is *coatomic* with respect to homomorphism $\Phi_i, \quad i = 1, 2, ...,$ defined in (5.15). □

Corollary 5.2. *Let operator $T : X_1 \to X_2$ satisfy the following conditions:*

1. *$T : X_1 \to X_2$ is full comemory.*
2. *$T \in I$ - condition.*
3. *$Comem_T(\emptyset_{\mu_1}) = \emptyset_{\mu_2}$.*
4. *$Mem_T(\emptyset_{\mu_2}) = \Sigma_1$.*

Then $T : X_1 \to X_2$ can be represented in the following form:

$$(Tx)(t) = (T_i x_i)(t), \quad t \in E_{2i}, \quad i = 1, 2, ...,$$

where operators $T_i : X_1(E_{1i}, \Sigma_1(E_{1i}), \mu_1; \mathcal{X}_1) \to X_2(E_{2i}, \Sigma_2(E_{2i}), \mu_2; \mathcal{X}_2), \quad i = 1, 2, ...,$ defined as the corresponding restrictions of operator T, are short memory, atomic *and* coatomic. *Here x_i is a restriction of function $x \in X_1$ on the set $E_{1i} = [\inf Mem_T E_{2i}]$.*

6. Representation of full comemory operators

Assume that $\Sigma_1 \subset \Sigma_1'$, where Σ_1' is the largest σ-algebra of the subsets of Ω_1. We will need the following definition which first appeared in [15].

Definition 6.1. Let $\Sigma_1 \subset \Sigma_1'$ be σ-algebrae of subsets of Ω_1. Then Σ_1 is said to satisfy Ω-condition with respect to Σ_1' (written $\Sigma_1 \in \Omega(\Sigma_1')$), if there is an at most countable cover of Ω_1 by pairwise disjoint sets $\Omega_1 = \sqcup_j \Omega_1^j$, $\Omega_1^j \in \Sigma_1'$, such that for each $j \in \mathbb{N}$ one has $\Sigma_1 \cap \Omega_1^j = \Sigma_1' \cap \Omega_1^j$.

Let us now formulate the following result on the representation of continuous *atomic* operators. We will start with two lemmas.

Lemma 6.1. [15] *Let $N : X_1 \to X_2$ be local operator, and $\Sigma_1 \in \Omega(\Sigma_1')$. Then N admits the unique extension to the local operator $N' : X_1' \to X_2$. Moreover, the extended operator N' preserves the continuity of N (in measure).*

The next lemma together with its' proof belong to E. Stepanov.

Lemma 6.2. *Let* $(\Omega_1, \Sigma_1', \mu_1)$ *and* $(\Omega_2, \Sigma_2, \mu_2)$ *be standard measure spaces,* $\Sigma_1 \subset \Sigma_1'$ *and* $F\colon \Sigma_1 \to \Sigma_2$ *be a* σ*-homomorphism. Then any continuous operator* $T\colon X_1 \to X_2$ *atomic with respect to* F *can be represented as*

$$(Tu)(t) = f(t, u(g(t))) \quad \text{for } \mu_2 - a.e.\ t \in \Omega_2$$

for some Carathéodory function $f\colon \Omega_2 \times \mathcal{X}_1 \to \mathcal{X}_2$*, a measurable function* $g\colon \Omega_2 \to \Omega_1$ *satisfying*

$$\mu_2(g^{-1}(e_1)) = 0 \text{ when } \mu_1(e_1) = 0$$

and every $u \in X_1$*, if and only if* $F(\Sigma_1) \in \Omega(\Sigma_2)$.

Proof. The theorem 32.3 of [19] implies the existence of a measurable function g: $\Omega_2 \to \Omega_1$ satisfying (2.1) and $[g^{-1}(e_1)] = F(e_1)$ for every $e_1 \in \Sigma_1'$. According to the representation theorem 3.1 from [3], one has $T = N \circ T_g$, where N is a local operator defined over $L^0(\Omega_2, F(\Sigma_1), \mu_2; \mathcal{X}_1)$. If $F(\Sigma_1) \in \Omega(\Sigma_2)$, then according to Lemma 6.1 the operator N admits a unique continuous local extension to the whole space $L^0(\Omega_2, \Sigma_2, \mu_2; \mathcal{X}_1)$, and hence is representable as

$$(Nv)(t) = f(t, v(t)) \text{ for } \mu_2 - \text{a.e. } t \in \Omega_2,$$

where f: $\Omega_2 \times \mathcal{X}_1 \to \mathcal{X}_2$ is a Carathéodory function.

Otherwise, if $F(\Sigma_1) \notin \Omega(\Sigma_2)$, then the theorem 7 from [15] asserts the existence of a continuous local operator N: $L^0(\Omega_2, F(\Sigma_1), \mu_2; \mathcal{X}_1) \to L^0(\Omega_2, \Sigma_2, \mu_2; \mathcal{X}_2)$ which cannot be represented as a Nemytskiĭ operator generated by a Carathéodory function. Therefore, the operator $T := N \circ T_g$ is atomic with respect to F but cannot be represented as indicated in the statement of the theorem.

□

Theorem 6.1. *Let* $(\Omega_1, \Sigma_1, \mu_1)$ *and* $(\Omega_2, \Sigma_2, \mu_2)$ *be standard measure spaces. Any continuous* full comemory *operator* $T; X_1(\Omega_1, \Sigma_1, \mu_1; \mathcal{X}_1) \to X_2(\Omega_2, \Sigma_2, \mu_2; \mathcal{X}_2)$*, satisfying the condition*

$$Comem_T(\emptyset_{\mu_1}) = \emptyset_{\mu_2},$$

together with I*-condition can be represented as*

$$(Tx)(t) = f(t, x(g(t))) \text{ for } \mu_2 - a.e.\ x \in \Omega_2 \tag{6.1}$$

for some Carathéodory function $f\colon \Omega_2 \times \mathcal{X}_1 \to \mathcal{X}_2$*, a measurable function* $g\colon \Omega_2 \to \Omega_1$ *satisfying (2.1) and every* $x \in X_1$.

Proof. The conditions of the theorem imply, in virtue of Theorem 5.1, that operator $T : X_1 \to X_2$ is *atomic* with respect to homomorphism $F : \Sigma_1 \to \Sigma_2$, defined by (5.4). Let us show that $F(\Sigma_1) \in \Omega(\Sigma_2)$. Since the $T \in I$-condition is valid, there exists a collection of mutually disjoint sets E_{2i}, $\quad E_{2i} \in \Sigma_2, \quad i = 1, 2, ...$, such that

$$\Omega_2 = \bigsqcup_i E_{2i}$$

and for any set $e_{2i} \in \Sigma_2(E_{2i}), \quad i = 1, 2, ...$, the equality

$$\max \mathrm{Comem}(E_{2i})_T[\inf \mathrm{Mem}_T(e_{2i})] = e_{2i}$$

holds.

Thus,

$$e_{2i} \in F(\Sigma_1) \cap E_{2i}.$$

This implies

$$F(\Sigma_1) \cap E_{2i} = \Sigma_2 \cap E_{2i} = \Sigma_2(E_{2i}).$$

The last equality means that $F(\Sigma_1) \in \Omega(\Sigma_2)$. Reference to the Lemma 6.2 completes the proof. □

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