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# FIXED POINTS OF ROTATIVE MAPPINGS IN BANACH SPACES

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Abstract. The aim of this paper is to give some conditions providing existence of fixed points for lipschitzian mappings in a Banach space which are $n$-rotative with $n \geqslant 3$.

## 1. Introduction

In general, to assure the fixed point property for nonexpansive mappings some assumptions concerning the geometry of space are added. Another way is to put some additional restrictions on the mapping itself. One of them is so-called *rotativness.* This property assures the existence of fixed points in the case of nonexpansive mappings and in the case of $k$-lipschitzian mappings provided that $k > 1$ is not to large.

Let $C$ be a nonempty closed convex subset of a Banach space $E$. A mapping $T\colon C \to C$ is called $(a,n)$*-rotative* if there exists an integer $n \geqslant 2$ and a real number $0 \leqslant a < n$ such that for any $x \in C$,

$$\|x - T^n x\| \leqslant a\|x - Tx\|. \tag{1}$$

Clearly $n$-periodic mappings (a mapping $T\colon C \to C$ is said to be *n-periodic* if $T^n = I$) are $(0,n)$-rotative. Note also that all contractions are rotative for all $n \geqslant 2$.

The following theorem (due to K. Goebel and M. Koter, [1]) shows that the condition of rotativness is actually quite strong; it assures the existence of fixed points of nonexpansive mappings even without weak compactness, or another special geometric structure of the set $C$.

**Theorem 1.** [1] *If $C$ is a nonempty closed convex subset of a Banach space $E$, then any nonexpansive rotative mapping $T\colon C \to C$ has a fixed point.*

Recall that $T\colon C \to C$ is called *k-lipschitzian* if for all $x, y \in C$,

$$\|Tx - Ty\| \leqslant k\|x - y\|.$$

If $k = 1$ such a mapping is said to be *nonexpansive.*

Rotativness is independent of nonexpansivness. If we consider $k$-lipschitzian mapping with $k > 1$, the condition of rotativness (1) assures the existence of fixed points provided $k$ is not too large. Namely, we have the following:

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**Theorem 2.** [2] *If $C$ is a nonempty closed convex subset of a Banach space $E$, then for any $n \geqslant 2$ and $a < n$ there exists $\gamma > 1$ such that any $(a,n)$-rotative and $k$-lipschitzian mapping $T\colon C \to C$ has a fixed point provided $k < \gamma$.*

Clearly, $\gamma$ which appears in the above theorem depends on $a, n$ and the space in which the set $C$ is contained. Thus it is convenient to define the function $\gamma_n^E(a)$ as follows:

$$\gamma_n^E(a) = \inf\{k\colon \text{there is a closed convex set } C \subset E \text{ and a fixed point free } k\text{-lipschitzian } (a,n)\text{-rotative selfmapping of } C\}.$$

In general, precise values of $\gamma_n^E(a)$ are unknown. If $n \geqslant 2$ is arbitrary, by Kirk's theorem [9] we know only the estimates from below of the value $\gamma_n^E(a)$ at $a = 0$. Namely,

$$\gamma_n^E(0) \geqslant \begin{cases} 2 & \text{for } n = 2, \\ \sqrt[n-1]{\frac{1}{n-2}\left(-1 + \sqrt{n(n-1) - \frac{1}{n-1}}\right)} & \text{for } n > 2. \end{cases}$$

Some better results can be obtained for $n = 2$ (see [8]).

**Theorem 3.** *In an arbitrary Banach space $E$,*

$$\gamma_2^E(a) \geqslant \max\left\{\frac{1}{2}\left(2 - a + \sqrt{(2-a)^2 + a^2}\right), \frac{1}{8}\left(a^2 + 4 + \sqrt{(a^2+4)^2 - 64a + 64}\right)\right\}, \quad a \in [0, 2).$$

It is interesting that the first term gives better evaluation for $a \in [0, 2(\sqrt{2} - 1)]$, while the second one for $a \in [2(\sqrt{2} - 1), 2)$.

In [4] J. Górnicki gives evaluations of $\gamma_2^{l^p}(a)$ and $\gamma_2^{L^p}(a)$. In [10] M. Koter-Mórgowska gives some estimate of $\gamma_n^H(a)$ in Hilbert space for $n = 3, 4, 5, 6$.

In [12] A. T. Plant and S. Reich extend the notion of rotativness onto nonlinear semigroups and establish the fixed point results for nonexpansive rotative semigroups.

For more accurate studies we refer to [8] where the current state of knowledge concerning such mappings and related topics are presented.

Note that much of the above results are based on finding a sequence which converges to a fixed point of investigated mapping. In this paper we use Halpern's idea of iterative procedure [7]. Halpern used his procedure to construct an infinite sequence in Hilbert space which converges to a fixed point of a nonexpansive mapping. We use his idea in a different way and in Banach spaces. Halpern's result was extended to Banach spaces in [13].

In the present paper we apply some three modifications of Halpern's iterative procedure to give a better estimate for the function $\gamma_n^E(a)$ in all Banach spaces for $n \geqslant 3$.

## 2. Lipschitzian rotative mappings

We will start with the following lemma:

**Lemma 1.** [3] *Let $C$ be a nonempty closed convex subset of a Banach space $E$ and let $T\colon C \to C$ be a $k$-lipschitzian. Assume that $A, B \in \mathbb{R}$ and $0 \leq A < 1$ and $0 < B$. If for an arbitrary $x \in C$ there exists $u \in C$ such that*

$$\|Tu - u\| \leqslant A\|Tx - x\|$$

*and*

$$\|u - x\| \leqslant B\|Tx - x\|,$$

*then $T$ has a fixed point in $C$.*

To make notation more simple, we write

$$\gamma_n^1(a) = \sup_{\alpha\in(0,1)} \Bigg\{ s\colon \alpha(1-\alpha)as + \alpha^2 s^n + (1-\alpha)^n s^{n-1} + \alpha^2 \sum_{j=2}^{n-1} (1-\alpha)^{j-1} s^j \frac{1-s^{n-j}}{1-s} - 1 = 0 \Bigg\},$$

$$\gamma_n^2(a) = \sup_{\alpha\in(0,1)} \Bigg\{ s\colon (1-\alpha)s\left[\alpha(1-\alpha)^{n-2}s^{n-2}a + (1-\alpha)^{n-1}s^{n-2}\right] + \alpha^2 s^n \sum_{j=0}^{n-2} (1-\alpha)^j \frac{1-s^{j+1}}{1-s} - 1 = 0 \Bigg\},$$

$$\gamma_n^3(a) = \sup_{\alpha\in(0,1)} \Bigg\{ s\colon \alpha a + \alpha^2 s \frac{1-s^{n-1}}{1-s} + (1-\alpha)^n s^{n-1} + \alpha^2 \sum_{j=2}^{n-1} (1-\alpha)^{j-1} s^j \frac{1-s^{n-j}}{1-s} - 1 = 0 \Bigg\},$$

$$\gamma_n^4(a) = \sup_{\alpha\in(0,1)} \Bigg\{ s\colon \alpha a + \alpha^2 s(s^{n-1} + a) + (1-\alpha)^n s^{n-1} + \alpha^2 \sum_{j=2}^{n-1} (1-\alpha)^{j-1} s^j \frac{1-s^{n-j}}{1-s} - 1 = 0 \Bigg\},$$

where $a \in [0, n)$ and $n \geqslant 3$.

