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ON A CLASS OF INFINITE HORIZON OPTIMAL CONTROL PROBLEMS WITH PERIODIC COST FUNCTIONS

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Abstract. In this paper we study discrete time and continuous time infinite horizon optimal control problems with periodic cost functions. For these problems we obtain the reduction to finite cost and the representation formula, and the existence of optimal solutions on infinite horizon.

1. Introduction

The study of optimal control problems defined on infinite intervals has recently been a rapidly growing area of research. These problems arise in engineering [1, 19, 20], in models of economic growth [3, 5, 9, 10, 13-15], in infinite discrete models of solid-state physics related to dislocations in one-dimensional crystals [2, 16] and in the theory of thermodynamical equilibrium for materials [4, 8, 11, 12, 17, 18]. In this paper we study discrete time and continuous time optimal control problems with periodic cost functions. Such problems arise, for example, in the analysis of infinite discrete models for crystals [2, 16].

We consider the infinite horizon problem of minimizing the expression $\sum_{i=0}^{N-1} v(x_i, x_{i+1})$ as N grows to infinity where $\{x_i\}_{i=0}^{\infty}$ is a sequence in the Euclidean n-dimensional space R^n and v is a lower semicontinuous function defined on $R^n \times R^n$. This provides a convenient setting for the study of various optimization problems, e.g., continuous time control systems which are represented by ordinary differential equations whose cost integrand contains a discounting factor [6], the infinite-horizon deterministic control problem of minimizing $\int_0^T L(z(t), z'(t))dt$ as $T \to \infty$ [7], the analysis of a long slender bar of a polymeric material under tension [4, 8, 11], the analysis of an infinite discrete model for crystals which undergo phase transitions [2, 16] and models of economic dynamics [9, 10, 13, 14]. Here we extend the results of [6] obtained for a function v defined on a set $K \times K$ where K is a compact subset of R^n. In our paper v is defined on $R^n \times R^n$ and is periodic.

The paper is organized as follows. The extentions of the results of [6] are obtained in Section 2. In Section 3 we consider variational problems with integrands which are periodic with respect to a state variable. In Section 4 we study the infinite horizon problem of minimizing the expression $\sum_{i=0}^{N-1} v_i(x_i, x_{i+1})$ as N grows to infinity where $\{x_i\}_{i=0}^{\infty}$ is a sequence in the Euclidean n-dimensional space R^n and $\{v_i\}_{i=0}^{\infty}$ is a sequence of lower semicontinuous periodic functions defined on $R^n \times R^n$. In Section 5 we present our main application which is devoted to continuous time

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periodic control systems. We establish that under certain conditions such infinite horizon systems have overtaking optimal solutions.

2. Autonomous discrete-time periodic control systems

Let R^n be the Euclidean n-dimensional space,

$$|x| = \max\{|x_i| : i = 1, \dots, n\} \text{ for all } x = (x_1, \dots, x_n) \in R^n$$

and let $\mathbf{Z}$ be the set of all integers. Assume that $v : R^n \times R^n \to R^1$ is a lower semi-continous function (i.e. $v(\lim_{k\to\infty}(x_k, y_k)) \le \liminf_{k\to\infty} v(x_k, y_k))$ which satisfy the following assumptions:

(2.1)
$$\sup\{v(x, y) : x, y \in R^n, \ 0 \le x_i \le 1 \text{ and } 0 \le y_i - x_i \le 1 \text{ for } i = 1, \dots, n\} = a < \infty,$$

$$\inf\{v(x, y) : x, y \in R^n\} = b > -\infty, \tag{2.2}$$

$$v(x + m, y + m) = v(x, y) \text{ for each } x, y \in R^n \text{ and each } m \in \mathbf{Z}^n, \tag{2.3}$$

there exists a number $\Gamma > 0$ such that

$$\inf\{v(x, y) : x, y \in R^n \text{ and } |x - y| \ge \Gamma\} \ge a. \tag{2.4}$$

We will prove the following result which is an extention of Theorem 3.1 of [6] established for a function $v : K \times K \to R^1$ where K is a compact subset of R^n.

Theorem 2.1. *There exists a constant μ such that:*

(1) *For every sequence $\{z_i\}_{i=0}^{\infty} \subset R^n$ and every integer $N \ge 0$ the inequality*

$$\sum_{i=0}^{N} [v(z_i, z_{i+1}) - \mu] \ge b - a$$

holds.

(2) *For every sequence $\{z_i\}_{i=0}^{\infty} \subset R^n$ the sequence $\left\{\sum_{i=0}^{N} [v(z_i, z_{i+1}) - \mu]\right\}_{N=0}^{\infty}$ is either bounded or it diverges to infinity.*

(3) *For every initial value z_0 there is a sequence $\{z_i^*\}_{i=0}^{\infty}$ with $z_0^* = z_0$ which satisfies*

$$\left| \sum_{i=0}^{N} [v(z_i^*, x_{i+1}^*) - \mu] \right| \le 4(a - b)$$

for all integers $N \ge 0$.

We preface the proof of the theorem by auxiliary lemmas.

Define

$$\mu = \inf\left\{ \liminf_{N\to\infty} N^{-1} \sum_{i=0}^{N-1} v(z_i, z_{i+1}) : \{z_i\}_{i=0}^{\infty} \subset R^n \right\}. \tag{2.5}$$

For any natural number N set

$$\lambda(N) = \inf\left\{ N^{-1} \sum_{i=0}^{N-1} v(z_i, z_{i+1}) : \{z_i\}_{i=0}^{N} \subset R^n \text{ and } z_N - z_0 \in \mathbf{Z}^n \right\}, \tag{2.6}$$

$$\rho(N) = \inf\left\{N^{-1}\sum_{i=0}^{N-1} v(z_i, z_{i+1}) : \{z_i\}_{i=0}^{N} \subset R^n\right\}. \tag{2.7}$$

Remark 2.1. Let N be a natural number and let $\{z_i\}_{i=0}^{N} \subset R^n$ satisfy $z_N - z_0 \in \mathbf{Z}^n$. We can associate with $\{z_i\}_{i=0}^{N}$ a sequence $\{y_i\}_{i=0}^{\infty} \subset R^n$ such that

$$y_i = z_i,\ i = 0, \dots, N,$$

$$y_{i+jN} = y_i + j(z_N - z_0) \text{ for all integers } i, j \geq 0.$$

Remark 2.1 and relations (2.2), (2.3), (2.5), (2.6) and (2.7) imply that

$$\rho(N) \leq \mu \leq \lambda(N),\ N = 1, 2, \dots \tag{2.8}$$

Set

$$A = \{(x + m, y + m) : \ x = (x_1, \dots, x_n),\ y = (y_1, \dots, y_n) \in R^n \text{ satisfy}$$
$$0 \leq x_i \leq 1,\ 0 \leq y_i - x_i \leq 1 \text{ for } i = 1, \dots, n \text{ and } m \in \mathbf{Z}^n\}.$$

Lemma 2.2. *$N(\lambda(N) - \rho(N)) \leq a - b$ for all natural numbers N.*

Proof. Let N be a natural number and $\{z_i\}_{i=0}^{N} \subset R^n$. Evidently there is a sequence $\{y_i\}_{i=0}^{N} \subset R^n$ such that

$$y_i = z_i,\ i = 0, \dots, N-1,\ y_N - y_0 \in \mathbf{Z}^n \text{ and } (y_{N-1}, y_N) \in A.$$

By (2.1)-(2.3) and (2.6)

$$N\lambda(N) \leq \sum_{i=0}^{N-1} v(y_i, y_{i+1}) \leq \sum_{i=0}^{N-1} v(z_i, z_{i+1}) - b + a.$$

Since this inequality holds for an arbitrary sequence $\{z_i\}_{i=0}^{N} \subset R^n$, this completes the proof of the lemma. □

Lemma 2.3. *Let $\{z_i\}_{i=0}^{\infty} \subset R^n$ and let q be a natural number such that $|z_q - z_{q-1}| \geq \Gamma$. Assume that a sequence $\{y_i\}_{i=0}^{\infty} \subset R^n$ satisfies*

$$y_i = z_i,\ i = 0, \dots, q-1,\ (y_{q-1}, y_q) \in A,\ y_i - z_i = y_q - z_q \in \mathbf{Z}^n \text{ for all integers } i \geq q.$$

Then $v(z_i, z_{i+1}) \geq v(y_i, y_{i+1})$ for all integers $i \geq 0$.

