



REGULARIZATION AND RESOLUTION OF MONOTONE VARIATIONAL INEQUALITIES WITH OPERATORS GIVEN BY HYPOMONOTONE APPROXIMATIONS

YAKOV ALBER, DAN BUTNARIU, AND IRINA RYAZANTSEVA

ABSTRACT. We study the stability in reflexive, smooth and strictly convex Banach spaces of the classical Tikhonov-Browder operator regularization method for monotone variational inequalities with data perturbations. We prove that this regularization method is stable even if the perturbed data contain operators which fail to be monotone, but are strongly hypomonotone. We use this stability result in order to prove convergence in smooth uniformly convex spaces of an iterative algorithm for approximating solutions of monotone variational inequalities. The algorithm we analyze involves in computations the perturbed data only and it converges even if the perturbed operators are not necessarily monotone, but strongly hypomonotone.

1. INTRODUCTION

Let X be a reflexive, smooth, strictly convex Banach space, let X^* be the dual space of X and let $A : X \rightarrow 2^{X^*}$ be a monotone operator which is demiclosed and convex valued on the interior of its domain. Recall that the operator A is called *demiclosed* if its graph,

$$\text{Graph } A := \{(x, \xi) \in X \times X^* : \xi \in Ax\},$$

is sequentially closed in $X \times X^*$ when X is provided with the strong topology and X^* is provided with the weak* topology.

Let Ω be a nonempty, closed, convex subset of $\text{Int}(\text{Dom } A)$ and let $\psi \in X^*$. We consider the variational inequality

$$(1) \quad \langle Ax - \psi, y - x \rangle \geq 0, \quad \forall y \in \Omega.$$

By a *solution* of the variational inequality (1) we mean a vector $x^* \in \Omega$ such that, for some $\xi^* \in Ax^*$, we have

$$(2) \quad \langle \xi^* - \psi, y - x^* \rangle \geq 0, \quad \forall y \in \Omega.$$

We denote by $\mathcal{S}(A, \psi, \Omega)$ the set of solutions of (1). We assume that the problem data A , ψ and Ω are given by sequences of approximations $\{A_k\}_{k \in \mathbb{N}}$, $\{\psi^k\}_{k \in \mathbb{N}}$ and $\{\Omega_k\}_{k \in \mathbb{N}}$, respectively.

The problem of finding a solution of (1) may be ill-posed in the sense that either $\mathcal{S}(A, \psi, \Omega)$ is not a singleton and/or small perturbations of the problem data lead

2000 *Mathematics Subject Classification*. Primary 47J06, 47A52; Secondary 47H14, 47J20.

Key words and phrases. Monotone operator, Maximal monotone operator, Strongly hypomonotone operator, Demiclosed operator, Variational inequality, Regularization method, Iterative algorithm, Epi-quasi-inverse, Modulus of uniform convexity, Bregman projection.

to relatively large perturbations of the solution set. In such situations, solving variational inequalities

$$(3) \quad \langle A_k x - \psi^k, y - x \rangle \geq 0, \quad \forall y \in \Omega_k$$

instead of (1) makes little sense because the variational inequality (3) may have no solutions or the problem of finding solutions of (3) can be as ill-posed as the original problem and its solutions may be far from the set $\mathcal{S}(A, \psi, \Omega)$. The aim of this paper is two folds. First we study the stability of a regularization method for situations in which the variational inequality (1) is ill-posed. Second, we use the stability properties of that regularization method in order to establish convergence of an iterative algorithm for finding solutions of the variational inequality (1) by using in computations the approximative data only.

The regularization method we consider in this paper goes back to Tikhonov [53] and Browder [20]. Its underlying idea is to associate to the variational inequality (1) the perturbed regularized variational inequalities

$$(4) \quad \langle (A_k + \alpha_k J) x - \psi^k, y - x \rangle \geq 0, \quad \forall y \in \Omega_k,$$

where J is the normalized duality mapping and $\{\alpha_k\}_{k \in \mathbb{N}}$ is a sequence of positive real numbers (regularization parameters) such that $\lim_{k \rightarrow \infty} \alpha_k = 0$. It is well known that, under various conditions concerning the original and the approximative data, each variational inequality (4) has a unique solution x^k and that the sequence $\{x^k\}_{k \in \mathbb{N}}$ converges weakly, and sometimes strongly, to a solution of (1) – see [2], [4], [8], [9], [13], [16], [20], [29], [30], [31], [32], [33], [34], [35], [36], [37], [38], [49], [53], [54]. In all these works it is presumed that the approximating operators A_k preserve the basic continuity and monotonicity features of the operator A involved in the original variational inequality (1). In this paper we consider the situation in which, *by contrast to A , the operators A_k may fail to be monotone, although they are required to satisfy a less demanding property which we call "strong hypomonotonicity" in what follows.* Precisely, we study the above described regularization method under the following assumptions on the approximative data:

(A1) For each $k \in \mathbb{N}$, we have that $\Omega_k \subseteq \text{Int}(\text{Dom } A_k)$, the operator A_k is demiclosed on Ω_k and, for any $x \in \Omega_k$, the set $A_k x$ is convex.

(A2) For each $k \in \mathbb{N}$, there exists a number $h_k \geq 0$ such that A_k is *strongly h_k -hypomonotone* on Ω_k , that is, for any $x, y \in \Omega_k$, the following inequality holds

$$(5) \quad \langle A_k x - A_k y, x - y \rangle \geq -h_k (\|x\| - \|y\|)^2;$$

(A3) There exist a sequence of positive real numbers $\{\alpha_k\}_{k \in \mathbb{N}}$, and the sequences of nonnegative real numbers $\{\delta_k\}_{k \in \mathbb{N}}$, $\{\omega_k\}_{k \in \mathbb{N}}$ and $\{\tau_k\}_{k \in \mathbb{N}}$, and there exist three bounded on bounded sets functions $a, b : X \rightarrow \mathbb{R}_+$ and $c : X^* \rightarrow \mathbb{R}_+$ such that $\lim_{k \rightarrow \infty} \alpha_k = 0$, $h_k < \alpha_k$ for all $k \in \mathbb{N}$,

$$(6) \quad \lim_{k \rightarrow \infty} \frac{\delta_k + \omega_k + \tau_k + h_k}{\alpha_k} = 0,$$

$$(7) \quad \limsup_{k \rightarrow \infty} \frac{b(x)}{\|x\|^2} < \infty,$$

and such that, for each $k \in \mathbb{N}$, the following proximity requirements are satisfied:

$$\mathbf{(A3-1)} \quad \psi^k \in X^* \text{ and } \|\psi - \psi^k\|_* \leq \delta_k;$$

$$\mathbf{(A3-2)} \quad \text{For any } x \in \Omega, \text{ there exists } z^k \in \Omega_k \text{ such that}$$

$$(8) \quad \left\| x - z^k \right\| \leq a(x)\omega_k,$$

and such that, for all $\zeta \in Ax$, we have

$$(9) \quad \text{dist}_*(\zeta, A_k z^k) \leq c(\zeta)\tau_k;$$

$$\mathbf{(A3-3)} \quad \text{For any } v^k \in \Omega_k, \text{ there exists } v \in \Omega \text{ such that}$$

$$(10) \quad \left\| v^k - v \right\| \leq b(v^k)\omega_k.$$

Assumption **(A1)** is commonly used in the study of the stability of the regularization method we discuss here. If the operators A_k are maximal monotone, as they are often presumed to be, and if $\Omega_k \subseteq \text{Int}(\text{Dom } A_k)$, then the other requirements of **(A1)** are automatically satisfied. The strong hypomonotonicity, defined by (5) in assumption **(A2)**, is a strengthened form of the hypomonotonicity concept used by Rockafellar [47] (see also [46, p. 548]). Its importance in our context stems from the fact that if A_k are strongly h_k -hypomonotone with $h_k < \alpha_k$, then the operators $A_k + \alpha_k J$ are strictly monotone and coercive – see Lemma 2.1 below. Assumption **(A3)** is a weaker version of the often used requirement (see, for instance, [13], [10], [11]) that $\text{Dom } A_k = \text{Dom } A$ for all $k \in \mathbb{N}$ and the operators A_k converge to A uniformly on bounded subsets of $\text{Dom } A$ and with respect to the Hausdorff metric topology on the family of closed convex nonempty subsets of X^* . Similar conditions have been previously used in [4] and [8]. It implies a form of graphical convergence to $A + N_\Omega$ of the operators $A_k + N_{\Omega_k}$ as shown in [7, Section 2].