Now we are ready to formulate the main theorem of this part of the paper.

**Theorem 4.** *Given an integer $n \geqslant 3$, let $C$ be a nonempty convex closed subset of a Banach space $E$. If $T\colon C \to C$ is $k$-lipschitzian $(k > 1)$ and $(a, n)$-rotative mapping such that*

$$k < \max\left\{\gamma_n^1(a), \gamma_n^2(a), \gamma_n^3(a), \gamma_n^4(a)\right\},$$

*then $T$ has a fixed point in $C$.*

*Proof.* We must take four cases into consideration.

***Case I***. We consider the following sequence: let $x$ be an arbitrary point in $C$, i.e. $x_0 = x \in C$ and

$$\begin{aligned} x_1 &= \alpha x_0 + (1-\alpha)Tx_0, \\ x_2 &= \alpha x_0 + (1-\alpha)Tx_1, \\ &\dots\dots\dots\dots\dots\dots \\ x_{n-2} &= \alpha x_0 + (1-\alpha)Tx_{n-3}, \\ x_{n-1} &= \alpha T^n x_0 + (1-\alpha)Tx_{n-2}, \end{aligned}$$

where $\alpha \in (0,1)$. Put $z = x_{n-1}$, then

$$\begin{aligned} \|z - Tz\| &= \|\alpha T^n x_0 + (1-\alpha)Tx_{n-2} - Tz\| \\ &= \|\alpha(T^n x_0 - Tz) + (1-\alpha)(Tx_{n-2} - Tz)\| \\ &\leqslant \alpha k\|T^{n-1}x_0 - z\| + (1-\alpha)k\|x_{n-2} - z\|. \end{aligned} \tag{2}$$

Now, we have the evaluation

$$\begin{aligned} \|T^{n-1}x_0 - z\| &= \|T^{n-1}x_0 - \alpha T^n x_0 - (1-\alpha)Tx_{n-2}\| \\ &= \|\alpha(T^{n-1}x_0 - T^n x_0) + (1-\alpha)(T^{n-1}x_0 - Tx_{n-2})\| \\ &\leqslant \alpha k^{n-1}\|x_0 - Tx_0\| + (1-\alpha)k\|T^{n-2}x_0 - x_{n-2}\|, \end{aligned} \tag{3}$$

where

$$\begin{aligned} &(1-\alpha)k\|T^{n-2}x_0 - x_{n-2}\| = \\ &\quad = (1-\alpha)k\|T^{n-2}x_0 - \alpha x_0 - (1-\alpha)Tx_{n-3}\| \\ &\quad = (1-\alpha)k\|\alpha(T^{n-2}x_0 - x_0) + (1-\alpha)(T^{n-2}x_0 - Tx_{n-3})\| \\ &\quad \leqslant (1-\alpha)\alpha k\|T^{n-2}x_0 - x_0\| + (1-\alpha)^2k^2\|T^{n-3}x_0 - x_{n-3}\| \leqslant \dots \\ &\quad \leqslant \alpha(1-\alpha)k\|T^{n-2}x_0 - x_0\| + \alpha(1-\alpha)^2k^2\|T^{n-3}x_0 - x_0\| \\ &\qquad + \alpha(1-\alpha)^3k^3\|T^{n-4}x_0 - x_0\| + \dots \\ &\qquad\quad + \alpha(1-\alpha)^{n-2}k^{n-2}\|Tx_0 - x_0\|. \end{aligned} \tag{4}$$

Finally from (4), using only the triangle inequality and the fact that $T$ is $k$-lipschitzian we get

$$(1-\alpha)k\|T^{n-2}x_0 - x_{n-2}\| \leqslant \alpha \sum_{j=2}^{n-1}(1-\alpha)^{j-1}k^{j-1}\frac{1-k^{n-j}}{1-k}\|Tx_0 - x_0\|, \tag{5}$$

and consequently from (3) and (5) we obtain

$$\|T^{n-1}x_0 - z\| \leqslant \left\{\alpha k^{n-1} + \alpha\sum_{j=2}^{n-1}(1-\alpha)^{j-1}k^{j-1}\frac{1-k^{n-j}}{1-k}\right\}\|Tx_0 - x_0\|. \tag{6}$$

For the next expression, using (1), we have the following evaluation

$$\begin{aligned} \|x_{n-2} - z\| &= \|\alpha x_0 + (1-\alpha)Tx_{n-3} - \alpha T^n x_0 - (1-\alpha)Tx_{n-2}\| \\ &= \|\alpha(x_0 - T^n x_0) + (1-\alpha)(Tx_{n-3} - Tx_{n-2})\| \\ &\leqslant \alpha a\|x_0 - Tx_0\| + (1-\alpha)k\|x_{n-3} - x_{n-2}\| \leqslant \dots \\ &\leqslant \alpha a\|x_0 - Tx_0\| + (1-\alpha)^{n-2}k^{n-2}\|x_0 - x_1\| \\ &= \left\{\alpha a + (1-\alpha)^{n-1}k^{n-2}\right\}\|x_0 - Tx_0\|. \end{aligned} \tag{7}$$

Inequalities (2), (6) and (7) yield

$$\|z - Tz\| \leqslant \left\{\alpha^2 k^n + \alpha^2 \sum_{j=2}^{n-1}(1-\alpha)^{j-1}k^j \frac{1-k^{n-j}}{1-k} + \right. \\ \left. +\alpha(1-\alpha)ak + (1-\alpha)^n k^{n-1}\right\}\|x_0 - Tx_0\|. \tag{8}$$

Moreover, we have

$$\begin{aligned} \|z - x_0\| &= \|\alpha T^n x_0 + (1-\alpha)Tx_{n-2} - x_0\| \\ &= \|\alpha(T^n x_0 - x_0) + (1-\alpha)(Tx_{n-2} - x_0)\| \\ &\leqslant \alpha\|T^n x_0 - x_0\| + (1-\alpha)\|Tx_{n-2} - T^n x_0 + T^n x_0 - x_0\| \\ &\leqslant \alpha\|T^n x_0 - x_0\| + (1-\alpha)k\|x_{n-2} - T^{n-1}x_0\| + (1-\alpha)(T^n x_0 - x_0)\| \\ &= \|x_0 - T^n x_0\| + (1-\alpha)k\|x_{n-2} - T^{n-1}x_0\|. \end{aligned} \tag{9}$$