The validity of Lemma 2.3 follows from relations (2.1), (2.3) and (2.4).

Proof of Theorem 2.1. Let $\{z_i\}_{i=0}^{\infty} \subset R^n$ and N be a natural number. There exists a sequence $\{y_i\}_{i=0}^{N} \subset R^n$ such that

$$y_i = z_i,\ i = 0, \dots, N-1,\ (y_{N-1}, y_N) \in A \text{ and } y_N - y_0 \in \mathbf{Z}^n.$$

It follows from (2.1), (2.2), (2.3), (2.6) and (2.8) that

$$\sum_{i=0}^{N-1} v(z_i, z_{i+1}) \geq \sum_{i=0}^{N-1} v(y_i, y_{i+1}) + b - a \geq N\lambda(N) + b - a \geq N\mu + b - a.$$

Assertion 1 of Theorem 2.1 is established.

Assertion 2 follows from Assertion 1. Let us prove Assertion 3. It is sufficient to establish the existence of a sequence $\{z_i\}_{i=0}^{\infty} \subset R^n$ such that

$$\left|\sum_{i=0}^{N}[v(z_i, z_{i+1}) - \mu]\right| \le 2(a-b) \text{ for all integers } N \ge 0.$$

We can assume without loss of generality that $\Gamma > 2$. Let N be a natural number. Lemma 2.3 implies that there is a sequence $\{z_i^N\}_{i=0}^{N} \subset R^n$ such that

$$|z_i^N - z_{i+1}^N| \le \Gamma,\ i = 0\dots, N-1,\ z_0^N - z_N^N \in \mathbf{Z}^n,\ |z_0^N| \le 1$$

and

$$\sum_{i=0}^{N-1} v(z_i^N, z_{i+1}^N) = N\lambda(N).$$

By Lemma 2.2 and (2.8)

$$\sum_{i=0}^{N-1}[v(z_i^N, z_{i+1}^N) - \mu] \le a - b,\ N = 1, 2, \dots. \tag{2.9}$$

Clearly there exists a strictly increasing sequence of natural numbers $\{N_j\}_{j=1}^{\infty}$ such that for every integer $i \ge 0$

$$z_i^{N_j} \to y_i \in R^n \text{ as } j \to \infty.$$

Fix a natural number N. For all large natural numbers j it follows from Assertion 1 and (2.9) that

$$\sum_{i=0}^{N_j-1}[v(z_i^{N_j}, z_{i+1}^{N_j}) - \mu] \le a - b,$$

$$\sum_{i=N}^{N_j-1}[v(z_i^{N_j}, z_{i+1}^{N_j}) - \mu] \ge -a + b,$$

$$\sum_{i=0}^{N-1}[v(z_i^{N_j}, z_{i+1}^{N_j}) - \mu] \le 2(a - b).$$

This relation implies that

$$\sum_{i=0}^{N-1}[v(y_i, y_{i+1}) - \mu] \le 2(a - b),$$

which completes the proof of the theorem. □

The next result is an extension of Proposition 5.1 of [6], which is concerned with obtained for a function $v : K \times K \to R^1$ where K is a compact subset of R^n.

Theorem 2.2. *Let v be a continuous function. We define*

$$\pi(x) = \inf\left\{\liminf_{N\to\infty}\sum_{i=0}^{N-1}[v(z_i, z_{i+1}) - \mu] :\ \{z_i\}_{i=0}^{\infty} \subset R^n,\ z_0 = x\right\}, \tag{2.10}$$

$$\theta(x, y) = v(x, y) - \mu + \pi(y) - \pi(x) \tag{2.11}$$

for each $x, y \in R^n$. *Then* $\pi : R^n \to R^1$ *and* $\theta : R^n \times R^n \to R^1$ *are continuous functions,*

$$\pi(x+m) = \pi(x),\ \theta(x+m, y+m) = \theta(x, y) \quad \text{for each } x, y \in R^n \text{ and each } m \in \mathbf{Z}^n, \tag{2.12}$$

the function θ *is nonnegative and*

$$E(x) = \{y \in R^n :\ \theta(x, y) = 0\}$$

is nonempty for any $x \in R^n$.

Proof. We can assume without loss of generality that $\Gamma > 2$. For $x \in R^n$ we set

$$\Lambda(x) = \{\{z_i\}_{i=0}^{\infty} \subset R^n :\ z_0 = x \text{ and } |z_1 - z_0| \le \Gamma\}.$$

It is easy to verify that relation (2.12) holds and

$$\pi(x) \le v(x, y) - \mu + \pi(y) \text{ for all } x, y \in R^n.$$

Thus θ is nonnegative. Lemma 2.3 implies that

$$\pi(x) = \inf\left\{ \liminf_{N \to \infty} \sum_{i=0}^{N-1} [v(z_i, z_{i+1}) - \mu] :\ \{z_i\}_{i=0}^{\infty} \in \Lambda(x) \right\},\ x \in R^n.$$

This relation and the uniform continuity of v on bounded subsets of $R^n \times R^n$ imply the continuity of the function π.

It only remains to prove that $E(x) \ne \emptyset$ for every $x \in R^n$. Suppose to the contrary that for some $x \in R^n$ we have $E(x) = \emptyset$. There is a sequence $\{x_i\}_{i=1}^{\infty} \subset R^n$ such that $\theta(x, x_i) \to \inf\{\theta(x, y) :\ y \in R^n\}$ as $i \to \infty$.

Let i be a natural number. If $|x_i - x| > \Gamma$ we choose $y_i \in R^n$ such that $(x, y_i) \in A$ and $y_i - x_i \in \mathbf{Z}^n$. If $|x_i - x| \le \Gamma$ we set $y_i = x_i$. Relations (2.1), (2.3) and (2.4) imply that

$$\theta(x, y_i) \le \theta(x, x_i),\ i = 1, 2, \dots.$$

Now it is easy to verify that there exists $\bar{x} \in R^n$ such that

$$\theta(x, \bar{x}) = \inf\{\theta(x, y) :\ y \in R^n\} = \delta > 0.$$

There is a sequence $\{z_i\}_{i=1}^{\infty} \subset R^n$ such that $z_0 = x$ and

$$\liminf_{N \to \infty} \sum_{i=0}^{N-1} [v(z_i, z_{i+1}) - \mu] \le \pi(x) + 2^{-1}\delta.$$

We have

$$\pi(x) + 2^{-1}\delta \ge [\theta(x, z_1) + \pi(x) - \pi(z_1)]$$

$$+ \liminf_{N \to \infty} \sum_{i=1}^{N} [v(z_i, z_{i+1}) - \mu] \ge [\delta + \pi(x) - \pi(z_1)] + \pi(z_1).$$

We obtained a contradiction, hence $E(x) \ne \emptyset$ for all $x \in R^n$. The theorem is proved. □

3. Variational problems with periodic integrands

Let $L : R^n \times R^n \to R^1$ be a bounded below Borel function which is bounded on any compact subset of R^{2n}. We assume that

$$L(x+m, v) = L(x, v) \text{ for all } x, v \in R^n \text{ and all } m \in \mathbf{Z}^n \tag{3.1}$$

and that there exist positive numbers c_1, c_2 such that

$$L(z, y) \geq c_1|y| \text{ for all } z, y \in R^n \text{ such that } |y| \geq c_2. \tag{3.2}$$

A trajectory is an absolutely continuous function $z : \Delta \to R^n$ where Δ is either $[a, b] \subset R^1$ or $[a, \infty)$.