We prove (see Theorem 2.1 below) that the assumptions **(A1)**-**(A3)** are sufficient to ensure well definedness, as well as convergence, of the sequence $\{x^k\}_{k \in \mathbb{N}}$ to a solution of the original variational inequality (1). Using this fact, we establish in Theorem 4.1 convergence of an iterative algorithm for finding solutions of the variational inequality (1). The algorithm we consider here (see Section 4) is the product of analyzing and refining a series of procedures for solving equations and variational inequalities due to Alber [1], Polyak [41], Bakushinskii and Polyak [17], Bruck [21], [22], Bakushinskii [15], Reich [42], [43], Vainikko [55], Ryazantseva [48], [49], [50], [51], Alber and Reich [14], Alber, Kartsatos and Litsyn [10], and Alber and Nashed [11]. It has the important feature that it involves in computations the approximative data only. The information on the original data needed for the application of the algorithm is exclusively qualitative (knowledge of the monotonicity and demiclosedness of A and of the existence of solutions to (1)). Convergence of variants of this algorithm was proved before for monotone operators A_k in uniformly convex and uniformly smooth Banach spaces whose moduli of uniform convexity were subjected to some restrictive requirement and under proximity assumptions somewhat stronger than those we make in this paper (compare with [10, Theorem 1] and [11, Theorem 3.3]). *We prove that the algorithm we propose in Section 4 converges in any smooth uniformly convex Banach space and, by contrast to previously studied variants, our convergence result does not require monotonicity of A_k , but strong hypomonotonicity* – see Theorem 4.1. Assumptions **(A1)**-**(A3)** by themselves are not

sufficient for ensuring convergence of our algorithm. In addition to these conditions we had to impose conditions **(A4)** and **(A5)**. Condition **(A4)** is a uniform boundedness on bounded sets requirement for the operators A_k . Condition **(A5)** requires that the step size parameters ε_k involved in the algorithm be chosen in a manner which is compatible with the geometric structure of the underlying space X . To see that conditions **(A1)**-**(A5)**, taken together, are non-contradictory one can verify that the operators defined in [8, Section 2.7] satisfy all these requirements.

The stability and convergence results (Theorems 2.1 and 4.1) proved in this paper were made possible by several newly established properties of the strongly h -hypomonotone operators (see Lemmas 2.1, 2.2 and 2.3) and by the improved "proximity theorem" (see Theorem 3.1) which may be of interest by themselves. In the context, the concept of epi-quasi-inverse of a real function, introduced and studied by Penot and Volle [40], proved to be very useful.

It was noted above that the main novelty in Theorems 2.1 and 4.1 is relaxing the assumption, made in all previous works we are aware of concerning the Tikhonov-Browder regularization method and the algorithmic scheme (79), that the approximative operators A_k should be monotone. Relaxing that assumption as a precondition for the implementation of the regularization method and of the algorithm discussed here is significant because, even if the data of a variational inequality are given by precise closed formulae, in practical computations of solutions random perturbations of data are inherent. The fact is that, in computations, we work with approximations of the data which change from an iteration to the other. Due to the random character of the data perturbations, there is no guarantee that the perturbed operators A_k involved in the computational process preserve the monotonicity of A . Theorem 4.1 is a certificate of robustness of the algorithm (79) under data perturbations and, specially, under alteration of the monotonicity of the operators. In informal terms, Theorem 4.1 essentially says that, in spite of the data perturbations which may alter monotonicity, the algorithm still produces strong approximations of solutions of the variational inequality, provided that the perturbations are kept between some boundaries (see condition **(A3)**) and that the variation of the 'noise' $N_k = A - A_k$ between close points is relatively small (see **(A2)**), that is,

$$\langle N_k x - N_k y, x - y \rangle \leq h_k (\|x\| - \|y\|)^2.$$

Classes of differential equations and optimization problems which can be equivalently written in the form of the variational inequality (1) and, then, can be solved by the algorithm we discuss here because their data, given by approximations, inherently satisfy the requirements of our results, are presented in a forthcoming report by Y. Alber and I. Ryazantseva.

2. OPERATOR REGULARIZATION OF VARIATIONAL INEQUALITIES

Let X be a reflexive, smooth, strictly convex Banach space and let $A : X \rightarrow 2^{X^*}$ be a monotone operator which is demiclosed and convex valued on the interior of its domain. Suppose that Ω be a nonempty, closed, convex subset of $\text{Int}(\text{Dom } A)$ and let $\psi \in X^*$. In this section we prove that, in these circumstances, the Tichonov-Browder regularization method is stable, that is, the following result holds:

Theorem 2.1. Consider the variational inequality (1) whose data A , ψ and Ω are given by sequences of approximations $\{A_k\}_{k \in \mathbb{N}}$, $\{\psi^k\}_{k \in \mathbb{N}}$ and $\{\Omega_k\}_{k \in \mathbb{N}}$, respectively.

(i) If conditions **(A1)** and **(A2)** are satisfied, and if $h_k < \alpha_k$ for all $k \in \mathbb{N}$, then, for each $k \in \mathbb{N}$, the variational inequality (4) has a unique solution x^k .

(ii) If conditions **(A1)**-**(A3)** are satisfied, then the variational inequality (1) has solutions if and only if the sequence $\{x^k\}_{k \in \mathbb{N}}$ is bounded. In this case, the sequence $\{x^k\}_{k \in \mathbb{N}}$ converges weakly to the minimal norm solution of (1). If, in addition to the previous requirements, the space X has the Kadec-Klee property, then the sequence $\{x^k\}_{k \in \mathbb{N}}$ converges strongly.

The proof of this result consists of a sequence of lemmas presented below. We start our proof by establishing the following technical result:

Lemma 2.1. If the operator $B : X \rightarrow 2^{X^*}$ is strongly h -hypomonotone on the nonempty subset Λ of $\text{Dom } B$ and if $\alpha \in (h, \infty)$, then the operator $B + \alpha J$ is strictly monotone on Λ . In these circumstances, the operator $B + \alpha J$ is also coercive on Λ in the sense that, if $z \in \Lambda$, then

$$\lim_{k \rightarrow \infty} \frac{\langle \zeta^k + \alpha J y^k, y^k - z \rangle}{\|y^k\|} = \infty,$$

for any sequence $\{(y^k, \zeta^k)\}_{k \in \mathbb{N}}$ with $y^k \in \Lambda$, $\zeta^k \in B y^k$ for all $k \in \mathbb{N}$ and such that $\|y^k\| \rightarrow \infty$.

Proof. If $x, y \in \Lambda$, then

$$\begin{aligned} (11) \quad & \langle (B + \alpha J)x - (B + \alpha J)y, x - y \rangle \\ &= \langle Bx - By, x - y \rangle + \alpha \langle Jx - Jy, x - y \rangle \\ &\geq -h(\|x\| - \|y\|)^2 + \alpha \langle Jx - Jy, x - y \rangle \\ &\geq -h(\|x\| - \|y\|)^2 + \alpha(\|x\| - \|y\|)^2. \end{aligned}$$

Suppose that $x \neq y$. If $\|x\| = \|y\|$, then the first inequality in (11) implies that

$$\langle (B + \alpha J)x - (B + \alpha J)y, x - y \rangle \geq \alpha \langle Jx - Jy, x - y \rangle > 0,$$

where the strict inequality results from the strict monotonicity of J . If $\|x\| \neq \|y\|$, then the last inequality in (11) implies that

$$(12) \quad \langle (B + \alpha J)x - (B + \alpha J)y, x - y \rangle \geq (\alpha - h)(\|x\| - \|y\|)^2 > 0$$

because $\alpha > h$. Hence, $B + \alpha J$ is strictly monotone.