Observe that

$$\begin{aligned} &(1-\alpha)k\|x_{n-2} - T^{n-1}x_0\| = \\ &\quad= (1-\alpha)k\|\alpha(x_0 - T^{n-1}x_0) + (1-\alpha)(Tx_{n-3} - T^{n-1}x_0)\| \\ &\quad\leqslant \alpha(1-\alpha)k\|x_0 - T^{n-1}x_0\| + (1-\alpha)^2k^2\|x_{n-3} - T^{n-2}x_0\| \leqslant \dots \\ &\quad\leqslant \alpha(1-\alpha)k\|x_0 - T^{n-1}x_0\| + \alpha(1-\alpha)^2k^2\|x_0 - T^{n-2}x_0\| \\ &\qquad + \alpha(1-\alpha)^3k^3\|x_0 - T^{n-3}x_0\| + \dots \\ &\qquad + \alpha(1-\alpha)^{n-2}k^{n-2}\|x_0 - T^2x_0\| + (1-\alpha)^{n-1}k^{n-1}\|x_0 - Tx_0\|. \end{aligned}$$

Now, using only the triangle inequality and the fact that $T$ is $k$-lipschitzian, we have

$$\begin{aligned} &(1-\alpha)k\|x_{n-2} - T^{n-1}x_0\| \\ &\qquad \leqslant \left\{(1-\alpha)^{n-1}k^{n-1} + \alpha\sum_{j=1}^{n-2}(1-\alpha)^j k^j \frac{1-k^{n-j}}{1-k}\right\}\|x_0 - Tx_0\|, \end{aligned}$$

which together with (9) gives

$$\begin{aligned} &\|z - x_0\| \\ &\leqslant \left\{a + (1-\alpha)^{n-1}k^{n-1} + \alpha\sum_{j=1}^{n-2}(1-\alpha)^j k^j \frac{1-k^{n-j}}{1-k}\right\}\|x_0 - Tx_0\|, \end{aligned} \tag{10}$$

Since

$$\alpha(1-\alpha)ak+\alpha^2k^n+(1-\alpha)^nk^{n-1}+\alpha^2\sum_{j=2}^{n-1}(1-\alpha)^{j-1}k^j\frac{1-k^{n-j}}{1-k}<1$$

for all $\alpha \in (0,1)$ and $k < \gamma_n^1(a)$, by inequalities (8) and (10), the Lemma 1 implies the existence of fixed points of $T$ in $C$.

***Case II***. We consider a sequence generated as follows:

$$(*)\qquad \begin{aligned} x_0 &= x \in C,\\ x_1 &= \alpha T^n x_0 + (1-\alpha)Tx_0,\\ x_2 &= \alpha T^n x_0 + (1-\alpha)Tx_1,\\ &\dots\dots\dots\dots\dots\dots\\ x_{n-2} &= \alpha T^n x_0 + (1-\alpha)Tx_{n-3},\\ x_{n-1} &= \alpha T^n x_0 + (1-\alpha)Tx_{n-2}, \end{aligned}$$

where $\alpha \in (0,1)$. Then for $z = x_{n-1}$, we get

$$(11)\qquad \begin{aligned} \|z-Tz\| &= \|\alpha T^n x_0 + (1-\alpha)Tx_{n-2} - Tz\|\\ &= \|\alpha(T^n x_0 - Tz) + (1-\alpha)(Tx_{n-2}-Tz)\|\\ &\leqslant \alpha k\|T^{n-1}x_0 - z\| + (1-\alpha)k\|x_{n-2}-z\|. \end{aligned}$$

Now, we have the evaluation

$$(12)\qquad \begin{aligned} \|T^{n-1}x_0 - z\| &= \|T^{n-1}x_0 - \alpha T^n x_0 - (1-\alpha)Tx_{n-2}\|\\ &= \|\alpha(T^{n-1}x_0 - T^n x_0) + (1-\alpha)(T^{n-1}x_0 - Tx_{n-2})\|\\ &\leqslant \alpha k^{n-1}\|x_0 - Tx_0\| + (1-\alpha)k\|T^{n-2}x_0 - x_{n-2}\|, \end{aligned}$$

where

$$(13)\qquad \begin{aligned} &(1-\alpha)k\|T^{n-2}x_0 - x_{n-2}\| \leqslant\\ &\quad\leqslant \alpha(1-\alpha)k^{n-1}\|x_0 - T^2x_0\| + (1-\alpha)^2k\|T^{n-3}x_0 - x_{n-3}\| \leqslant \dots\\ &\quad\leqslant \alpha(1-\alpha)k^{n-1}\|x_0 - T^2x_0\| + \alpha(1-\alpha)^2k^{n-1}\|x_0 - T^3x_0\| + \dots\\ &\qquad + \alpha(1-\alpha)^{n-3}k^{n-1}\|x_0 - T^{n-2}x_0\|\\ &\qquad + \alpha(1-\alpha)^{n-2}k^{n-1}\|x_0 - T^{n-1}x_0\|\\ &\quad\leqslant \alpha(1-\alpha)k^{n-1}(1+k)\|x_0 - Tx_0\|\\ &\qquad + \alpha(1-\alpha)^2k^{n-1}(1+k+k^2)\|x_0 - Tx_0\| + \dots\\ &\qquad + \alpha(1-\alpha)^{n-3}k^{n-1}(1+k+\dots+k^{n-3})\|x_0 - Tx_0\|\\ &\qquad + \alpha(1-\alpha)^{n-2}k^{n-1}(1+k+\dots+k^{n-2})\|x_0 - Tx_0\|\\ &\quad= \left\{\alpha k^{n-1}\sum_{j=1}^{n-2}(1-\alpha)^j\frac{1-k^{j+1}}{1-k}\right\}\|x_0 - Tx_0\|. \end{aligned}$$

Finally by (12) and (13) we get

$$\|T^{n-1}x_0 - z\| \leqslant \left\{ \alpha k^{n-1} + \alpha k^{n-1} \sum_{j=1}^{n-2} (1-\alpha)^j \frac{1-k^{j+1}}{1-k} \right\} \|x_0 - Tx_0\| \tag{14}$$
$$= \left\{ \alpha k^{n-1} \sum_{j=0}^{n-2} (1-\alpha)^j \frac{1-k^{j+1}}{1-k} \right\} \|x_0 - Tx_0\|.$$

For the next expression in (11) we have the following evaluation (using (1)):

$$\begin{aligned} \|x_{n-2} - z\| &= \|\alpha T^n x_0 + (1-\alpha)Tx_{n-3} - \alpha T^n x_0 - (1-\alpha)Tx_{n-2}\| \\ &\leqslant (1-\alpha)k\|x_{n-3} - x_{n-2}\| \leqslant \dots \\ &\leqslant (1-\alpha)^{n-2}k^{n-2}\|x_0 - x_1\| \\ &= (1-\alpha)^{n-2}k^{n-2}\|\alpha(x_0 - T^n x_0) + (1-\alpha)(x_0 - Tx_0)\| \\ &\leqslant \left\{\alpha(1-\alpha)^{n-2}k^{n-2}a + (1-\alpha)^{n-1}k^{n-2}\right\} \|x_0 - Tx_0\|. \end{aligned} \tag{15}$$