We will establish the following result which extends Theorem 4.1 of [7], established for integrands $L : K \times R^n \to R^1$, where K is a compact subset of R^n, which satisfy a Lipschitzian condition with respect to the state variable.

Theorem 3.1. *There exist numbers $M(L) > 0$ and $\mu(L)$ such that:*

(1) *For any trajectory $z : [0, \infty) \to R^n$ and any number $T > 0$*

$$\int_0^T [L(z(t), z'(t)) - \mu(L)]dt \geq -M(L).$$

(2) *For any trajectory $z : [0, \infty) \to R^n$ the function*

$$T \to \int_0^T [L(z(t), z'(t)) - \mu(L)]dt, \; T \in (0, \infty)$$

is either bounded or diverges to infinity as $T \to \infty$.

(3) *For any $z_0 \in R^n$ there exists a trajectory $z : [0, \infty) \to R^n$ such that $z(0) = z_0$ and for any $T > 0$*

$$\left| \int_0^T [L(z(t), z'(t)) - \mu(L)]dt \right| \leq M(L).$$

We preface the proof of Theorem 3.1 by several preliminary propositions. Set

$$d_L = \inf\{L(x, y) : \; x, y \in R^n\}. \tag{3.3}$$

For $x, y \in K$ we set

$$u(x, y) = \inf \left\{ \int_0^1 L(z(t), z'(t))dt : \; z : [0, 1] \to R^n \text{ is a trajectory, } z(0) = x, \; z(1) = y \right\}.$$

It is easy to verify that

$$\inf\{u(x, y) : \; x, y \in R^n\} \geq d_L, \tag{3.4}$$

the function $u : R^n \times R^n \to R^1$ is bounded on any compact subset of R^{2n} and that

$$u(x+m, y+m) = u(x, y) \text{ for all } x, y \in R^n \text{ and all } m \in \mathbf{Z}^n. \tag{3.5}$$

Lemma 3.1. *For any positive number K there exists a number $\Gamma \geq 0$ such that*

$$u(x, y) \geq K \text{ for all } x, y \in R^n \text{ satisfying } |x - y| \geq \Gamma.$$

Proof. Let K be a positive number. Choose a positive number Γ such that

$$\Gamma \geq c_2 + c_1^{-1}(K + \sup\{|L(x,y)| : \ x,y \in R^n, \ |y| \leq c_2\}). \tag{3.6}$$

Let $x, y \in R^n$ satisfy $|x-y| \geq \Gamma$ and let $z : [0,1] \to R^n$ be a trajectory satisfying $z(0) = x, \ z(1) = y$. Set

$$F_1 = \{t \in [0,1] : \ |z'(t)| < c_2\}, \ F_2 = [0,1] \setminus F_1.$$

By (3.2) and (3.6)

$$\int_0^1 L(z(t), z'(t))dt \geq \int_{F_2} L(z(t), z'(t))dt - \sup\{|L(\xi,\eta)| : \ \xi,\eta \in R^n, \ |\eta| \leq c_2\},$$

$$\Gamma \leq |x-y| \leq \int_0^1 |z'(t)|dt \leq c_2 + \int_{F_2} |z'(t)|dt \leq c_2 + \int_{F_2} c_1^{-1} L(z(t), z'(t))dt \leq$$

$$\leq c_2 + c_1^{-1}\left[\int_0^1 L(z(t), z'(t))dt + \sup\{|L(\xi,\eta)| : \ \xi,\eta \in R^n, \ |\eta| \leq c_2\}\right],$$

$$\int_0^1 L(z(t), z'(t))dt \geq K.$$

Hence $u(x,y) \geq K$ and the lemma is proved. □

For $x, y \in R^n$ we define

$$v(x,y) = \liminf_{(\xi,\eta)\to(x,y)} u(\xi,\eta)$$

where $\xi, \eta \in R^n$. Evidently $v : R^n \times R^n \to R^1$ is bounded from below, lower semicontinuous function which is bounded on any compact subset of R^{2n}. Relation (3.5) implies that

$$v(x+m, y+m) = v(x,y) \text{ for all } x, y \in R^n \text{ and all } m \in \mathbf{Z}^n.$$

Set

$$b = \inf\{v(x,y) : \ x, y \in R^n\},$$

$$a = \sup\{v(x,y) : \ x = (x_1, \dots, x_n), \ y = (y_1, \dots, y_n) \in R^n,$$

$$0 \leq x_i \leq 1, \ 0 \leq y_i - x_i \leq 1 \text{ for } i = 1, \dots, n\},$$

$$\mu = \inf\{\liminf_{N\to\infty} N^{-1} \sum_{i=0}^{N-1} v(x_i, x_{i+1}) : \ \{x_i\}_{i=0}^{\infty} \subset R^n\}.$$

By Lemma 3.1 there exists a positive number Γ such that

$$\inf\{v(x,y) : \ x, y \in R^n, \ |x-y| \geq \Gamma\} \geq a+1.$$

It is easy to see that Theorem 2.1 is valid with v, μ, a, b.

Lemma 3.2. *Let $x, y \in R^n$ and $\epsilon \in (0, 1/2)$. Then there exists $\gamma \in (0,\epsilon)$ and a trajectory $z : [0, 1+\gamma] \to R^n$ such that*

$$z(0) = x, \ z(1+\gamma) = y, \ \int_0^{1+\gamma} L(z(t), z'(t))dt \leq v(x,y) + \epsilon.$$

Proof. Set

$$K = \sup\{|L(z,v)| : \ z, v \in R^n, \ |v| \le 16\}. \tag{3.7}$$

Choose $\gamma \in (0, \epsilon)$ such that

$$\gamma K < \epsilon/8. \tag{3.8}$$

It is easy to see that there are $x_1, y_1 \in R^n$ such that

$$|x - x_1| \le 8^{-1}\gamma, \ |y - y_1| \le \gamma/8, \tag{3.9}$$
$$u(x_1, y_1) < v(x, y) + \gamma/8.$$

There exists a trajectory $z_0 : [0,1] \to R^n$ such that

$$z(0) = x_1, \ z(1) = y_1, \ \int_0^1 L(z_0(t), z_0'(t))dt < v(x,y) + \gamma/8. \tag{3.10}$$

Define a trajectory $z : [0, 1+\gamma] \to R^n$ such that

$$z(t) = x + 2\gamma^{-1}t(x_1 - x), \ t \in [0, \gamma/2],$$
$$z(t) = z_0(t - \gamma/2), \ t \in [\gamma/2, 1 + \gamma/2],$$
$$z(t) = y_1 + 2\gamma^{-1}(t - 1 - \gamma/2)(y - y_1), \ t \in [1 + \gamma/2, 1 + \gamma].$$