In order to prove that $B + \alpha J$ is coercive, fix $z \in \Lambda$ and $\xi \in Bz$. Suppose that $y \in \Lambda$ and $\zeta \in By$. Then we have that

$$\begin{aligned} \langle \zeta + \alpha Jy, y - z \rangle &= \langle \zeta + \alpha Jy - \xi - \alpha Jz, y - z \rangle \\ &\quad + \langle \xi + \alpha Jz, y - z \rangle \\ &\geq (h - \alpha)(\|y\| - \|z\|)^2 \\ &\quad - \|\xi + \alpha Jz\|_* \|y - z\|, \end{aligned}$$

where the inequality is results from (12). This implies that, whenever $y^k \neq 0$ and $\zeta^k \in By^k$, we have

$$\begin{aligned} \frac{\langle \zeta^k + \alpha Jy^k, y^k - z \rangle}{\|y^k\|} &\geq (h - \alpha) \|y^k\| \left(1 - \frac{\|z\|}{\|y^k\|}\right)^2 \\ &\quad - \|\xi + \alpha Jz\|_* \left(1 + \frac{\|z\|}{\|y^k\|}\right). \end{aligned}$$

Letting $k \rightarrow \infty$ in this inequality, the coercivity condition results. \square

The following lemma shows that maximal monotone extensions of a demiclosed, convex valued, monotone operator B may differ at points of the boundary of $\text{Dom } B$ only. This does not mean that such an operator B is necessarily maximal monotone. For instance, the operator $B : \mathbb{R} \rightarrow 2^{\mathbb{R}}$ given by $B(x) = -1$ if $-1 \leq x < 0$, $B(0) = [-1, 1]$, $B(x) = 1$ if $0 < x \leq 1$, and $B(x) = \emptyset$ if $|x| > 1$ is monotone, demiclosed and convex valued on its domain $\text{Dom } B = [-1, 1]$. However, B is not maximal monotone since the operator $\bar{B} : \mathbb{R} \rightarrow 2^{\mathbb{R}}$ defined by $\bar{B}(x) = B(x)$ if $|x| \neq 1$ and $\bar{B}(-1) = (-\infty, -1]$, $\bar{B}(1) = [1, +\infty)$ is a maximal monotone extension of B .

Lemma 2.2. *Let $B : X \rightarrow 2^{X^*}$ be a monotone demiclosed operator. If $x \in \text{Int}(\text{Dom } B)$ and if Bx is convex, then for any maximal monotone extension \bar{B} of B we have $\bar{B}x = Bx$.*

Proof. The operator B is monotone and, therefore, it is locally bounded at x . Consequently, if $\{x^k\}_{k \in \mathbb{N}}$ is a sequence in X such that $\lim_{k \rightarrow \infty} x^k = x$, and if $\{\zeta^k\}_{k \in \mathbb{N}}$ is a sequence such that $\zeta^k \in Bx^k$ for all $k \in \mathbb{N}$, then $\{\zeta^k\}_{k \in \mathbb{N}}$ is bounded. Since X^* is reflexive, it results that $\{\zeta^k\}_{k \in \mathbb{N}}$ has weak accumulation points. From the demiclosedness of B it follows that any such weak accumulation point belongs to Bx . Denote by Rx the closed convex hull of the set of weak accumulation points of all sequences $\{\zeta^k\}_{k \in \mathbb{N}}$ as described above. The set Bx is convex. Due to the demiclosedness of B , the set Bx is weakly closed and, therefore, closed. Hence, we have that $Rx \subseteq Bx$. Let \bar{B} be a maximal monotone extension of B . Obviously, $Rx \subseteq Bx \subseteq \bar{B}x$. We claim that the inclusion $Rx \supseteq \bar{B}x$ holds too. Suppose by contradiction that this is not the case. Then, there exists $\eta \in \bar{B}x$ such that $\eta \notin Rx$. According to the strong separation theorem (see, for instance, [27, p. 64]), there exists a vector $u \in X$ such that

$$(13) \quad \langle \zeta - \eta, u \rangle < 0, \quad \forall \zeta \in Rx.$$

Let $\{t_k\}_{k \in \mathbb{N}}$ be a sequence of positive real numbers converging to zero such that, for any $k \in \mathbb{N}$, we have that $z^k := x + t_k u \in \text{Dom } B$. For each $k \in \mathbb{N}$ let η^k be an arbitrary element of Bz^k . The sequence $\{\eta^k\}_{k \in \mathbb{N}}$ is bounded because $\lim_{k \rightarrow \infty} z^k = x$ and B is locally bounded at x . Thus, there exists a subsequence $\{\eta^{i_k}\}_{k \in \mathbb{N}}$ of $\{\eta^k\}_{k \in \mathbb{N}}$ which converges weakly to some $\bar{\eta}$ in X^* . Clearly, $\bar{\eta} \in Rx$. By the monotonicity of \bar{B} we have

$$\langle \eta^{i_k} - \eta, z^{i_k} - x \rangle \geq 0, \quad \forall k \in \mathbb{N}.$$

Hence,

$$\langle \eta^{i_k} - \eta, u \rangle \geq 0, \quad \forall k \in \mathbb{N}.$$

Letting here $k \rightarrow \infty$ we obtain that $\langle \bar{\eta} - \eta, u \rangle \geq 0$ and this contradicts (13). \square

Observe that, under the assumptions of Theorem 2.1(i), the operators A_k are demiclosed, strongly h_k -hypomonotone and convex valued on the closed, convex, nonempty subset Ω_k of $\text{Int}(\text{Dom } A_k)$. Therefore, application of the following lemma completes the proof of Theorem 2.1(i).

Lemma 2.3. *Suppose that the operator $B : X \rightarrow 2^{X^*}$ is demiclosed, convex valued and strongly h -hypomonotone on the nonempty, closed, convex set Λ contained in $\text{Int}(\text{Dom } B)$ and that $\alpha \in (h, \infty)$. Then the variational inequality*

$$(14) \quad \langle (B + \alpha J)x - \psi, y - x \rangle \geq 0, \quad \forall y \in \Lambda,$$

has a unique solution.

Proof. According to Lemma 2.1, the operator $B + \alpha J$ is strictly monotone on Λ . The following standard argument shows that, since $B + \alpha J$ is strictly monotone on Λ , the variational inequality (14) can not have more than one solution. Suppose by contradiction that $x, x' \in \Lambda$ are different solutions of (14). Then, for some $v \in Bx'$, we have

$$\langle v + \alpha Jx' - \psi, y - x' \rangle \geq 0, \quad \forall y \in \Lambda.$$

In particular, we have

$$\langle v + \alpha Jx' - \psi, x - x' \rangle \geq 0.$$

Similarly, for some $\xi \in Bx$ we have

$$\langle \xi + \alpha Jx - \psi, x' - x \rangle \geq 0.$$

Summing up the two inequalities we deduce that

$$\langle (\xi + \alpha Jx) - (v + \alpha Jx'), x' - x \rangle \geq 0.$$

Since the operator $B + \alpha J$ is monotone, this implies

$$\langle (\xi + \alpha Jx) - (v + \alpha Jx'), x - x' \rangle = 0.$$

The operator $B + \alpha J$ is strictly monotone and, consequently, the last equality cannot hold unless $x = x'$, hence, a contradiction.

We are going to prove now that the variational inequality (14) has at least one solution. To this end, observe that the solution set of (14) and the solution set of the inclusion

$$(15) \quad \psi \in (B + \alpha J + N_\Lambda)x,$$

where N_Λ stands for the normal cone operator of the set Λ , coincide. The operator $B + \alpha J$ is demiclosed on $\Lambda \subseteq \text{Int}(\text{Dom } B) = \text{Int}(\text{Dom } B + \alpha J)$. Clearly, $B + \alpha J$ is also convex valued on Λ because B is so. Let \bar{B} be a maximal monotone extension of B . Then $\bar{B} + \alpha J$ is a maximal monotone extension of $B + \alpha J$. According to Lemma 2.2, we have that $(B + \alpha J)x = (\bar{B} + \alpha J)x$ for all $x \in \Lambda$. Hence, the inclusion (15) is equivalent to the inclusion

$$(16) \quad \psi \in (\bar{B} + \alpha J + N_\Lambda)x$$

where the operator $\bar{B} + \alpha J + N_\Lambda$ is maximal monotone. As shown above, the operator $B + \alpha J$ is coercive on Λ . This implies that $\bar{B} + \alpha J + N_\Lambda$ is coercive too. Hence, Corollary 32.27 in [56] applies to the inclusion (16) and it shows that this inclusion has solution. Clearly, that implies that the equivalent inclusion (15) has solution and, consequently, the variational inequality (14) has solution too. \square

From now and until the end of this section, we assume that the variational inequality (1) has solutions and that conditions **(A1)**-**(A3)** are satisfied. By the already proved Theorem 2.1(i), under these conditions, the variational inequality (4) has a unique solution x^k . The following lemma is a first step in the proof of Theorem 2.1(ii).