Inequalities (11), (14) and (15) yield

$$\|z - Tz\| \leqslant \left\{ (1-\alpha)k \left[\alpha(1-\alpha)^{n-2}k^{n-2}a + (1-\alpha)^{n-1}k^{n-2}\right] \right. \tag{16}$$
$$\left. + \alpha^2 k^n \sum_{j=0}^{n-2} (1-\alpha)^j \frac{1-k^{j+1}}{1-k} \right\} \|x_0 - Tx_0\|.$$

Moreover, we have

$$\begin{aligned} \|z - x_0\| &= \|\alpha T^n x_0 + (1-\alpha)Tx_{n-2} - x_0\| \\ &= \|\alpha(T^n x_0 - x_0) + (1-\alpha)(Tx_{n-2} - x_0)\| \\ &\leqslant \alpha\|T^n x_0 - x_0\| + (1-\alpha)\|Tx_{n-2} - T^n x_0 + T^n x_0 - x_0\| \\ &= \|x_0 - T^n x_0\| + (1-\alpha)k\|x_{n-2} - T^{n-1}x_0\|. \end{aligned} \tag{17}$$

Observe that

$$\begin{aligned} &(1-\alpha)k\|x_{n-2} - T^{n-1}x_0\| = \\ &\quad = (1-\alpha)k\|\alpha(T^n x_0 - T^{n-1}x_0) + (1-\alpha)(Tx_{n-3} - T^{n-1}x_0)\| \\ &\quad \leqslant \alpha(1-\alpha)k\|T^n x_0 - T^{n-1}x_0\| + (1-\alpha)^2k^2\|x_{n-3} - T^{n-2}x_0\| \leqslant \dots \\ &\quad \leqslant \alpha(1-\alpha)k\|T^n x_0 - T^{n-1}x_0\| + \alpha(1-\alpha)^2k^2\|T^n x_0 - T^{n-2}x_0\| + \dots \\ &\qquad + \alpha(1-\alpha)^{n-2}k^{n-2}\|T^n x_0 - T^2 x_0\| + (1-\alpha)^{n-1}k^{n-1}\|x_0 - Tx_0\| \\ &\quad \leqslant \alpha(1-\alpha)kk^{n-1}\|Tx_0 - x_0\| \\ &\qquad + \alpha(1-\alpha)^2k^2(k^{n-1} + k^{n-2})\|Tx_0 - x_0\| + \dots \\ &\qquad\quad + \alpha(1-\alpha)^{n-2}k^{n-2}(k^{n-1} + \dots + k^2)\|Tx_0 - x_0\| \\ &\qquad\qquad + (1-\alpha)^{n-1}k^{n-1}\|x_0 - Tx_0\| \end{aligned}$$

$$= \alpha k^n \sum_{j=1}^{n-2} (1-\alpha)^j \frac{1-k^j}{1-k} \|x_0 - Tx_0\| + (1-\alpha)^{n-1} k^{n-1} \|x_0 - Tx_0\|,$$

which together with (17) gives

$$\|z - x_0\| \leqslant \left\{ a + \alpha k^n \sum_{j=1}^{n-2} (1-\alpha)^j \frac{1-k^j}{1-k} + (1-\alpha)^{n-1} k^{n-1} \right\} \|x_0 - Tx_0\|. \tag{18}$$

Since

$$(1-\alpha)k \left[ \alpha(1-\alpha)^{n-2} k^{n-2} a + (1-\alpha)^{n-1} k^{n-2} \right] + \alpha^2 k^n \sum_{j=0}^{n-2} (1-\alpha)^j \frac{1-k^{j+1}}{1-k} < 1$$

for all $\alpha \in (0,1)$ and $k < \gamma_n^2(a)$, by inequalities (16) and (18), the Lemma 1 implies the existence of fixed points of $T$ in $C$.

***Case III***. We consider a sequence generated as follows: let $x$ be an arbitrary point in $C$, i.e. $x_0 = x \in C$ and

$$\begin{aligned} x_1 &= \alpha x_0 + (1-\alpha) T x_0, \\ x_2 &= \alpha x_0 + (1-\alpha) T x_1, \\ &\dots\dots\dots\dots\dots\dots \\ x_{n-2} &= \alpha x_0 + (1-\alpha) T x_{n-3}, \\ x_{n-1} &= \alpha x_0 + (1-\alpha) T x_{n-2}, \end{aligned}$$

where $\alpha \in (0,1)$. Then for $z = x_{n-1}$, we have

$$\begin{aligned} \|z - Tz\| &= \|\alpha x_0 + (1-\alpha) T x_{n-2} - Tz\| \\ &= \|\alpha(x_0 - Tz) + (1-\alpha)(Tx_{n-2} - Tz)\| \\ &\leqslant \alpha \|x_0 - T^n x_0 + T^n x_0 - Tz\| + (1-\alpha) k \|x_{n-2} - z\| \\ &\leqslant \alpha \|x_0 - T^n x_0\| + \alpha k \|T^{n-1} x_0 - z\| + (1-\alpha) k \|x_{n-2} - z\|. \end{aligned} \tag{19}$$

Now, we have the evaluation

$$\begin{aligned} \|T^{n-1} x_0 - z\| &= \|T^{n-1} x_0 - \alpha x_0 - (1-\alpha) T x_{n-2}\| \\ &= \|\alpha(T^{n-1} x_0 - x_0) + (1-\alpha)(T^{n-1} x_0 - T x_{n-2})\| \\ &\leqslant \alpha \|T^{n-1} x_0 - x_0\| + (1-\alpha) k \|T^{n-2} x_0 - x_{n-2}\|. \end{aligned} \tag{20}$$

Using only the triangle inequality and the fact that $T$ is $k$-lipschitzian we have

$$\begin{aligned} \alpha \|T^{n-1} x_0 - x_0\| &\leqslant \alpha (k^{n-2} + k^{n-3} + \cdots + 1) \|T x_0 - x_0\| \\ &= \alpha \frac{1-k^{n-1}}{1-k} \|T x_0 - x_0\|. \end{aligned} \tag{21}$$

Now, by (5) and (21) we obtain

$$\|T^{n-1}x_0 - z\| \leqslant \left\{ \alpha \frac{1-k^{n-1}}{1-k} + \alpha \sum_{j=2}^{n-1} (1-\alpha)^{j-1} k^{j-1} \frac{1-k^{n-j}}{1-k} \right\} \|Tx_0 - x_0\|. \tag{22}$$

For the next expression in (19) we have the following evaluation

$$\begin{aligned} \|x_{n-2} - z\| &= \|\alpha x_0 + (1-\alpha)Tx_{n-3} - \alpha x_0 - (1-\alpha)Tx_{n-2}\| \\ &= \|(1-\alpha)(Tx_{n-3} - Tx_{n-2})\| \leqslant (1-\alpha)k\|x_{n-3} - x_{n-2}\| \leqslant \dots \\ &\leqslant (1-\alpha)^{n-2}k^{n-2}\|x_0 - x_1\| = (1-\alpha)^{n-1}k^{n-2}\|x_0 - Tx_0\|. \end{aligned} \tag{23}$$