Clearly the trajectory z is well defined and satisfies $z(0) = x$, $z(1+\gamma) = y$. By (3.7) and (3.9)

$$|L(z(t), z'(t))| \le K, \ t \in (0, \gamma/2)$$

and

$$L(z(t), z'(t)) \le \Delta, \ t \in (1 + \gamma/2, 1 + \gamma).$$

These relations together with (3.8) and (3.10) imply the validity of the lemma. □

Proof of Theorem 3.1. Set

$$\mu(L) = \mu, \ M(L) = 5(a - b) + |d_L| + |\mu| + 1.$$

Note that Theorem 2.1 is valid with v, μ, a, b. Let $z : [0, \infty) \to R^n$ be a trajectory. By Theorem 2.1

$$\int_0^N [L(z(t), z'(t)) - \mu]dt \ge \sum_{i=0}^{N-1}[v(z(i), z(i+1)) - \mu] \ge b - a \text{ for all natural numbers } N.$$

Let T be a positive number. There is an integer $N \ge 0$ such that $N < T \le N + 1$. In view of (3.3)

$$\int_0^T [L(z(t), z'(t)) - \mu]dt \ge \int_N^T [L(z(t), z'(t)) - \mu]dt + b - a \tag{3.11}$$
$$\ge b - a - |d_L| - |\mu|.$$

Thus Assertion 1 of Theorem 3.1 is proved.

Assertion 2 follows from Assertion 1. We will prove Assertion 3. Let $z_0 \in R^n$. By Theorem 2.1 there exists a sequence $\{x_i\}_{i=0}^{\infty} \subset R^n$ such that $x_0 = z_0$ and

$$\left|\sum_{i=0}^{N}[v(x_i, x_{i+1}) - \mu]\right| \le 4(a - b) \text{ for all integers } N \ge 0. \tag{3.12}$$

Set

$$\epsilon_i = 2^{-i}(1+|\mu|)^{-1},\ i = 1, 2, \dots .$$

By induction using Lemma 3.2 we construct a sequence of numbers $\gamma_i \in (0, \epsilon_i)$, $i = 1, 2 \dots$ and a trajectory $z : [0, \infty) \to R^n$ such that for all integers $N \geq 0$

$$z(\beta_N) = x_N,\ \int_{\beta_N}^{\beta_{N+1}} L(z(t), z'(t))dt \leq v(x_N, x_{N+1}) + \epsilon_{N+1},$$

where $\beta_0 = 0$, $\beta_N = \sum_{i=1}^{N} \gamma_i + N$ for all natural numbers N. By these relations and by relation (3.12) for $N = 1, 2, \dots$

$$\int_0^{\beta_N} [L(z(t), z'(t)) - \mu]dt \leq -\mu\beta_N + \sum_{i=0}^{N-1} [v(x_i, x_{i+1}) + \epsilon_{i+1}]$$

$$\leq \sum_{i=0}^{N-1} [v(x_i, x_{i+1}) - \mu] - \mu(\beta_N - N) + \sum_{i=1}^{N} \epsilon_i$$

$$\leq 4(a-b) + (1+|\mu|) \sum_{i=1}^{N} \epsilon_i \leq 4(a-b) + 1.$$

Let T be a positive number. Choose a natural number N such that $\beta_N > T + 1$. Then by relation (3.11) which holds for any trajectory

$$\int_0^T [L(z(t), z'(t)) - \mu]dt = \int_0^{\beta_N} [L(z(t), z'(t)) - \mu]dt$$

$$- \int_T^{\beta_N} [L(z(t), z'(t)) - \mu]dt \leq 4(a-b) + 1 + (a - b + |d_L| + |\mu|) \leq M(L).$$

The proof of the theorem is complete. □

For $x \in R^n$ we set

$$\pi(x) = \inf \left\{ \liminf_{T\to\infty} \int_0^T [L(z(t), z'(t)) - \mu(L)]dt : \right.$$

$$\left. z : [0, \infty) \to R^n \text{ is a trajectory and } z(0) = x \right\}.$$

By Theorem 3.1 the function $\pi : R^n \to R^1$ is bounded,

$$|\pi(x)| \leq M(L) \text{ for each } x \in R^n,$$

$$\pi(x + m) = \pi(x) \text{ for each } x \in R^n \text{ and each } m \in \mathbf{Z}^n.$$

Let δ be a positive number. A trajectory $s : [0, \infty) \to R^n$ is called δ-weakly optimal [6] if there exists a strictly increasing sequence of positive numbers $\{T_i\}_{i=1}^{\infty}$ such that $T_i \to \infty$ as $i \to \infty$ and that for any trajectory $z : [0, \infty) \to R^n$ satisfying $z(0) = s(0)$ the relation

$$\int_0^{T_i} [L(s(t), s'(t)) - L(z(t), z'(t))]dt \leq \delta$$

holds for all large i.

Proposition 3.1. *For any $x \in R^n$ and any $\delta > 0$ there exists a δ-weakly optimal trajectory $s : [0,\infty) \to R^n$ satisfying $s(0) = x$.*

Proof. There is a trajectory $s : [0,\infty) \to R^n$ such that $s(0) = x$ and

$$\liminf_{T\to\infty} \int_0^T [L(s(t), s'(t)) - \mu(L)]dt \le \pi(x) + \delta/4.$$

To complete the proof we should only note that there exists a strictly increasing sequence of positive numbers $\{T_i\}_{i=1}^{\infty}$ such that $T_i \to \infty$ and

$$\lim_{i\to\infty} \int_0^{T_i} [L(s(t), s'(t)) - \mu(L)]dt \le \pi(x) + \delta/2. \qquad \square$$

Proposition 3.2. *$\pi : R^n \to R^1$ is a Lipschitzian function.*

Proof. Set

$$K = \sup\{|L(z,v)| : \; z, v \in R^n \text{ and } |v| \le 16\}.$$

Let $x, y \in R^n$ satisfy $0 < |x-y| \le 1$ and let $z(\cdot) : [0,\infty) \to R^n$ be a trajectory such that $z(0) = y$. We define a trajectory $z_1 : [0,\infty) \to R^n$ by

$$z_1(t) = x + t|x-y|^{-1}(y-x), \; t \in [0, |x-y|],$$

$$z_1(t + |x-y|) = z(t), \; t \in [0,\infty).$$

Evidently z_1 is well defined and

$$\begin{aligned}\pi(x) &\le \liminf_{T\to\infty} \int_0^T [L(z_1(t), z_1'(t)) - \mu(L)]dt \\ &= \int_0^{|x-y|} [L(z_1(t), z_1'(t)) - \mu(L)]dt + \\ &+ \liminf_{T\to\infty} \int_0^T [L(z(t), z'(t)) - \mu(L)]dt \\ &\le \liminf_{T\to\infty} \int_0^T [L(z(t), z'(t)) - \mu(L)]dt + |x-y|(|\mu(L)| + K).\end{aligned}$$