Lemma 2.4. *If the variational inequality (1) has at least one solution, then the sequence $\{x^k\}_{k \in \mathbb{N}}$ is bounded.*

Proof. Since x^k is a solution of (4), there exists a vector $\xi^k \in A_k x^k$ such that

$$(17) \quad \langle \xi^k + \alpha_k Jx^k - \psi^k, x - x^k \rangle \geq 0, \quad \forall x \in \Omega_k.$$

Let x^* be a solution of (1) and let $\xi^* \in Ax^*$ be such that (2) is satisfied. According to condition **(A3)** there exists $v^k \in \Omega_k$ and $\zeta^k \in A_k v^k$ such that

$$(18) \quad \|x^* - v^k\| \leq a(x^*)\omega_k,$$

and

$$(19) \quad \|\xi^* - \zeta^k\|_* \leq \tau_k c(\xi^*).$$

Also, for some $w^k \in \Omega$, we have that

$$(20) \quad \|x^k - w^k\| \leq b(x^k)\omega_k.$$

Taking $x = v^k$ in (17) and $y = w^k$ in (2) and adding the resulting inequalities we obtain

$$\langle \xi^* - \psi, w^k - x^* \rangle + \langle \xi^k + \alpha_k Jx^k - \psi^k, v^k - x^k \rangle \geq 0.$$

This implies that

$$\begin{aligned} 0 &\leq \alpha_k \langle Jx^k, v^k - x^k \rangle - \langle \xi^k - \zeta^k, x^k - v^k \rangle + \langle \zeta^k - \xi^*, v^k - x^k \rangle \\ &\quad + \langle \xi^* - \psi, v^k - x^k + w^k - x^* \rangle + \langle \psi - \psi^k, v^k - x^k \rangle. \end{aligned}$$

Note that

$$\begin{aligned} \langle Jx^k, x^k - v^k \rangle &= \langle Jx^k, x^k - x^* \rangle + \langle Jx^k, x^* - v^k \rangle \\ &\geq \|x^k\|^2 - \|x^k\| \|x^*\| - \|x^k\| \|x^* - v^k\| \\ &\geq \|x^k\|^2 - \|x^k\| (\|x^*\| + a(x^*)\omega_k), \end{aligned}$$

where the last inequality follows from (18). From the strong h_k -hypomonotonicity of A_k combined with the fact that $\xi^k \in A_k x^k$ and $\zeta^k \in A_k v^k$ we deduce that

$$-\langle \xi^k - \zeta^k, x^k - v^k \rangle \leq h_k \|x^k - v^k\|^2.$$

The last three relations combined give

$$\alpha_k \left[\|x^k\|^2 - \|x^k\| (\|x^*\| + a(x^*)\omega_k) \right]$$

$$\begin{aligned} &\leq h_k \left\| x^k - v^k \right\|^2 + \left\langle \zeta^k - \xi^*, v^k - x^k \right\rangle \\ &\quad + \left\langle \xi^* - \psi, v^k - x^k + w^k - x^* \right\rangle + \left\langle \psi - \psi^k, v^k - x^k \right\rangle \end{aligned}$$

By (19) and (20) we deduce that

$$\begin{aligned} (21) \quad &\alpha_k \left[\left\| x^k \right\|^2 - \left\| x^k \right\| (\|x^*\| + a(x^*)\omega_k) \right] \\ &\leq h_k \left\| x^k - v^k \right\|^2 + \tau_k c(\xi^*) \left(\left\| v^k - x^* \right\| + \left\| x^* - x^k \right\| \right) \\ &\quad + \left\langle \xi^* - \psi, v^k - x^k + w^k - x^* \right\rangle + \left\langle \psi - \psi^k, v^k - x^k \right\rangle \\ &\leq h_k \left\| x^k - v^k \right\|^2 + \tau_k c(\xi^*) \left(a(x^*)\omega_k + \|x^*\| + \left\| x^k \right\| \right) \\ &\quad + \|\xi^* - \psi\|_* \left(\left\| v^k - x^* \right\| + \left\| w^k - x^k \right\| \right) + \|\psi - \psi^k\|_* \left\| v^k - x^k \right\| \\ &\leq h_k \left\| x^k - v^k \right\|^2 + \tau_k c(\xi^*) \left(a(x^*)\omega_k + \|x^*\| + \left\| x^k \right\| \right) \\ &\quad + \|\xi^* - \psi\|_* \left(a(x^*) + b(x^k) \right) \omega_k + \delta_k \left(a(x^*)\omega_k + \|x^*\| + \left\| x^k \right\| \right). \end{aligned}$$

Suppose, by contradiction, that the sequence $\{x^k\}_{k \in \mathbb{N}}$ is unbounded. Then, for some unbounded subsequence of $\{x^k\}_{k \in \mathbb{N}}$, still denoted $\{x^k\}_{k \in \mathbb{N}}$, and for $k \in \mathbb{N}$ sufficiently large, we deduce from (21) that

$$\begin{aligned} (22) \quad &1 - \frac{\|x^*\| + a(x^*)\omega_k}{\|x^k\|} \leq \frac{h_k}{\alpha_k} \left(1 + \frac{\|v^k\|^2}{\|x^k\|^2} + 2 \frac{\|v^k\|}{\|x^k\|} \right) \\ &\quad + \frac{\tau_k c(\xi^*)}{\alpha_k \|x^k\|^2} \left(a(x^*)\omega_k + \|x^*\| + \left\| x^k \right\| \right) \\ &\quad + \|\xi^* - \psi\|_* \left(\frac{a(x^*)}{\|x^k\|^2} + \frac{b(x^k)}{\|x^k\|^2} \right) \frac{\omega_k}{\alpha_k} \\ &\quad + \frac{\delta_k}{\alpha_k} \left(\frac{a(x^*)}{\|x^k\|^2} \omega_k + \frac{\|x^*\|}{\|x^k\|^2} + \left\| x^k \right\|^{-1} \right). \end{aligned}$$

Taking lim sup on both sides of (22), observing that the sequence $\{v^k\}_{k \in \mathbb{N}}$ is bounded and using (7), we obtain a contradiction. \square

Lemma 2.4 shows that, if the variational inequality (1) has a solution, then the sequence $\{x^k\}_{k \in \mathbb{N}}$ is bounded. Since the space X is reflexive, if the sequence $\{x^k\}_{k \in \mathbb{N}}$ is bounded, then it has weakly convergent subsequences. Therefore, the following lemma shows that the variational inequality (1) has solutions if and only if the sequence $\{x^k\}_{k \in \mathbb{N}}$ is bounded.

Lemma 2.5. *If the sequence $\{x^k\}_{k \in \mathbb{N}}$ is bounded, then any weak accumulation point of the sequence $\{x^k\}_{k \in \mathbb{N}}$ is a solution of (1).*

Proof. Let $\{x^{i_k}\}_{k \in \mathbb{N}}$ be a weakly convergent subsequence of the sequence $\{x^k\}_{k \in \mathbb{N}}$ and let \bar{x} be the weak limit of this subsequence. According to (20), for each $k \in \mathbb{N}$, there exists $w^{i_k} \in \Omega$ such that $\|x^{i_k} - w^{i_k}\| \leq b(x^{i_k})\omega_{i_k}$, where b is bounded on bounded sets and the sequence $\{\omega_{i_k}\}_{k \in \mathbb{N}}$ converges to zero (cf. (6)). Hence, the sequences $\{x^{i_k}\}_{k \in \mathbb{N}}$ and $\{w^{i_k}\}_{k \in \mathbb{N}}$ have the same weak limit \bar{x} . Since $\{w^{i_k}\}_{k \in \mathbb{N}}$ is a sequence in Ω and Ω is weakly closed as being closed and convex, it follows that $\bar{x} \in \Omega$. Taking into account that $\Omega_k \subseteq \text{Int}(\text{Dom } A_k)$ one can apply the version of Minty's lemma presented in [9, Lemma 2.3] and deduce that for any $k \in \mathbb{N}$, for any $y \in \Omega_k$ and for any $\varphi^k \in A_k y$ we have

$$(23) \quad \langle \varphi^k + \alpha_k Jy - \psi^k, y - x^k \rangle \geq 0.$$

Let z be an arbitrary vector in Ω and let $\zeta \in Az$. By assumption **(A3-2)**, there exists a sequence $\{z^k\}_{k \in \mathbb{N}}$ in X and a sequence $\{\zeta^k\}_{k \in \mathbb{N}}$ in X^* such that $\lim_{k \rightarrow \infty} z^k = z$, $\lim_{k \rightarrow \infty} \zeta^k = \zeta$, $z^k \in \Omega_k$ and $\zeta^k \in A_k z^k$ for all $k \in \mathbb{N}$. According to (23), we have

$$\langle \zeta^k + \alpha_k Jz^k - \psi^k, z^k - x^k \rangle \geq 0,$$

for all $k \in \mathbb{N}$. Replacing k by i_k in this inequality and letting $k \rightarrow \infty$ we obtain

$$\langle \zeta - \psi, z - \bar{x} \rangle \geq 0.$$

Since the last inequality holds for any $z \in \Omega$ and $\zeta \in Az$, application of the Minty type lemma from [9, Lemma 2.3] implies that \bar{x} is a solution of (1). \square

Suppose from now on that the variational inequality (1) has at least one solution and, hence, that the sequence $\{x^k\}_{k \in \mathbb{N}}$ is bounded. Note that the Minty type lemma quoted above also implies that the solutions set $\mathcal{S}(A, \psi, \Omega)$ is convex and closed. Since $\mathcal{S}(A, \psi, \Omega)$ is nonempty, it follows that $\mathcal{S}(A, \psi, \Omega)$ has a unique minimal norm element.