By combining (19) with (22) and (23) we get

$$\begin{aligned} \|z - Tz\| \leqslant \Bigg\{ &\alpha a + \alpha^2 k \frac{1-k^{n-1}}{1-k} + (1-\alpha)^n k^{n-1} \\ &+ \alpha^2 \sum_{j=2}^{n-1} (1-\alpha)^{j-1} k^j \frac{1-k^{n-j}}{1-k} \Bigg\} \|x_0 - Tx_0\|. \end{aligned} \tag{24}$$

Moreover, we have

$$\begin{aligned} \|z - x_0\| &= \|\alpha x_0 + (1-\alpha)Tx_{n-2} - x_0\| = (1-\alpha)\|Tx_{n-2} - x_0\| \\ &\leqslant (1-\alpha)\left\{\|Tx_{n-2} - T^n x_0\| + \|T^n x_0 - x_0\|\right\} \\ &\leqslant (1-\alpha)k\|x_{n-2} - T^{n-1}x_0\| + (1-\alpha)a\|Tx_0 - x_0\|. \end{aligned} \tag{25}$$

Observe that

$$\begin{aligned} &(1-\alpha)k\|x_{n-2} - T^{n-1}x_0\| = \\ &\quad = (1-\alpha)k\|\alpha x_0 + (1-\alpha)Tx_{n-3} - T^{n-1}x_0\| \\ &\quad = (1-\alpha)k\|\alpha(x_0 - T^{n-1}x_0) + (1-\alpha)(Tx_{n-3} - T^{n-1}x_0)\| \\ &\quad \leqslant \alpha(1-\alpha)k\|x_0 - T^{n-1}x_0\| + (1-\alpha)^2k^2\|x_{n-3} - T^{n-2}x_0\| \leqslant \dots \\ &\quad \leqslant \alpha(1-\alpha)k\|x_0 - T^{n-1}x_0\| + \alpha(1-\alpha)^2k^2\|x_0 - T^{n-2}x_0\| \\ &\qquad + \alpha(1-\alpha)^3k^3\|x_0 - T^{n-3}x_0\| + \dots \\ &\qquad\quad + (1-\alpha)^{n-1}k^{n-1}\|x_0 - Tx_0\|. \end{aligned}$$

Now, using only the triangle inequality and the fact that $T$ is $k$-lipschitzian, we have

$$\begin{aligned} &(1-\alpha)k\|x_{n-2} - T^{n-1}x_0\| \\ &\qquad \leqslant \left\{ (1-\alpha)^{n-1}k^{n-1} + \alpha \sum_{j=1}^{n-2} (1-\alpha)^j k^j \frac{1-k^{n-j}}{1-k} \right\} \|x_0 - Tx_0\|, \end{aligned}$$

which together with (25) gives

$$\|z - x_0\| \leqslant \Big\{(1-\alpha)a + (1-\alpha)^{n-1}k^{n-1} + \alpha \sum_{j=1}^{n-2}(1-\alpha)^j k^j \frac{1-k^{n-j}}{1-k}\Big\}\|x_0 - Tx_0\|. \tag{26}$$

Since

$$\alpha a + \alpha^2 k \frac{1-k^{n-1}}{1-k} + (1-\alpha)^n k^{n-1} + \alpha^2 \sum_{j=2}^{n-1}(1-\alpha)^{j-1}k^j \frac{1-k^{n-j}}{1-k} < 1$$

for all $\alpha \in (0,1)$ and $k < \gamma_n^3(a)$, by inequalities (24) and (26) the Lemma 1 implies the existence of fixed points of $T$ in $C$.

***Case IV***. Using (1) we can also evaluate the expression (21) in a different manner. Namely,

$$\begin{aligned}\alpha\|T^{n-1}x_0 - x_0\| &\leqslant \alpha(\|T^{n-1}x_0 - T^n x_0\| + \|T^n x_0 - x_0\|) \\ &\leqslant \alpha(k^{n-1} + a)\|Tx_0 - x_0\|.\end{aligned} \tag{27}$$

Now, by (5) and (27), we get from (20):

$$\|T^{n-1}x_0 - z\| \leqslant \Big\{\alpha(k^{n-1}+a) + \alpha \sum_{j=2}^{n-1}(1-\alpha)^{j-1}k^{j-1}\frac{1-k^{n-j}}{1-k}\Big\}\|Tx_0 - x_0\|. \tag{28}$$

Inequalities (19), (28) and (23) yield

$$\|z - Tz\| \leqslant \Big\{\alpha a + \alpha^2 k(k^{n-1}+a) + (1-\alpha)^n k^{n-1} + \alpha^2 \sum_{j=2}^{n-1}(1-\alpha)^{j-1}k^j \frac{1-k^{n-j}}{1-k}\Big\}\|x_0 - Tx_0\|. \tag{29}$$

Since

$$\alpha a + \alpha^2 k(k^{n-1}+a) + (1-\alpha)^n k^{n-1} + \alpha^2 \sum_{j=2}^{n-1}(1-\alpha)^{j-1}k^j \frac{1-k^{n-j}}{1-k} < 1$$

for all $\alpha \in (0,1)$ and $k < \gamma_n^4(a)$, by inequalities (29) and (26), the Lemma 1 implies the existence of fixed points of $T$ in $C$. This completes the proof. □

*Remark* 1. Begin with observation that the function given in the case II (inequality (16))

$$g(\alpha,k) = (1-\alpha)k\left[\alpha(1-\alpha)^{n-2}k^{n-2}a + (1-\alpha)^{n-1}k^{n-2}\right] + \alpha^2 k^n \sum_{j=0}^{n-2}(1-\alpha)^j \frac{1-k^{j+1}}{1-k}$$

is continuous for $\alpha \in (0,1)$ and $k > 1$ and $n \geqslant 3$. Now, for $\alpha = \frac{1}{n}$, note that

$$\lim_{k \to 1^+} g(\frac{1}{n}, k) = g(\frac{1}{n}) = 1 + \left(\frac{a}{n} - 1\right)\left(\frac{n-1}{n}\right)^{n-1},$$

where the number $g(\frac{1}{n})$ is obtained in the paper [5]. Thus, since $g(\frac{1}{n}) < 1$ for $a < n$, then $g(\frac{1}{n}, k) < 1$ if $k$ is sufficently near 1 (but with $k > 1$) and $0 \leqslant a < n$. For such $k$, $a \in [0, n)$ and $\alpha = \frac{1}{n}$ the sequence generated by iteration procedure $(*)$ for any $x_0 \in C$ converges to a fixed point of $T$. This implies that $\gamma_n^E(a) > 1$ for any Banach space $E$ and $0 \leqslant a < n$, which presents another proof of the theorem 2.

*Remark* 2. It follows from Theorem 4 that

$$\gamma_n^E(a) \geqslant \max\left\{\gamma_n^1(a), \gamma_n^2(a), \gamma_n^3(a), \gamma_n^4(a)\right\},$$

where $n \geqslant 3$, $a \in [0, n)$.