This relation holds for any trajectory $z : [0,\infty) \to R^n$ satisfying $z(0) = y$. Hence

$$\pi(x) \le \pi(y) + |x-y|(|\mu(L)| + \delta).$$

This completes the proof of the proposition. $\square$

4. Discrete time nonautonomous problems

Let $v_i : R^n \times R^n \to R^1 \cup \{\infty\}$, $i = 0, 1, 2, \dots$ be a sequence of functions such that for each integer $i \ge 0$ the following conditions hold:

$$a_i = \sup\{v_i(x,y) : \; x = (x_1, \dots, x_n), \; y = (y_1, \dots, y_n) \in R^n, \quad 0 \le x_j \le 1, \; 0 \le y_j - x_j \le 1 \text{ for all } j = 1, \dots, n\} < \infty, \tag{4.1}$$

$$b_i = \inf\{v_i(x,y) : \; x, y \in R^n\} > -\infty, \tag{4.2}$$

$$v_i(x+m, y+m) = v_i(x,y) \text{ for each } x, y \in R^n \text{ and each } m \in \mathbf{Z}^n, \tag{4.3}$$

there exists a number $\Gamma_i > 0$ such that

$$\inf\{v_i(x, y) : \ x, y \in R^n \text{ and } |x - y| \geq \Gamma_i\} \geq a_i. \tag{4.4}$$

We assume that

$$a = \sup\{a_i : \ i = 0, 1, \dots\} < \infty, \tag{4.5}$$

$$b = \inf\{b_i : \ i = 0, 1, \dots\} > -\infty. \tag{4.6}$$

We may assume without loss of generality that

$$\Gamma_i \geq 2 \text{ for all integers } i \geq 0. \tag{4.7}$$

For $x \in R^n$ and a natural number N we set

$$S(x, N) = \inf \left\{ \sum_{i=0}^{N-1} v_i(z_i, z_{i+1}) : \ \{z_i\}_{i=0}^{N} \subset R^n, \ z_0 = x \right\}.$$

Also we set

$$A = \{(x, y) \in R^n \times R^n : \ x = (x_1, \dots, x_n), \ y = (y_1, \dots, y_n), \\ 0 \leq y_i - x_i \leq 1 \text{ for } i = 1, \dots, n\}. \tag{4.8}$$

Relations (4.1), (4.3) and (4.4) imply the following lemma.

Lemma 4.1. *Let $\{z_i\}_{i=0}^{\infty} \subset R^n$ and let q be a natural number for which $|z_q - z_{q-1}| \geq \Gamma_{q-1}$. We define a sequence $\{y_i\}_{i=0}^{\infty} \subset R^n$ by*

$$y_i = z_i, \ i = 0, \dots, q-1, \ y_q - z_q \in \mathbf{Z}^n, \ (y_{q-1}, y_q) \in A,$$

$$y_i = z_i + y_q - z_q \text{ for all integers } i \geq q.$$

Then $v_i(z_i, z_{i+1}) \geq v_i(y_i, y_{i+1}), \ i = 0, 1, \dots$.

Theorem 4.1. *Let $v_i, \ i = 0, 1, \dots$ be a sequence of lower semicontinuous functions. Then for any $x \in R^n$ there exists a sequence $\{x_i\}_{i=0}^{\infty} \subset R^n$ such that*

$$x_0 = x, \ |x_i - x_{i+1}| \leq \Gamma_i, \ i = 0, 1, \dots.$$

$$\sum_{i=0}^{N-1} v_i(x_i, x_{i+1}) \leq S(x, N) + a_N - b_N, \ N = 1, 2, \dots.$$

Proof. Let $x \in K$. Lemma 4.1 implies that for any natural number N there is a sequence $\{z_i^N\}_{i=0}^{N} \subset R^n$ such that

$$z_0^N = x, \ |z_{i+1}^N - z_i^N| \leq \Gamma_i, \ i = 0, \dots, N-1,$$

$$\sum_{i=0}^{N-1} v_i(z_i^N, z_{i+1}^N) = S(x, N).$$

Let m, N be natural numbers such that $m < N$. Clearly there is a sequence $\{z_i\}_{i=0}^{N} \subset R^n$ such that

$$z_i = z_i^m, \ i = 0, \dots, m, \ z_{m+1} - z_{m+1}^N \in \mathbf{Z}^n, \ (z_m, z_{m+1}) \in A,$$

$$z_i = z_i^N + z_{m+1} - z_{m+1}^N, \ i = m+1, \dots, N.$$

In view of (4.1), (4.2), (4.8) and (4.3)

$$0 \le \sum_{i=0}^{N-1} [v_i(z_i, z_{i+1}) - v_i(z_i^N, z_{i+1}^N)] = S(x, m)$$

$$- \sum_{i=0}^{m-1} v_i(z_i^N, z_{i+1}^N) + v_m(z_m, z_{m+1}) - v_m(z_m^N, z_{m+1}^N),$$

$$\sum_{i=0}^{m-1} v_i(z_i^N, z_{i+1}^N) \le S(x, m) + a_m - b_m \tag{4.9}$$

for each pair of natural numbers m, N satisfying $m < N$. There exists a strictly increasing sequence of natural numbers $\{N_k\}_{k=1}^{\infty}$ such that $z_i^{N_k} \to x_i$ as $k \to \infty$ for any integer $i \ge 0$. Relation (4.9) implies that

$$\sum_{i=0}^{m-1} v_i(x_i, x_{i+1}) \le S(x, m) + a_m - b_m, \; m = 1, 2, \dots.$$

The theorem is proved. □

Theorem 4.1 inplies the following result.

Theorem 4.2. *Let v_i, $i = 0, 1, \dots$ be a sequence of lower semicontinuous functions and $a_i - b_i \to 0$ as $i \to \infty$. Then for any $x \in R^n$ there exists a sequence $\{x_i\}_{i=1}^{\infty} \subset R^n$ such that*

$$x_0 = x, \; |x_i - x_{i+1}| \le \Gamma_i, \; i = 0, 1, \dots,$$

$$S(x, N) - \sum_{i=0}^{N-1} v_i(x_i, x_{i+1}) \to 0 \text{ as } N \to \infty.$$

Theorem 4.3. *Let $a_i - b_i \to 0$ as $i \to \infty$. Then for every $x \in R^n$ for every $\epsilon > 0$ there exists a sequence $\{y_i\}_{i=0}^{\infty} \subset R^n$ such that*

$$y_0 = x, \; |y_i - y_{i+1}| \le \Gamma_i, \; i = 0, 1, \dots$$

and that

$$\sum_{i=0}^{N-1} v_i(y_i, y_{i+1}) \le S(x, N) + \epsilon$$

for all sufficient large N.

Proof. Let $x \in R^n$ and $\epsilon > 0$. Set $\epsilon_i = 2^{-i-3}\epsilon, \; i = 1, 2, \dots$. Lemma 4.1 implies that for any natural number N there exists a sequence $\{z_i^N\}_{i=0}^{N} \subset R^n$ such that $z_0^N = x$,

$$|z_i^N - z_{i+1}^N| \le \Gamma_i, \; i = 0, \dots, N-1, \tag{4.10}$$

and that

$$\sum_{i=0}^{N-1} v_i(z_i^N, z_{i+1}^N) \le S(x, N) + \epsilon_N. \tag{4.11}$$

Let m, N be natural numbers satisfying $m < N$. There exists a sequence $\{z_i(m,N)\}_{i=0}^N$ such that

$$z_i(m,N) = z_i^m,\ i = 0,\dots,m,\ (z_m(m,N), z_{m+1}(m,N)) \in A,$$
$$z_{m+1}(m,N) - z_{m+1}^N \in \mathbf{Z}^n,\ z_i(m,N) - z_i^N = z_{m+1}(m,N) - z_{m+1}^N,$$
$$i = m+1,\dots,N.$$