Lemma 2.6. *The sequence $\{x^k\}_{k \in \mathbb{N}}$ converges weakly to the minimal norm element of $\mathcal{S}(A, \psi, \Omega)$. If the space has the Kadec-Klee property, then $\{x^k\}_{k \in \mathbb{N}}$ converges strongly.*

Proof. By Lemma 2.3, since $\mathcal{S}(A, \psi, \Omega)$ is nonempty, the sequence $\{x^k\}_{k \in \mathbb{N}}$ is bounded. We show that the unique weak accumulation point of $\{x^k\}_{k \in \mathbb{N}}$ is exactly the minimal norm element of the set $\mathcal{S}(A, \psi, \Omega)$. Let $\{x^{i_k}\}_{k \in \mathbb{N}}$ be a weakly convergent subsequence of the sequence $\{x^k\}_{k \in \mathbb{N}}$ and let \bar{x} be its weak limit. By Lemma 2.5, $\bar{x} \in \mathcal{S}(A, \psi, \Omega)$. Clearly, if $\bar{x} = 0$, then \bar{x} is the minimal norm solution of (1). Suppose that $\bar{x} \neq 0$. Then, by taking a subsequence if necessary, we may assume without loss of generality that $x^{i_k} \neq 0$ for all $k \in \mathbb{N}$. Now, replace k by i_k in (21), divide by $\|x^{i_k}\| \alpha_{i_k}$ the resulting inequality and take lim sup on both sides. In this way one obtains that, for any $x^* \in \mathcal{S}(A, \psi, \Omega)$, we have that $\limsup_{k \rightarrow \infty} \|x^{i_k}\| \leq \|x^*\|$ and, consequently,

$$(24) \quad \|\bar{x}\| \leq \liminf_{k \rightarrow \infty} \|x^{i_k}\| \leq \limsup_{k \rightarrow \infty} \|x^{i_k}\| \leq \|x^*\|.$$

Hence, \bar{x} is the minimal norm element of $\mathcal{S}(A, \psi, \Omega)$ in this case too.

Suppose now that the space X has the Kadec-Klee property. In this situation, for proving that the sequence $\{x^k\}_{k \in \mathbb{N}}$ converges strongly, it is sufficient to show that the sequence $\{\|x^k\|\}_{k \in \mathbb{N}}$ converges to $\|\bar{x}\|$. We distinguish again two situations. Suppose that $\bar{x} = 0$, but $\limsup_{k \rightarrow \infty} \|x^k\| = q > 0$. Then there exists a subsequence $\{x^{i_k}\}_{k \in \mathbb{N}}$ of the sequence $\{x^k\}_{k \in \mathbb{N}}$ consisting of non-null vectors and such that $\lim_{k \rightarrow \infty} \|x^{i_k}\| = q$. For this subsequence formula (24) still holds and it also holds true when $x^* = \bar{x}$ because the vector x^* involved there is an arbitrary element of $\mathcal{S}(A, \psi, \Omega)$. Hence, we get

$$0 < q = \lim_{k \rightarrow \infty} \|x^{i_k}\| \leq \|\bar{x}\| = 0,$$

that is, a contradiction. Consequently, $\limsup_{k \rightarrow \infty} \|x^k\| = 0$ and this implies $\lim_{k \rightarrow \infty} \|x^k\| = 0 = \|\bar{x}\|$. If $\bar{x} \neq 0$, then all but possibly finitely many terms of the sequence $\{x^k\}_{k \in \mathbb{N}}$ are not zero. Hence, there exists a positive integer k_0 such that, for any $k \geq k_0$, one can divide (21) by $\|x^k\| \alpha_k$. Taking \limsup as $k \rightarrow \infty$ on both sides of the resulting inequality one deduces

$$\|\bar{x}\| \leq \liminf_{k \rightarrow \infty} \|x^k\| \leq \limsup_{k \rightarrow \infty} \|x^k\| \leq \|x^*\|,$$

for any $x^* \in \mathcal{S}(A, \psi, \Omega)$. Writing the previous inequality for $x^* = \bar{x}$ we deduce that $\lim_{k \rightarrow \infty} \|x^k\| = \|\bar{x}\|$ in this case too. \square

3. A PROXIMITY THEOREM

In this section we prove a preliminary result which is needed in the build up of the algorithm for solving variational inequalities we are going to present in the next section. This result is a development of a series of previous similar theorems known as ‘‘proximity lemmas’’. Our proximity theorem is a generalization and improvement of the proximity lemmas due to Alber, Kartsatos and Litsyn [10] and to Alber and Nashed [11]. It improves upon these results in the sense that our proximity theorem applies to operators which are not necessarily monotone. Also, by contrast with the proximity lemmas mentioned above, which are proved under restrictive conditions on the moduli of convexity and smoothness of the underlying space X , our proximity theorem only requires that the space X is uniformly convex, that is, the modulus of convexity of X , $\delta_X : \mathbb{R} \rightarrow [0, +\infty]$ given by¹

$$\delta_X(t) := \inf \left\{ 1 - \frac{1}{2} \|x + y\| : \|x\| = \|y\| = 1, \|x - y\| \geq t \right\},$$

is positive for any $t > 0$. In that follows we assume that the space X is uniformly convex.

Let $\bar{g}_X : \mathbb{R} \rightarrow [0, +\infty]$ be the function defined by

$$(25) \quad \bar{g}_X(t) := \begin{cases} \frac{\delta_X(t)}{t} & \text{if } t \neq 0, \\ 0 & \text{if } t = 0. \end{cases}$$

This function is nondecreasing (cf. [26, Proposition 3]), vanishes when $t \leq 0$ and is positive whenever $t > 0$ because X is uniformly convex. From now on, we assume

¹We make here the usual assumption that $\inf \emptyset = +\infty$ and $\sup \emptyset = -\infty$.

that $g_X : \mathbb{R} \rightarrow [0, +\infty]$ is a nondecreasing function which vanishes on $(-\infty, 0]$ and such that

$$(26) \quad 0 < g_X(t) \leq \bar{g}_X(t), \quad \forall t > 0.$$

Clearly, one can choose $g_X = \bar{g}_X$. The relevance of choosing a function g_X other than \bar{g}_X is related with the possibility of using lower evaluations of δ_X instead of δ_X itself which may be hard to precisely compute. This aspect will become clear in the next section. To the function g_X we associate its *epi-quasi-inverse* $g_X^\sharp : \mathbb{R} \rightarrow [-\infty, +\infty]$ given by (cf. [40, p. 126])

$$(27) \quad g_X^\sharp(s) = \sup \{t \in \mathbb{R} : g_X(t) < s\}.$$

Clearly, the function g_X^\sharp is nondecreasing too. If the restriction of g_X to $[0, 2]$ is invertible, then $g_X^\sharp(s) = g_X^{-1}(s)$ for any $s \in g_X([0, 2])$ (cf. [40]).