For $n = 3$ the graph of the lower bound of $\gamma_3^E(a)$ for an arbitrary Banach space $E$ is illustrated by means of computer graphic in Figure 1 (thick line). It is interesting that the solution of the equation

$$3k^3 + 10k^2 + 6ak - 27 = 0$$

(obtained form $\gamma_3^1(a)$ for $\alpha = \frac{1}{3}$) gives better evaluation for $a \in [0, \xi]$, while the solution of the equation

$$2k^4 + 5k^3 + (8 + 4a)k^2 - 27 = 0$$

(obtained form $\gamma_3^2(a)$ for $\alpha = \frac{1}{3}$) for $a \in [\xi, 3)$, where $\xi \approx 1.71$.

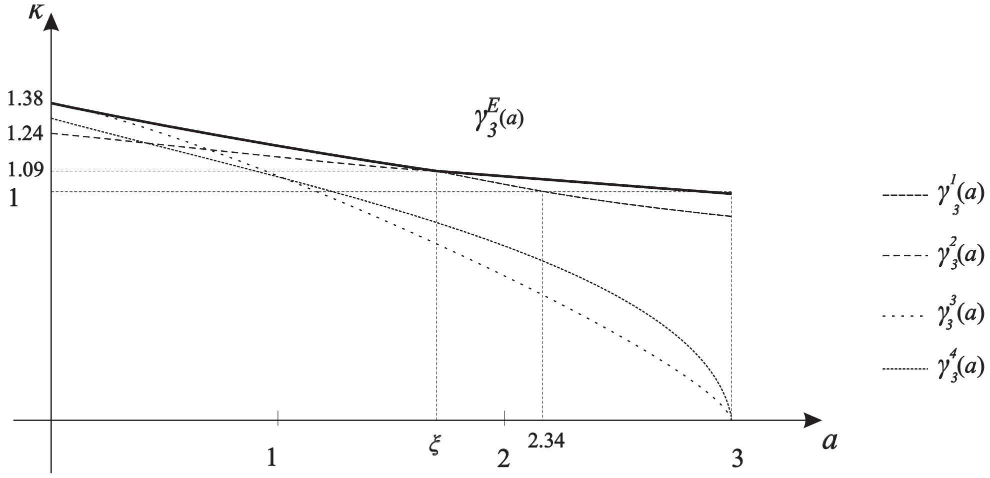


Figure 1.

*Remark* 3. It follows from Theorem 4 that $\gamma_3^E(0) \geqslant 1.3821$ (this evaluation is obtained from $\gamma_3^1(a = 0)$ for $\alpha = 0.345$); $\gamma_4^E(0) \geqslant 1.2524$ (from $\gamma_4^1(a = 0)$ for $\alpha = 0.283$); $\gamma_5^E(0) \geqslant 1.1777$ (from $\gamma_5^1(a = 0)$ for $\alpha = 0.224$); $\gamma_6^E(0) \geqslant 1.1329$ (from $\gamma_6^1(a = 0)$ for $\alpha = 0.185$); $\ldots$ . All these evaluations are better than those obtained by W. A. Kirk [9] and J. Linhart [11] and better even than those which are obtained by M. Koter-Mórgowska [10] for Hilbert spaces.

## 3. Uniformly lipschitzian rotative mappings

Recall, that a mapping $T\colon C \to C$ is called *uniformly k-lipschitzian* if for all $n \in \mathbb{N}$ and $x, y \in C$,

$$\|T^n x - T^n y\| \leqslant k\|x - y\|.$$

If such a mapping is $(a, 2)$-rotative, we have exactly the same situation as in the general case of lipschitzian mappings. However, if we consider mappings which are uniformly $k$-lipschitzian and $(a, n)$-rotative with $n \geqslant 3$, we obtain new conditions.

In [9] W. A. Kirk has proved in Banach spaces, that $n$-periodic mapping $T$ such that $\|T^i x - T^i y\| \leqslant k\|x - y\|$ for $x, y \in C$, $i = 1, 2, \ldots, n-1$, $k > 1$, has a fixed point if

$$\frac{1}{n^2}\left[(n-1)(n-2)k^2 + 2(n-1)k\right] < 1. \tag{30}$$

It follows form (30) that for $n = 3$, $k < 1.3452$; for $n = 4$, $k < 1.2078$; for $n = 5$, $k < 1.1280$; for $n = 6$, $k < 1.1147$.

Theorem 2 gives a possibility to define the function $\widetilde{\gamma}_n^E(a)$ as follows:

$$\widetilde{\gamma}_n^E(a) = \inf\{k\colon \text{there is a closed convex set } C \subset E \text{ and a fixed point free uniformly } k\text{-lipschitzian } (a,n)\text{-rotative selfmapping of } C\}.$$

Obviously, $\widetilde{\gamma}_n^E(a) \geqslant \gamma_n^E(a)$.

In the next theorem we give some estimate of the function $\widetilde{\gamma}_n^E(a)$ for $n \geqslant 3$.

To make notation more simple, we write

$$\widetilde{\gamma}_n^1(a) = \sup_{\alpha\in(0,1)} \Big\{ s\colon \alpha(1-\alpha)as + \alpha^2 s^2 + (1-\alpha)^n s^{n-1} + \alpha^2 \sum_{j=1}^{n-2} (1-\alpha)^j s^{j+1}[1 + (n-j-2)s] - 1 = 0 \Big\},$$

$$\widetilde{\gamma}_n^2(a) = \sup_{\alpha\in(0,1)} \Big\{ s\colon (1-\alpha)s\left[\alpha(1-\alpha)^{n-2}s^{n-2}a + (1-\alpha)^{n-1}s^{n-2}\right] + \alpha^2 s^2 + \alpha^2 \sum_{j=1}^{n-2} (1-\alpha)^j s^{j+2}(1 + js) - 1 = 0 \Big\},$$

$$\widetilde{\gamma}_n^3(a) = \sup_{\alpha\in(0,1)} \Big\{ s\colon \alpha a + \alpha^2 s\left[(n-2)s + 1\right] + (1-\alpha)^n s^{n-1} + \alpha^2 \sum_{j=1}^{n-2} (1-\alpha)^j s^{j+1}[(n-j-2)s + 1] - 1 = 0 \Big\},$$

$$\widetilde{\gamma}_n^4(a) = \sup_{\alpha\in(0,1)} \Big\{ s\colon \alpha a + \alpha^2 s(s+a) + (1-\alpha)^n s^{n-1} + \alpha^2 \sum_{j=1}^{n-2} (1-\alpha)^j s^{j+1}[(n-j-2)s+1] - 1 = 0 \Big\},$$

where $a \in [0, n)$ and $n \geqslant 3$.