(4.11), (4.8), (4.2), (4.3) and (4.1) imply that

$$\epsilon_N \ge \sum_{i=0}^{N-1} [v_i(z_i^N, z_{i+1}^N) - v_i(z_i(m,N), z_{i+1}(m,N))]$$
$$\ge \sum_{i=0}^{m-1} v_i(z_i^N, z_{i+1}^N) - S(x,m) - \epsilon_m + b_m - v_m(z_m(m,N), z_{m+1}(m,N))$$
$$\ge \sum_{i=0}^{m-1} v_i(z_i^N, z_{i+1}^N) - S(x,m) - \epsilon_m - a_m + b_m$$
$$\ge b_m - a_m - \epsilon_m, \tag{4.12}$$

$$\sum_{i=0}^{m-1} v_i(z_i^N, z_{i+1}^N) \le S(x,m) + \epsilon_m + \epsilon_N + a_m - b_m \tag{4.13}$$

for each pair of natural numbers m, N satisfying $m < N$. Choose a strictly increasing sequence of nonnegative integers $\{N_i\}_{i=0}^\infty$ such that $N_0 = 0$, $N_{i+1} - N_i \ge 10$ for all integers $i \ge 0$ and that

$$\sum_{i=1}^{\infty} (a_{N_i} - b_{N_i}) < \epsilon/8. \tag{4.14}$$

It is easy to see that there exists a sequence $\{y_i\}_{i=0}^\infty \subset R^n$ such that

$$y_0 = x,\ y_i = z_i^{N_1},\ i = 1,\dots,N_1$$

and that for all natural numbers k

$$(y_{N_k}, y_{N_k+1}) \in A,\ y_{N_{k+1}} - z_{N_k+1}^{N_k+1} \in \mathbf{Z}^n,$$
$$y_i = z_i^{N_k+1} + y_{N_k+1} - z_{N_k+1}^{N_k+1},\ i = N_k+1,\dots,N_{k+1}.$$

We will show that $\{y_i\}_{i=0}^\infty$ is the required sequence.

By induction we will prove that for all integers $k \ge 2$ the following relation holds:

$$\sum_{i=0}^{N_k-1} [v_i(y_i, y_{i+1}) - v_i(z_i^{N_k}, z_{i+1}^{N_k})] \le \sum_{j=1}^{k-1} 2(\epsilon_{N_j} + a_{N_j} - b_{N_j}). \tag{4.15}$$

We verify that (4.15) holds for $k = 2$. It is easy to see that

$$\sum_{i=0}^{N_2-1} [v_i(y_i, y_{i+1}) - v_i(z_i(N_1,N_2), z_{i+1}(N_1,N_2))] \le v_{N_1}(y_{N_1}, y_{N_1+1}) - b_{N_1}$$
$$\le a_{N_1} - b_{N_1}$$

and this relation together with (4.12) implies (4.15) for $k = 2$. Assume now that relation (4.15) holds for some integer $k \geq 2$. By (4.12)

$$\sum_{i=0}^{N_{k+1}-1} [v_i(y_i, y_{i+1}) - v_i(z_i^{N_k+1}, z_{i+1}^{N_k+1})]$$

$$= \sum_{i=0}^{N_{k+1}-1} [v_i(y_i, y_{i+1}) - v_i(z_i(N_k, N_{k+1}), z_{i+1}(N_k, N_{k+1}))]$$

$$+ \sum_{i=0}^{N_{k+1}-1} [v_i(z_i(N_k, N_{k+1}), z_{i+1}(N_k, N_{k+1})) - v_i(z_i^{N_{k+1}}, z_{i+1}^{N_{k+1}})]$$

$$\leq \sum_{i=0}^{N_k-1} [v_i(y_i, y_{i+1}) - v_i(z_i^{N_k}, z_{i+1}^{N_k})] + v_{N_k}(y_{N_k}, y_{N_k+1}) - b_{N_k} + a_{N_k} - b_{N_k} + \epsilon_{N_k}$$

$$\leq \sum_{j=1}^{k-1} 2(\epsilon_{N_j} + a_{N_j} - b_{N_j}) + 2a_{N_k} - 2b_{N_k} + \epsilon_{N_k} \leq \sum_{j=1}^{k} 2(\epsilon_{N_j} + a_{N_j} - b_{N_j}).$$

Thus relation (4.15) holds for every integer $k \geq 2$. Let $j > N_2$ be an integer. There is an integer $k \geq 2$ such that $N_k < j \leq N_{k+1}$. Then in view of (4.13)-(4.15)

$$\sum_{i=0}^{j-1} v_i(z_i^{N_{k+1}}, z_{i+1}^{N_{k+1}}) \leq S(x, j) + \epsilon_j + \epsilon_{N_{k+1}} + a_j - b_j,$$

$$\sum_{i=0}^{j-1} [v_i(y_i, y_{i+1}) - v_i(z_i^{N_k+1}, z_{i+1}^{N_{k+1}})]$$

$$= \sum_{i=0}^{N_{k+1}-1} [v_i(y_{i+1}, y_{i+1}) - v_i(z_i^{N_{k+1}}, z_{i+1}^{N_{k+1}})] \leq \sum_{i=1}^{k} 2(\epsilon_{N_i} + a_{N_i} - b_{N_i}),$$

$$\sum_{i=0}^{j-1} v_i(y_i, y_{i+1}) \leq \sum_{i=1}^{k} 2(\epsilon_{N_i} + a_{N_i} - b_{N_i}) + S(x, j)$$

$$+ \epsilon_j + \epsilon_{N_{k+1}} + a_j - b_j \leq S(x, j) + 3\epsilon/4 + a_j - b_j.$$

This completes the proof of the theorem. □

5. Periodic optimal control problems

We consider a system

$$C_T(u) = \int_0^T f_0(z(t), u(t), t) dt,$$

$$z' = f(z, u) \tag{5.1}$$

where $z(t) \in R^n$, $u(t) \in \Omega$ for all $t \in [0, T]$, $\Omega \subset R^m$ is closed and $f_0 : R^n \times \Omega \times [0, \infty)$ and $f : R^n \times \Omega \to R^n$ are continuous functions. The admissible controls are all the measurable functions $u(t)$ for which the constraints $u(t) \in \Omega$ and $z(t) \in R^n$ are satisfied (where z and u are related as in (5.1)).

We assume the following:

(1)

$$f(z+q,u)=f(z,u)$$

for all $z \in R^n$, $u \in \Omega$ and each $q \in \mathbf{Z}^n$ and

$$f_0(z+q,u,t)=f_0(z,u,t)$$

for all $z \in R^n$, $u \in \Omega$, $q \in \mathbf{Z}^n$ and all $t \in [0,\infty)$.

(2) For any bounded set $E \subset \Omega$ the function f_0 is bounded on the set $R^n \times E \times [0,\infty)$ and the function f is bounded on the set $R^n \times E$.