In order to state our proximity theorem, let $T_i : X \rightarrow 2^{X^*}$ be an operator, let $\varphi^i \in X^*$ and let Λ_i be a nonempty subset of $\text{Dom } T_i$, where $i \in \{1, 2\}$. Suppose that, for some positive real numbers α and β , the variational inequalities

$$(28) \quad \langle (T_1 + \alpha J)x - \varphi^1, y - x \rangle \geq 0, \quad \forall y \in \Lambda_1$$

and

$$(29) \quad \langle (T_2 + \beta J)x - \varphi^2, y - x \rangle \geq 0, \quad \forall y \in \Lambda_2,$$

have solutions x^α and x^β , respectively. Then there exist $\xi^\alpha \in T_1 x^\alpha$ and $\xi^\beta \in T_2 x^\beta$ such that

$$(30) \quad \langle \xi^\alpha + \alpha Jx^\alpha - \varphi^1, y - x^\alpha \rangle \geq 0, \quad \forall y \in \Lambda_1$$

and

$$(31) \quad \langle \xi^\beta + \beta Jx^\beta - \varphi^2, y - x^\beta \rangle \geq 0, \quad \forall y \in \Lambda_2.$$

With these notations we state the following result which gives an evaluation of the distance between x^α and x^β . In the statement we use a constant L which occurs in Figiel's paper [26, Proposition 10]. It is a lower bound for the speed of variation of the function $\rho_X(t)/t^2$, where $\rho_X(t)$ is the modulus of smoothness of the space X . It follows from [52] and [5] that $L \in (1, 1.7)$.

Theorem 3.1 (The Proximity Theorem). *Suppose that the space X is uniformly convex and the operator T_2 is strongly h -hypomonotone. Let \bar{a} , \bar{b} , \bar{c} , δ , τ and ω be nonnegative real numbers such that*

$$(32) \quad \|\varphi^1 - \varphi^2\|_* \leq \delta$$

and the following conditions hold:

(i) *There exists $\bar{x}^\alpha \in \Lambda_2$ such that*

$$(33) \quad \|x^\alpha - \bar{x}^\alpha\| \leq \bar{a}\omega \text{ and } \text{dist}_*(\xi^\alpha, T_2 \bar{x}^\alpha) \leq \bar{c}\tau;$$

(ii) *There exists $\bar{x}^\beta \in \Lambda_1$ is such that*

$$(34) \quad \|x^\beta - \bar{x}^\beta\| \leq \bar{b}\omega.$$

If

$$(35) \quad M \geq \max \left\{ \|x^\alpha\|, \|x^\beta\| \right\} \text{ and } M_1 \geq \max \left\{ \|\xi^\alpha\|_*, \|\xi^\beta\|_* \right\},$$

then

$$(36) \quad \|x^\alpha - x^\beta\| \leq K_0 g_X^\sharp \left(K_1 \frac{|\alpha - \beta|}{\alpha} + K_2 \frac{\tau + \delta}{\alpha} + K_3 \sqrt{\frac{\omega + h}{\alpha}} \right),$$

where

$$(37) \quad K_0 = 2 \max \{1, M\}, \quad K_1 = 2LMK_0, \quad K_2 = 2LK_0 \max\{\bar{c}, 1\},$$

and

$$(38) \quad K_3 = \max \{1, 4L(K_4 + K_5)\},$$

with

$$(39) \quad K_4 = (M_1 + \|\varphi^1\|_* + \alpha M)\bar{b} + (3M_1 + \|\varphi^2\|_* + \tau\bar{c} + \beta M)\bar{a},$$

$$(40) \quad K_5 = (\omega\bar{a} + 2M)^2,$$

and L is the Figiel constant.

Proof. The inequality (36) clearly holds when $x^\alpha = x^\beta$. In that follows we assume that $x^\alpha \neq x^\beta$. Condition (i) implies that there exists $\bar{\xi}^\alpha \in T_2 \bar{x}^\alpha$ such that

$$(41) \quad \|\xi^\alpha - \bar{\xi}^\alpha\|_* \leq \tau\bar{c}.$$

Denote

$$D = \left\langle \xi^\alpha + \alpha Jx^\alpha - \varphi^1 - \xi^\beta - \beta Jx^\beta + \varphi^2, x^\alpha - x^\beta \right\rangle.$$

Then

$$(42) \quad \begin{aligned} D &= \alpha \left\langle Jx^\alpha - Jx^\beta, x^\alpha - x^\beta \right\rangle + (\alpha - \beta) \left\langle Jx^\beta, x^\alpha - x^\beta \right\rangle \\ &\quad + \left\langle \xi^\alpha - \bar{\xi}^\alpha, x^\alpha - x^\beta \right\rangle + \left\langle \bar{\xi}^\alpha - \xi^\beta, x^\alpha - \bar{x}^\alpha \right\rangle \\ &\quad + \left\langle \bar{\xi}^\alpha - \xi^\beta, \bar{x}^\alpha - x^\beta \right\rangle - \left\langle \varphi^1 - \varphi^2, x^\alpha - x^\beta \right\rangle. \end{aligned}$$

The strong h -hypomonotonicity of T_2 implies that

$$(43) \quad \left\langle \bar{\xi}^\alpha - \xi^\beta, \bar{x}^\alpha - x^\beta \right\rangle \geq -h \left(\|\bar{x}^\alpha\| - \|x^\beta\| \right)^2.$$

According to [13, Lemma 2.1] we have that

$$(44) \quad \left\langle Jx^\alpha - Jx^\beta, x^\alpha - x^\beta \right\rangle \geq (2L)^{-1} \delta_X \left(\frac{\|x^\alpha - x^\beta\|}{C} \right),$$

where $L > 0$ is the Figiel constant and

$$C := 2 \max \left\{ 1, \|x^\alpha\|, \|x^\beta\| \right\}.$$

Combining (41), (42), (43) and (44) and taking into account (32) and (33), we obtain that

$$(45) \quad D \geq \alpha(2L)^{-1} \delta_X \left(\frac{\|x^\alpha - x^\beta\|}{C} \right)$$

$$\begin{aligned}
& - |\alpha - \beta| \left\| \|x^\beta\| \right\| \|x^\alpha - x^\beta\| - \tau \bar{c} \|x^\alpha - x^\beta\| \\
& - \delta \left\| \|x^\alpha - x^\beta\| - \omega \bar{a} \left\| \bar{\xi}^\alpha - \xi^\beta \right\|_* - h \left(\|\bar{x}^\alpha\| - \|x^\beta\| \right)^2 \right. \\
& \geq \alpha (2L)^{-1} \delta_X \left(\frac{\|x^\alpha - x^\beta\|}{C} \right) \\
& - |\alpha - \beta| \left\| \|x^\beta\| \right\| \|x^\alpha - x^\beta\| - \tau \bar{c} \|x^\alpha - x^\beta\| \\
& - \delta \left\| \|x^\alpha - x^\beta\| - \omega \bar{a} \left\| \bar{\xi}^\alpha - \xi^\beta \right\|_* - h \|\bar{x}^\alpha - x^\beta\| \right\|^2.
\end{aligned}$$

Note that, by (41) and (35), we have

$$\begin{aligned}
\left\| \bar{\xi}^\alpha - \xi^\beta \right\|_* & \leq \left\| \bar{\xi}^\alpha - \xi^\alpha \right\|_* + \left\| \xi^\alpha - \xi^\beta \right\|_* \\
& \leq \left\| \bar{\xi}^\alpha - \xi^\alpha \right\|_* + \left\| \xi^\alpha \right\|_* + \left\| \xi^\beta \right\|_* \leq \tau \bar{c} + 2M_1.
\end{aligned}$$

Similarly, by (33) and (35), we obtain that

$$\|\bar{x}^\alpha\|_* \leq \|\bar{x}^\alpha - x^\alpha\| + \|x^\alpha\| \leq \omega \bar{a} + M.$$

Therefore, inequality (45) leads to

$$\begin{aligned}
(46) \quad D & \geq \alpha (2L)^{-1} \delta_X \left(\frac{\|x^\alpha - x^\beta\|}{C} \right) - \omega \bar{a} (2M_1 + \tau \bar{c}) \\
& - (M |\alpha - \beta| + \tau \bar{c} + \delta) \|x^\alpha - x^\beta\| - h K_5,
\end{aligned}$$

where K_5 is given by (40). Observe that, according to (30), we have that

$$\begin{aligned}
(47) \quad \left\langle \xi^\alpha + \alpha Jx^\alpha - \varphi^1, x^\alpha - x^\beta \right\rangle & = \left\langle \xi^\alpha + \alpha Jx^\alpha - \varphi^1, x^\alpha - \bar{x}^\beta \right\rangle \\
& + \left\langle \xi^\alpha + \alpha Jx^\alpha - \varphi^1, \bar{x}^\beta - x^\beta \right\rangle \\
& \leq \left\langle \xi^\alpha + \alpha Jx^\alpha - \varphi^1, \bar{x}^\beta - x^\beta \right\rangle.
\end{aligned}$$