**Theorem 5.** *Given an integer $n \geqslant 3$, let $C$ be a nonempty closed convex subset of a Banach space $E$. If $T\colon C \to C$ is uniformly $k$-lipschitzian ($k > 1$) and $(a, n)$-rotative mapping such that*

$$k < \max\left\{\widetilde{\gamma}_n^1(a), \widetilde{\gamma}_n^2(a), \widetilde{\gamma}_n^3(a), \widetilde{\gamma}_n^4(a)\right\},$$

*then $T$ has a fixed point in $C$.*

*Proof.* As in the proof of Theorem 4 we take four cases into consideration.
***Case I***. Let $x_0$ be an arbitrary point in $C$ and let $z$ be the same as in Case I in the proof of Theorem 4. Then from (2) and

$$\begin{aligned}\|T^{n-1}x_0 - z\| &= \|\alpha(T^{n-1}x_0 - T^n x_0) + (1-\alpha)(T^{n-1}x_0 - Tx_{n-2})\| \\ &\leqslant \alpha k\|x_0 - Tx_0\| + (1-\alpha)k\|T^{n-2}x_0 - x_{n-2}\|\end{aligned}$$

and from (4) using only the triangle inequality and the fact that $T$ is uniformly $k$-lipschitzian we get evaluation

$$\begin{aligned}(1-\alpha)k\|T^{n-2}x_0 - x_{n-2}\| &\leqslant \alpha(1-\alpha)k[1+(n-3)k]\|Tx_0 - x_0\| \\ &\quad + \alpha(1-\alpha)^2k^2[1+(n-4)k]\|Tx_0 - x_0\| + \dots \\ &\quad + \alpha(1-\alpha)^{n-2}k^{n-2}\|Tx_0 - x_0\| \\ &= \alpha\sum_{j=1}^{n-2}(1-\alpha)^j k^j[(n-j-2)k+1]\|Tx_0 - x_0\|,\end{aligned} \tag{31}$$

which together with (2) and (7) gives

$$\|z - Tz\| \leqslant \Big\{ \alpha^2k^2 + \alpha^2\sum_{j=1}^{n-2}(1-\alpha)^j k^{j+1}[(n-j-2)k+1] + \alpha(1-\alpha)ak + (1-\alpha)^n k^{n-1} \Big\}\|x_0 - Tx_0\|. \tag{32}$$

The estimate for the expression $\|z - x_0\|$ is the same as in (10). Since

$$\alpha(1-\alpha)ak + \alpha^2k^2 + (1-\alpha)^n k^{n-1} + \alpha^2\sum_{j=1}^{n-2}(1-\alpha)^j k^{j+1}[(n-j-2)k+1] < 1$$

for all $\alpha \in (0,1)$ and $k < \widetilde{\gamma}_n^1(a)$, by inequalities (32) and (10), the Lemma 1 implies the existence of fixed points of $T$ in $C$.

***Case II***. Let $x_0$ be an arbitrary point in $C$ and let $z$ be the same as in Case II in the proof of Theorem 4. From the evaluation

$$\begin{aligned}\|T^{n-1}x_0 - z\| &= \|T^{n-1}x_0 - \alpha T^n x_0 - (1-\alpha)Tx_{n-2}\| \\ &= \|\alpha(T^{n-1}x_0 - T^n x_0) + (1-\alpha)(T^{n-1}x_0 - Tx_{n-2})\| \\ &\leqslant \alpha k\|x_0 - Tx_0\| + (1-\alpha)k\|T^{n-2}x_0 - x_{n-2}\|,\end{aligned} \tag{33}$$

where

$$\begin{aligned}&(1-\alpha)k\|T^{n-2}x_0 - x_{n-2}\| \\ &\quad\leqslant \alpha(1-\alpha)k^2\|x_0 - T^2x_0\| + (1-\alpha)^2k^2\|T^{n-3}x_0 - x_{n-3}\| \\ &\quad\leqslant \alpha(1-\alpha)k^2\|x_0 - T^2x_0\| + \alpha(1-\alpha)^2k^3\|x_0 - T^3x_0\| + \dots \\ &\qquad + \alpha(1-\alpha)^{n-3}k^{n-2}\|x_0 - T^{n-2}x_0\| \\ &\qquad\quad + \alpha(1-\alpha)^{n-2}k^{n-1}\|x_0 - T^{n-1}x_0\| \\ &\quad\leqslant \alpha(1-\alpha)k^2[1+k]\|x_0 - Tx_0\| \\ &\qquad + \alpha(1-\alpha)^2k^3[1+2k]\|x_0 - Tx_0\| + \dots \\ &\qquad\quad + \alpha(1-\alpha)^{n-2}k^{n-1}[1+(n-2)k]\|x_0 - Tx_0\| \\ &\quad= \left\{\alpha\sum_{j=1}^{n-2}(1-\alpha)^j k^{j+1}(1+jk)\right\}\|x_0 - Tx_0\|,\end{aligned} \tag{34}$$

we get (by (33) and (34))

$$\|T^{n-1}x_0 - z\| \leqslant \left\{\alpha k + \alpha\sum_{j=1}^{n-2}(1-\alpha)^j k^{j+1}(1+jk)\right\}\|x_0 - Tx_0\|. \tag{35}$$

By combining (11) with (35) and (15) we get

$$\begin{aligned}\|z - Tz\| \leqslant \Bigg\{&(1-\alpha)k\left[\alpha(1-\alpha)^{n-2}k^{n-2}a + (1-\alpha)^{n-1}k^{n-2}\right] \\ &+ \alpha^2k^2 + \alpha^2\sum_{j=1}^{n-2}(1-\alpha)^j k^{j+2}(1+jk)\Bigg\}\|x_0 - Tx_0\|.\end{aligned} \tag{36}$$

Moreover, from (17) and

$$\begin{aligned}&(1-\alpha)k\|x_{n-2} - T^{n-1}x_0\| \\ &\quad\leqslant \alpha(1-\alpha)k^2\|Tx_0 - x_0\| + (1-\alpha)^2k^2\|T^{n-2}x_0 - x_{n-3}\| \\ &\quad\leqslant \alpha(1-\alpha)k^2\|Tx_0 - x_0\| + \alpha(1-\alpha)^2k^3\|T^2x_0 - x_0\| \\ &\qquad + \alpha(1-\alpha)^3k^4\|T^3x_0 - x_0\| + \dots \\ &\qquad\quad + \alpha(1-\alpha)^{n-2}k^{n-1}\|T^{n-2}x_0 - x_0\| + (1-\alpha)^{n-1}k^{n-1}\|x_0 - Tx_0\| \\ &\quad\leqslant \left\{\alpha\sum_{j=1}^{n-2}(1-\alpha)^j k^{j+1}\left[(j-1)k+1\right] + (1-\alpha)^{n-1}k^{n-1}\right\}\|x_0 - Tx_0\|,\end{aligned}$$

we get the following estimate:

$$\|z - x_0\| \leqslant \Big\{ a + \alpha \sum_{j=1}^{n-2} (1-\alpha)^j k^{j+1} [(j-1)k+1] + (1-\alpha)^{n-1} k^{n-1} \Big\} \|x_0 - Tx_0\|. \tag{37}$$

Since

$$(1-\alpha)k\big[\alpha(1-\alpha)^{n-2}k^{n-2}a + (1-\alpha)^{n-1}k^{n-2}\big] + \alpha^2 k^2 + \alpha^2 \sum_{j=1}^{n-2} (1-\alpha)^j k^{j+2}(1+jk) < 1$$

for all $\alpha \in (0,1)$ and $k < \widetilde{\gamma}_n^2(a)$, by inequalities (36) and (37), the Lemma 1 implies the existence of fixed points of $T$ in $C$.