(3) For any bounded set $E \subset \Omega$ the function $f_0(z,u,t) \to 0$ as $t \to \infty$ uniformly on $R^n \times E$ (for any $\epsilon > 0$ there $t_\epsilon > 0$ such that

$$|f_0(z,u,t)| \le \epsilon \text{ for each } z \in R^n,\ u \in E \text{ and each } t \in [t_\epsilon,\infty).$$

(4) There exist a number $d_0 > 0$ and a bounded function

$$\phi_0 :\ [0,\infty) \to [0,\infty)$$

such that $\phi_0(t) \to 0$ as $t \to \infty$ and that

$$f_0(z,u,t) \ge -d_0\phi_0(t) \text{ for each } z \in R^n,\ u \in \Omega \text{ and each } t \in [0,\infty).$$

(5) There exists a number $d_1 > 0$ such that for each $x = (x_1,\dots,x_n)$, $y = (y_1,\dots,y_n) \in R^n$ satisfying

$$0 \le x_i \le 1,\ 0 \le y_i - x_i \le 1,\ i = 1,\dots,n$$

there is an admissible control $u(t)$, $0 \le t \le 1$ with a corresponding trajectory $z(t)$, $t \in [0,1]$ such that

$$z(0)=x,\ z(1)=y,\ |u(t)| \le d_1,\ 0 \le t \le 1.$$

(6) For any number $T > 0$ there exist $\alpha_T > 0$, $\beta_T > 0$ such that

$$\alpha_T|f(z,u)| \le f_0(z,u,t) \text{ for all } z \in R^n,\ t \in [0,T]$$
$$\text{and each } u \in \Omega \text{ satisfying } |u| \ge \beta_T.$$

For $x \in R^n$, $T > 0$ we set

$$\sigma(x,T) = \inf\left\{ \int_0^T f_0(z(t),u(t),t)dt :\right.$$
$$\left. z' = f(z,u),\ z(t) \in R^n,\ u(t) \in \Omega,\ t \in [0,T],\ z(0) = x \right\}.$$

By Assumptions 1, 2, 4 and 5 the number $\sigma(x,T)$ is well defined.

For $x,y \in R^n$, $T \ge 0$ denote by $H(x,y,T)$ the set of all pairs of functions $(z(t),u(t)),\ t \in [T,T+1]$ such that

$$z' = f(z,u),\ z(t) \in R^n,\ u(t) \in \Omega \text{ for all } t \in [T,T+1] \text{ and } z(T) = x,\ z(T+1) = y,$$

and set

$$v_T(x,y) = \inf\left\{ \int_T^{T+1} f_0(z(t),u(t),t)dt :\ (z,u) \in H(x,y,T) \right\} \text{ if } H(x,y,T) \ne \emptyset,$$

otherwise $v_T(x,y) = \infty$.

For any integer $i \geq 0$ set

$$a_i = \sup\{v_i(x,y) : x = (x_1,\dots,x_n),\ y = (y_1,\dots,y_n) \in R^n,$$
$$0 \leq x_i \leq 1,\ 0 \leq y_i - x_i \leq 1 \text{ for each } i = 1,\dots,n\},$$
$$b_i = \inf\{v_i(x,y) : x,y \in R^n\}.$$

By the assumptions we made

$$b_i > -\infty,\ a_i < \infty,\ i = 0,1,\dots \tag{5.2}$$

(5.3)
$$v_i(x+q, y+q) = v_i(x,y) \text{ for each } x,y \in R^n \text{ each } q \in \mathbf{Z}^n \text{ and each } i = 0,1,\dots,$$

$$\sup\{a_i : i = 0,1,\dots\} < \infty, \tag{5.4}$$

$$\inf\{b_i : i = 0,1,\dots\} > -\infty, \tag{5.5}$$

$$a_i \to 0,\ b_i \to 0 \text{ as } i \to \infty. \tag{5.6}$$

Lemma 5.1. *For any integer $i \geq 0$ there exists a number $\Gamma_i \geq 0$ such that for each $x, y \in R^n$ satisfying $|x-y| \geq \Gamma_i$ the relation $v_i(x,y) \geq a_i$ holds.*

Proof. Let $i \in \{0,1,\dots\}$. By Assumption 6

$$\alpha_{i+1}|f(z,u)| \leq f_0(z,u,t) \tag{5.7}$$

for each $z \in R^n$, each $t \in [0, i+1]$ and each $u \in \Omega$ satisfying $|u| \geq \beta_{i+1}$ where $\alpha_{i+1} > 0$, $\beta_{i+1} > 0$. In view of Assumption 2 there exists a number $\gamma > 0$ such that

$$|f(z,u)| \leq \gamma \text{ for each } z \in R^n \text{ and each } u \in \Omega \text{ satisfying } |u| \leq \beta_{i+1}. \tag{5.8}$$

Choose a number $\Gamma_i > 0$ for which

$$\Gamma_i > \gamma + (\alpha_{i+1})^{-1}[|a_i| + \sup\{|\phi_0(t)| : t \in [i, i+1]\}](|d_0| + 1). \tag{5.9}$$

Let $x, y \in R^n$ satisfy $|x-y| \geq \Gamma_i$ and let $(z(t), u(t))(i \leq t \leq i+1) \in H(x,y,i)$. We have

$$\Gamma_i \leq |x-y| \leq \int_i^{i+1} |z'(t)|dt = \int_i^{i+1} |f(z(t),u(t))|dt.$$

Set

$$E_1 = \{t \in [i,i+1] : |u(t)| < \beta_{i+1}\},\ E_2 = [i,i+1] \setminus E_1.$$

Relation (5.8) implies that

$$\Gamma_i \leq \int_{E_1} |f(z(t),u(t))|dt + \int_{E_2} |f(z(t),u(t))| \leq \int_{E_2} |f(z(t),u(t))| + \gamma. \tag{5.10}$$

On the other hand by (5.7), Assumption 4, (5.10) and (5.9)

$$\int_i^{i+1} f_0(z(t),u(t),t)dt = \int_{E_1} f_0(z(t),u(t),t)dt + \int_{E_2} f_0(z(t),u(t),t)dt$$
$$\geq \int_{E_2} \alpha_{i+1}|f(z(t),u(t))|dt - \sup\{|\phi_0(t)| : t \in [i,i+1]\}d_0$$
$$\geq \alpha_{i+1}(\Gamma_i - \gamma) - \sup\{|\phi_0(t)| : t \in [i,i+1]\}|d_0| \geq a_i.$$

This completes the proof of the lemma. □

Lemma 5.2. *For any* $x \in R^n$

$$\sup\{|\sigma(x,T) - \sigma(x,i)| : T \in [i, i+1]\} \to 0 \text{ as } i \to \infty$$

(here $i \geq 0$ *is an integer).*

Proof. Let $x \in R^n$, $i \geq 0$ be an integer and let $T \in (i, i+1]$. Let furthermore, $u(t)$, $t \in [0,T]$ be an admissible control with a corresponding trajectory $z(t)$, $t \in [0,T]$ such that $z(0) = x$. By Assumption 4

$$\int_0^T f_0(z(t),u(t),t)dt \geq \int_0^i f_0(z(t),u(t),t)dt - \sup\{|\phi_0(t)| : t \in [i,i+1]\}|d_0|$$

$$\geq \sigma(x,i) - \sup\{|\phi_0(t)| : t \in [i,i+1]\}|d_0|.$$

This relation implies that

$$\sigma(x,T) - \sigma(x,i) \geq -\sup\{|\phi_0(t)| : t \in [i,i+1]\}|d_0|. \tag{5.11}$$

Let now $u(t)$, $t \in [0,i]$ be an admissible control with a corresponding trajectory $z(t)$, $t \in [0,i]$ such that $z(0) = x$. In view of Assumptions 1 and 5 there exists an admissible control $u_1(t)$, $t \in [0,T]$ with a corresponding trajectory $z_1(t)$, $t \in [0,T]$ such that $u_1(t) = u(t)$, $z_1(t) = z(t)$, $t \in [0,i]$ and that $|u_1(t)| \leq d_1$, for all $t \in [i,T]$. We have

$$\int_0^i f_0(z(t),u(t),t)dt \geq \int_0^T f_0(z_1(t),u_1(t),t)dt -$$

$$-\sup\{f_0(y,h,\tau) : y \in R^n, h \in \Omega, |h| \leq d_1, \tau \in [i,\infty)\}$$

$$\geq \sigma(x,T) - \sup\{|f_0(y,h,\tau)| : y \in R^n, h \in \Omega, |h| \leq d_1, \tau \in [i,\infty)\}.$$