Thus, by taking into account (47), (35) and (34), we obtain

$$\begin{aligned}
(48) \quad \left\langle \xi^\alpha + \alpha Jx^\alpha - \varphi^1, x^\alpha - x^\beta \right\rangle & \leq (\|\xi^\alpha - \varphi^1\|_* + \alpha \|x^\alpha\|) \|\bar{x}^\beta - x^\beta\| \\
& \leq (M_1 + \|\varphi^1\|_* + \alpha M) \omega \bar{b}.
\end{aligned}$$

Analogously, using (31), we deduce that

$$(49) \quad \left\langle \xi^\beta + \beta Jx^\beta - \varphi^2, x^\beta - x^\alpha \right\rangle \leq (M_1 + \|\varphi^2\|_* + \beta M) \omega \bar{a}.$$

From (42), (48) and (49) taken together, we obtain that

$$(50) \quad D \leq \omega \left[(M_1 + \|\varphi^1\|_* + \alpha M) \bar{b} + (M_1 + \|\varphi^2\|_* + \beta M) \bar{a} \right].$$

Denote by K_6 be the quantity occurring in (50) between square brackets. By (39) we have

$$K_4 = K_6 + (2M_1 + \tau \bar{c}) \bar{a}.$$

Then, according to (46) and (50), we obtain that

$$(51) \quad \begin{aligned} & \omega K_4 + K_5 h + (M |\alpha - \beta| + \tau \bar{c} + \delta) \|x^\alpha - x^\beta\| \\ & \geq \alpha (2L)^{-1} \delta_X \left(\frac{\|x^\alpha - x^\beta\|}{K_0} \right), \end{aligned}$$

where K_0 is given by (37). Dividing this inequality by $\|x^\alpha - x^\beta\| > 0$ we deduce that

$$(52) \quad \begin{aligned} & \frac{\omega K_4 + K_5 h}{\|x^\alpha - x^\beta\|} + M |\alpha - \beta| + \tau \bar{c} + \delta \\ & \geq \alpha (2LK_0)^{-1} \delta_X \left(\frac{\|x^\alpha - x^\beta\|}{K_0} \right) \frac{K_0}{\|x^\alpha - x^\beta\|} \\ & = \alpha (2LK_0)^{-1} \bar{g}_X \left(\frac{\|x^\alpha - x^\beta\|}{K_0} \right) \\ & > \alpha (4LK_0)^{-1} \bar{g}_X \left(\frac{\|x^\alpha - x^\beta\|}{K_0} \right) \\ & \geq \alpha (4LK_0)^{-1} g_X \left(\frac{\|x^\alpha - x^\beta\|}{K_0} \right), \end{aligned}$$

where \bar{g}_X is the function defined by (25) and g_X is the nondecreasing function satisfying (26). Note that, according to (27), if $u, v \in \mathbb{R}$, then

$$(53) \quad u > g_X(v) \Rightarrow g_X^\sharp(u) \geq v.$$

Thus, using (52), we obtain that

$$(54) \quad \|x^\alpha - x^\beta\| \leq K_0 g_X^\sharp \left(\frac{4LK_0 \omega K_4 + K_5 h}{\alpha \|x^\alpha - x^\beta\|} + K_1 \frac{|\alpha - \beta|}{\alpha} + K_2 \frac{\tau + \delta}{\alpha} \right),$$

where K_1 and K_2 are given by (37). Now we distinguish the following complementary situations.

Case 1: Suppose that

$$\frac{\|x^\alpha - x^\beta\|}{K_0} \leq g_X^\sharp \left(\sqrt{\frac{\omega + h}{\alpha}} + K_1 \frac{|\alpha - \beta|}{\alpha} + K_2 \frac{\tau + \delta}{\alpha} \right).$$

In this case, we have that

$$\begin{aligned} & g_X^\sharp \left(\sqrt{\frac{\omega + h}{\alpha}} + K_1 \frac{|\alpha - \beta|}{\alpha} + K_2 \frac{\tau + \delta}{\alpha} \right) \\ & \leq g_X^\sharp \left(K_3 \sqrt{\frac{\omega + h}{\alpha}} + K_1 \frac{|\alpha - \beta|}{\alpha} + K_2 \frac{\tau + \delta}{\alpha} \right) \end{aligned}$$

because g_X^\sharp is nondecreasing and $K_3 \geq 1$ (see (38)). Thus, combining the last two inequalities, we deduce (36).

Case 2: Suppose that

$$(55) \quad \frac{\|x^\alpha - x^\beta\|}{K_0} > g_X^\sharp \left(\sqrt{\frac{\omega + h}{\alpha}} + K_1 \frac{|\alpha - \beta|}{\alpha} + K_2 \frac{\tau + \delta}{\alpha} \right).$$

Recall (see, for instance, [28, Theorem 2.7.8]) that, since X is a uniformly convex Banach space, for any Hilbert space H we have that

$$\delta_X(t) \leq \delta_H(t) \leq \frac{t^2}{4}, \quad \forall t \in [0, 2].$$

Consequently,

$$\frac{1}{4} \left(\frac{\|x^\alpha - x^\beta\|}{K_0} \right)^2 \geq \delta_X \left(\frac{\|x^\alpha - x^\beta\|}{K_0} \right),$$

because $K_0 \geq 2M \geq \|x^\alpha - x^\beta\|$ as follows from (35) and (37). This implies that

$$(56) \quad \frac{1}{4} \frac{\|x^\alpha - x^\beta\|}{K_0} \geq \bar{g}_X \left(\frac{\|x^\alpha - x^\beta\|}{K_0} \right) \geq g_X \left(\frac{\|x^\alpha - x^\beta\|}{K_0} \right).$$

Observe that, by (27), if $u, v \in \mathbb{R}$, then

$$(57) \quad u > g_X^\sharp(v) \Rightarrow g_X(u) \geq v.$$

From (55) and (57) we deduce that

$$g_X \left(\frac{\|x^\alpha - x^\beta\|}{K_0} \right) \geq \sqrt{\frac{\omega + h}{\alpha}} + K_1 \frac{|\alpha - \beta|}{\alpha} + K_2 \frac{\tau + \delta}{\alpha} > \sqrt{\frac{\omega + h}{\alpha}}.$$

This and (56) combined imply

$$(58) \quad \frac{\|x^\alpha - x^\beta\|}{K_0} > 4\sqrt{\frac{\omega + h}{\alpha}}.$$

Taking into account (38), (58), the fact that g_X^\sharp is nondecreasing and the inequality (54), we deduce that

$$\begin{aligned} & K_0 g_X^\sharp \left[K_3 \frac{\omega + h}{\alpha} \left(4\sqrt{\frac{\omega + h}{\alpha}} \right)^{-1} + K_1 \frac{|\alpha - \beta|}{\alpha} + K_2 \frac{\tau + \delta}{\alpha} \right] \\ & \geq K_0 g_X^\sharp \left[4L(K_4 + K_5) \frac{\omega + h}{\alpha} \frac{K_0}{\|x^\alpha - x^\beta\|} + K_1 \frac{|\alpha - \beta|}{\alpha} + K_2 \frac{\tau + \delta}{\alpha} \right] \\ & \geq K_0 g_X^\sharp \left[4L \frac{K_4 \omega + K_5 h}{\alpha} \frac{K_0}{\|x^\alpha - x^\beta\|} + K_1 \frac{|\alpha - \beta|}{\alpha} + K_2 \frac{\tau + \delta}{\alpha} \right] \\ & \geq \|x^\alpha - x^\beta\| \end{aligned}$$

and this implies (36). □

4. AN ITERATIVE ALGORITHM FOR SOLVING VARIATIONAL INEQUALITIES

In this section we present an iterative algorithm for approximating solutions of the variational inequality (1), presuming that such a solution exists. All over this section we assume that the space X is smooth and uniformly convex and that the assumptions **(A1)**-**(A3)**, as well as the following assumption, are satisfied:

(A4) There exists a continuous nondecreasing function $\Phi : [0, \infty) \rightarrow \mathbb{R}$ such that, for any $k \in \mathbb{N}$, we have that

$$(59) \quad x \in \Omega_k \text{ and } \zeta \in A_k x \Rightarrow \|\zeta\|_* \leq \Phi(\|x\|).$$

In order to describe our algorithm in a consistent manner, we need some notations and preparations. Note that, since the space X is uniformly convex, it is also strictly convex and has the Kadec-Klee property. This guarantees applicability of Theorem 2.1(ii) in this space. Since the variational inequality (1) is presumed to have solutions, Theorem 2.1 guarantees that the sequence $\{x^k\}_{k \in \mathbb{N}}$ of solutions of the variational inequalities (4) exists and converges strongly to the minimal norm solution of (1). Second, recall that each operator A_k is strongly h_k -hypomonotone (by condition **(A2)**). Consider the variational inequalities

$$(60) \quad \langle (A + \alpha_k J)x - \psi, y - x \rangle \geq 0, \quad \forall y \in \Omega.$$