***Case III***. Let $x_0$ be an arbitrary point in $C$ and let $z$ be the same as in Case III in the proof of Theorem 4. For the expression $\|z - Tz\|$ we have the evaluations (19) and (20). Using only the triangle inequality and the fact that $T$ is uniformly $k$-lipschitzian we obtain

$$\alpha\|T^{n-1}x_0 - x_0\| \leqslant \alpha\,[(n-2)k+1]\,\|Tx_0 - x_0\|. \tag{38}$$

Now, combining (19) with (1) by (20), (38) and (31), we obtain

$$\|z - Tz\| \leqslant \Big\{ \alpha a + \alpha^2 k\,[(n-2)k+1] + (1-\alpha)^n k^{n-1} + \alpha^2 \sum_{j=1}^{n-2} (1-\alpha)^j k^{j+1} [(n-j-2)k+1] \Big\}\, \|x_0 - Tx_0\|. \tag{39}$$

Since

$$\alpha a + \alpha^2 k[(n-2)k+1] + (1-\alpha)^n k^{n-1} + \alpha^2 \sum_{j=1}^{n-2} (1-\alpha)^j k^{j+1} [(n-j-2)k+1] < 1$$

for all $\alpha \in (0,1)$ and $k < \widetilde{\gamma}_n^3(a)$, by inequalities (39) and (26), the Lemma 1 implies the existence of fixed points of $T$ in $C$.

***Case IV***. In this case using the condition of $(a,n)$-rotativness of $T$ we can also evaluate the expression (38) in a different manner. Namely,

$$\begin{aligned} \alpha\|T^{n-1}x_0 - x_0\| &\leqslant \alpha(\|T^{n-1}x_0 - T^n x_0\| + \|T^n x_0 - x_0\|) \\ &\leqslant \alpha(k+a)\|Tx_0 - x_0\|. \end{aligned} \tag{40}$$

Now, combining (19) with (1) by (20), (40) and (31), we obtain

$$\|z - Tz\| \leqslant \Big\{ \alpha a + \alpha^2 k(k+a) + (1-\alpha)^n k^{n-1} + \alpha^2 \sum_{j=1}^{n-2} (1-\alpha)^j k^{j+1}[(n-j-2)k+1] \Big\} \|x_0 - Tx_0\|. \tag{41}$$

Since

$$\alpha a + \alpha^2 k(k+a) + (1-\alpha)^n k^{n-1} + \alpha^2 \sum_{j=1}^{n-2} (1-\alpha)^j k^{j+1}[(n-j-2)k+1] < 1$$

for all $\alpha \in (0,1)$ and $k < \widetilde{\gamma}_n^4(a)$, by inequalities (41) and (26), the Lemma 1 implies the existence of fixed points of $T$ in $C$. This completes the proof. □

*Remark* 4. From Theorem 5 it follows that

$$\widetilde{\gamma}_n^E(a) \geqslant \max\left\{\widetilde{\gamma}_n^1(a), \widetilde{\gamma}_n^2(a), \widetilde{\gamma}_n^3(a), \widetilde{\gamma}_n^4(a)\right\},$$

where $n \geqslant 3$, $a \in [0, n)$.

For $n = 3$ the graph of the lower bound of $\widetilde{\gamma}_3^E(a)$ for an arbitrary Banach space $E$ is illustrated by means of computer graphic in Figure 2 (thick line). It is interesting that the solution of the equation

$$13k^2 + 6ak - 27 = 0$$

(obtained form $\widetilde{\gamma}_3^1(a)$ for $\alpha = \frac{1}{3}$) gives a better evaluation for $a \in [0, \xi]$, while the solution of the equation

$$2k^4 + 2k^3 + (11 + 4a)k^2 - 27 = 0$$

(obtained form $\widetilde{\gamma}_3^2(a)$ for $\alpha = \frac{1}{3}$) for $a \in [\xi, 3)$, where $\xi \approx 1.74$.

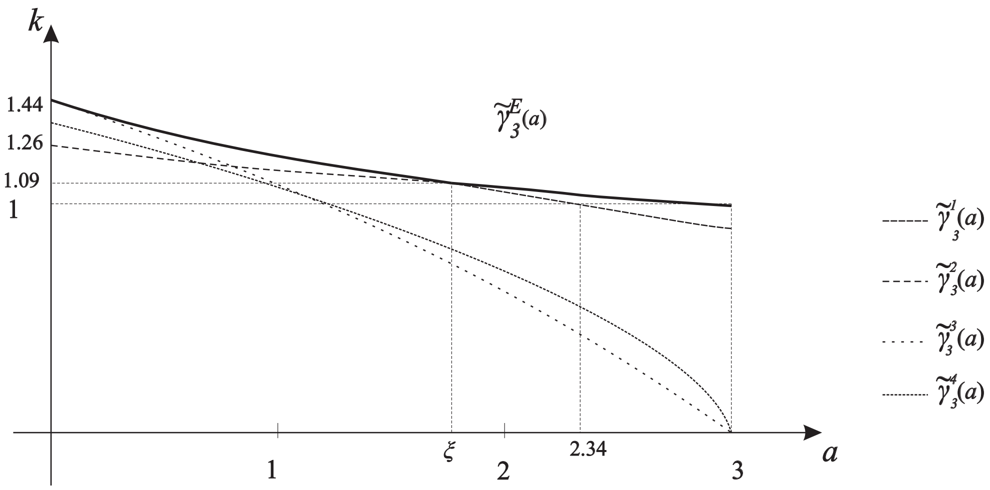


Figure 2.

*Remark* 5. It follows from Theorem 5 that $\widetilde{\gamma}_3^E(0) \geqslant 1.4558$ (this evaluation is obtained from $\widetilde{\gamma}_3^1(a = 0)$ for $\alpha = 0.393$); $\widetilde{\gamma}_4^E(0) \geqslant 1.2917$ (from $\widetilde{\gamma}_4^1(a = 0)$ for $\alpha = 0.322$); $\widetilde{\gamma}_5^E(0) \geqslant 1.2001$ (from $\widetilde{\gamma}_5^1(a = 0)$ for $\alpha = 0.255$); $\widetilde{\gamma}_6^E(0) \geqslant 1.1482$ (from $\widetilde{\gamma}_6^1(a = 0)$ for $\alpha = 0.206$); ... . All these evaluations are better than those obtained by W. A. Kirk [9] in Banach spaces and even than those which are obtained by M. Koter-Mórogowska [10] in Hilbert spaces.

*Remark* 6. All fixed point theorems for $n$-periodic ($n \geqslant 3$) mappings, presented in the paper [6] follow from Theorems 4 and 5.

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