This relation implies that

$$\sigma(x,i) - \sigma(x,T) \geq -\sup\{|f_0(y,h,\tau)| : y \in R^n, h \in \Omega, |h| \leq d_1, \tau \in [i,\infty)\}. \tag{5.12}$$

Now the validity of the lemma follows from relations (5.11), (5.12) which hold for every $i \in \{0,1,\dots,\}$ and every $T \in (i,i+1]$, and from Assumptions 3 and 4. □

It is easy to verify that for $x \in R^n$, $N \in \{0,1,\dots\}$

$$\sigma(x,N) = \inf\left\{\sum_{i=0}^{N-1} v_i(x_i,x_{i+1}) : \{x_i\}_{i=0}^N \subset R^n, x_0 = x\right\}. \tag{5.13}$$

Theorem 5.1. *For any* $x \in R^n$ *and any* $\epsilon > 0$ *there exists an admissible control* $u(t)$, $t \in [0,\infty)$ *with a corresponding trajectory* $z(t)$, $t \in [0,\infty)$ *such that* $z(0) = y$ *and that*

$$\int_0^T f_0(z(t),u(t),t)dt \leq \sigma(x,T) + \epsilon$$

for all sufficient large T.

Proof. By relations (5.2)-(5.6) and Lemma 5.1, Theorem 4.3 is valid for the functions v_i, $i = 0,1,\dots$. Let $x \in R^n$ and let ϵ be a positive number.

By Theorem 4.3 and relation (5.13) there exists a sequence $\{y_i\}_{i=0}^{\infty} \subset R^n$ such that $y_0 = x$ and that for large integers N the following relation holds:

$$\sum_{i=0}^{N-1} v_i(y_i, y_{i+1}) \le \sigma(x, N) + \epsilon/4.$$

Evidently there exists an admissible control $u(t)$, $t \in [0, \infty)$ with a corresponding trajectory $z(t)$, $t \in [0, \infty)$ such that for any integer $i \ge 0$

$$z(i) = y_i, \ \int_i^{i+1} f_0(z(t), u(t), t)dt \le v_i(y_i, y_{i+1}) + 2^{-i-4}\epsilon.$$

It is easy to see that for large integers N

$$\int_0^N f_0(z(t), u(t), t)dt \le \sigma(x, N) + \epsilon/2. \tag{5.14}$$

Let $N \ge 0$ be an integer and let $T \in [N, N+1)$. It follows from (5.14), Assumption 4 and Lemma 5.2 that for large integers N

$$\int_0^T f_0(z(t), u(t), t)dt - \sigma(x, T) - \epsilon/2 =$$

$$= \int_0^{N+1} f_0(z(t), u(t), t)dt - \sigma(x, N+1) - \epsilon/2$$

$$+\sigma(x, N+1) - \sigma(x, T) - \int_T^{N+1} f_0(z(t), u(t), t)dt$$

$$\le \sigma(x, N+1) - \sigma(x, T) - \int_T^{N+1} f_0(z(t), u(t), t)dt \le \mu_1 + \mu_2,$$

where

$$\mu_1 = 2\sup\{|\sigma(x, N) - \sigma(x, \tau)| : \ \tau \in [N, N+1]\}$$

and

$$\mu_2 = \sup\{|\phi_0(\tau)| : \ \tau \in [N, N+1]\}|d_0|.$$

But $\mu_1, \mu_2 \to 0$ as $N \to \infty$, which completes the proof of the theorem. □

The following is our main result which extends Theorem 6.2 of [6] to the case of periodic integrands. In particular, it asserts the existence of overtaking optimal solutions which we define as follows.

We say that a pair (z^*, u^*), where $u^*(\cdot)$ is an admissible control on $[0, \infty)$ with a corresponding trajectory $z^*(\cdot)$, is *overtaking optimal* if for any $\epsilon > 0$ there exists $T_\epsilon > 0$ such that

$$\int_0^T f_0(z^*(t), u^*(t), t)dt < \int_0^T f_0(z(t), u(t), t)dt + \epsilon$$

for each $T > T_\epsilon$ and each admissible pair (z, u) on the interval $[0, T]$ satisfying $z(0) = z^*(0)$.

Note that in the definition above T_ϵ depends only on ϵ. In the usual definition of an overtaking optimal trajectory used in the literature (see [3]) the pair (z, u) is defined on the interval $[0, \infty)$ and T_ϵ depends on ϵ and (z, u). Here we can use the strong version of the overtaking optimality criterion because of assumption 3).

It should be mentioned that Theorem 6.2 of [6] asserts the existence of overtaking optimal solutions in the usual sense for optimal control problems with state variables belonging to a compact subset of R^n and with an integrand $f_0(z,u,t) = \psi(t)g(z,u)$.

Theorem 5.2. *Assume that for every integer $i \geq 0$ the function $v_i(x,y)$ is well defined (namely the minimum is attained by a certain admissible control) and is lower semicontinuous on $R^n \times R^n$. Then for every $x \in R^n$ there exists an admissible control $u(t)$ with a corresponding trajectoy $z(t)$, $t \in [0,\infty)$ such that $z(0) = x$ and*

$$\lim_{t\to\infty}\left[\int_0^T f_0(z(t),u(t),t)dt - \sigma(x,T)\right] = 0.$$

In particular, this admissible pair (z,u) is overtaking optimal.

Proof. Let $x \in R^n$. By relations (5.2)-(5.6) and Lemma 5.1, Theorem 4.2 is valid for the functions v_i, $i = 0,1,\ldots,$. In view of Theorem 4.2 and relation (5.13) there exists a sequence $\{y_i\}_{i=0}^{\infty} \subset R^n$ such that $y_0 = x$ and that

$$\sum_{i=0}^{N-1} v_i(y_i,y_{i+1}) - \sigma(x,N) \to 0 \text{ as } N \to \infty.$$

Evidently there exists an admissible control $u(t)$, $t \in [0,\infty)$ with a corresponding trajectory $z(t)$, $t \in [0,\infty)$ such that for each integer $i \geq 0$ $z(i) = y_i$ and

$$\int_i^{i+1} f_0(z(t),u(t),t)dt = v_i(y_i,y_{i+1}).$$

Then

$$\int_0^N f_0(z(t),u(t),t)dt - \sigma(x,N) \to 0 \text{ as } N \to \infty. \tag{5.15}$$

(here N is a natural number).

Let $N \geq 0$ be an integer and $T \in [N,N+1)$. By Assumption 4, Lemma 5.2 and relation (5.15)

$$\int_0^T f_0(z(t),u(t),t) - \sigma(x,T) = \int_0^{N+1} f_0(z(t),u(t),t)dt - \sigma(x,N+1)$$

$$+\sigma(x,N+1) - \sigma(x,T) - \int_T^{N+1} f_0(z(t),u(t),t)dt$$

$$\leq \int_0^{N+1} f_0(z(t),u(t),t)dt - \sigma(x,N+1) \leq \mu_1 + \mu_2,$$

where

$$\mu_1 = 2\sup\{|\sigma(x,\tau) - \sigma(x,N)| : \ \tau \in [N,N+1]\}$$

and

$$\mu_2 = \sup\{|\phi_0(t)| : \ t \in [N, N+1]\}|d_0| \to 0 \text{ as } N \to \infty$$

and we have $\mu_1 + \mu_2 \to 0$ as $N \to 0$. This completes the proof of the theorem. □

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