According to Lemma 2.3 (which is applicable to (60) because A is monotone and, therefore, strongly 0-hypomonotone), the variational inequality (60) has a unique solution which we denote by u^k in what follows. Applying Theorem 2.1(ii) to the particular situation when $A_k = A$, $\Omega_k = \Omega$ and $\psi^k = \psi$ for all $k \in \mathbb{N}$, we deduce that the sequence $\{u^k\}_{k \in \mathbb{N}}$ converges to the same solution as $\{x^k\}_{k \in \mathbb{N}}$, that is, to the minimal norm solution \bar{x} of (1). Since both sequences $\{x^k\}_{k \in \mathbb{N}}$ and $\{u^k\}_{k \in \mathbb{N}}$ are bounded (as being convergent), there exists a positive real number M such that

$$(61) \quad M \geq \max \left\{ \|u^k\|, \|x^k\| \right\}, \quad \forall k \in \mathbb{N}.$$

For each $k \in \mathbb{N}$, let $\xi^k \in A_k x^k$ be such that (17) holds. According to **(A4)**, we have that $\|\xi^k\|_* \leq \Phi(\|x^k\|)$, for all $k \in \mathbb{N}$. Since Φ is bounded on bounded sets, it results that the sequence $\{\xi^k\}_{k \in \mathbb{N}}$ is bounded. Let $\zeta^k \in A u^k$ be such that

$$(62) \quad \langle \zeta^k + \alpha_k J u^k - \psi, y - u^k \rangle \geq 0, \quad \forall y \in \Omega.$$

Since $\{u^k\}_{k \in \mathbb{N}}$ is a sequence in $\text{Int}(\text{Dom } A)$ which converges to $\bar{x} \in \Omega \subseteq \text{Int}(\text{Dom } A)$ and A is locally bounded at \bar{x} (as being monotone), it results that $\{\zeta^k\}_{k \in \mathbb{N}}$ is bounded too. Consequently, there exists a positive real number M_1 such that

$$(63) \quad M_1 \geq \max \left\{ \|\zeta^k\|_*, \|\xi^k\|_* \right\}, \quad \forall k \in \mathbb{N}.$$

Finally, observe that, according to **(A3-1)**, the sequence $\{\psi^k\}_{k \in \mathbb{N}}$ is convergent. Therefore, there exists a positive number M_2 such that

$$(64) \quad M_2 \geq \|\psi^k\|_*, \quad \forall k \in \mathbb{N}.$$

The functions a , b and c being those involved in condition **(A3)**, we denote

$$(65) \quad \bar{a} := \sup \{a(x) : \|x\| \leq M\},$$

$$(66) \quad \bar{b} := \sup \{b(x) : \|x\| \leq M\},$$

$$(67) \quad \bar{c} := \sup \{c(\zeta) : \|\zeta\|_* \leq M_1\},$$

where M and M_1 are numbers satisfying (61) and (63), respectively. Clearly, \bar{a} , \bar{b} and \bar{c} are finite because of the boundedness on bounded sets of a , b and c . In what follows, we denote by K_0 , K_1 , K_2 and K_3 the numbers defined by (37) and (38) where, instead of K_4 and K_5 given by (39) and, respectively, (40), we take

$$(68) \quad K_4 := (M_1 + M_2 + \bar{\delta} + \bar{\alpha}M)\bar{b} + (3M_1 + M_2 + \bar{\tau}\bar{c} + \bar{\alpha}M)\bar{a},$$

$$(69) \quad K_5 := (M + \bar{\omega}\bar{a})^2 + M^2,$$

with $\bar{\alpha}$, $\bar{\delta}$, $\bar{\tau}$, $\bar{\omega}$ being positive upper bounds of the sequences $\{\alpha_k\}_{k \in \mathbb{N}}$, $\{\delta_k\}_{k \in \mathbb{N}}$, $\{\tau_k\}_{k \in \mathbb{N}}$, $\{\omega_k\}_{k \in \mathbb{N}}$, respectively (such upper bounds exists because, by **(A3)**, these sequences are convergent).

Let g_X be the nonnegative, nondecreasing function which satisfies (26) and vanishes on $(-\infty, 0)$ and let g_X^\sharp be its epi-quasi-inverse defined by (27). For each $k \in \mathbb{N}$, we denote

$$(70) \quad G_k := K_0 g_X^\sharp \left(K_2 \frac{\tau_k + \delta_k}{\alpha_k} + K_3 \sqrt{\frac{\omega_k + h_k}{\alpha_k}} \right),$$

$$(71) \quad \bar{G}_k := K_0 g_X^\sharp \left(K_1 \frac{|\alpha_k - \alpha_{k+1}|}{\alpha_k} \right),$$

where the numbers K_0, \dots, K_3 are those defined above (with K_4 and K_5 given by (68) and (69)). Let $\rho_{X^*} : [0, \infty) \rightarrow [0, \infty)$ be the modulus of smoothness of the dual space X^* , i.e., the function which, according to a theorem of J. Lindenstrauss (see [28, Theorem 2.7.5]), is given by

$$(72) \quad \rho_{X^*}(t) = \sup \left\{ \frac{ts}{2} - \delta_X(s) : s \in [0, 2] \right\}.$$

Since the space X is uniformly convex, its dual X^* is uniformly smooth, that is, $\lim_{t \rightarrow 0^+} t^{-1} \rho_{X^*}(t) = 0$. We assume that the problem data and the geometry of the space X in which our variational inequality (1) is set are interconnected in the sense of the following condition:

(A5) The sequence of regularization parameters $\{\alpha_k\}_{k \in \mathbb{N}}$ has the property that there exists a bounded sequence of positive real numbers $\{\varepsilon_k\}_{k \in \mathbb{N}}$ such that $\alpha_k \varepsilon_k < 1$ for all $k \in \mathbb{N}$,

$$(73) \quad \sum_{k=0}^{\infty} \alpha_k \varepsilon_k = \infty \text{ and } \lim_{k \rightarrow \infty} (\alpha_k \varepsilon_k)^{-1} [\rho_{X^*}(\varepsilon_k) + G_k + G_{k+1} + \bar{G}_k] = 0.$$

Condition **(A5)** represents the rule of choosing the step size parameters in the algorithm described below when the sequences involved in **(A3)** are a priori given. Implementation of this rule requires evaluations of ρ_{X^*} as well as of g_X^\sharp . In some spaces such evaluations are readily available. That is the case of the spaces which

are p -convex for some $p > 1$ as, for instance, the Lebesgue spaces, the Sobolev spaces, the Orlicz spaces (cf. [12, p. 613]). Recall that the space X is called p -convex if there exists a positive real number \tilde{c} such that $\delta_X(t) \geq \tilde{c}t^p$ for all $t > 0$. If X is p -convex for some $p > 1$, then, according to (72), we obtain that

$$\rho_{X^*}(t) \leq \sup \left\{ \frac{ts}{2} - \tilde{c}s^p : s \in [0, 2] \right\} = ct^q,$$

where c is a positive constant and $q = p/(p-1)$. In this case, one can define $g_X(t) = ct^{p-1}$ if $t \geq 0$, and $g_X(t) = 0$, otherwise. This implies that $g_X^\sharp(t) = (c^{-1}t)^{q/p}$ for any $t \geq 0$.

To any closed convex nonempty subset Λ of X we associate the operator $\Gamma_\Lambda : X^* \rightarrow \Lambda$ given by

$$\Gamma_\Lambda \xi := \arg \min \{W(\xi, u) : u \in \Lambda\},$$

where, for any $\xi \in X^*$ and for any $u \in X$, the function $W : X^* \times X \rightarrow \mathbb{R}$ is defined by

$$W(\xi, u) = \|u\|^2 - 2\langle \xi, u \rangle + \|\xi\|_*^2.$$

As shown in [5, p. 31], where this operator was introduced and studied, Γ_Λ is well defined. The algorithm for finding solutions of variational inequalities we present below requires computing values of Γ_Λ (see (79)) for various nonempty convex closed sets Λ . In this respect, recall (see [5]) that

$$(74) \quad \Gamma_\Lambda = (\Gamma_\Lambda \circ J) \circ J^* = P_\Lambda \circ J^*,$$

where J^* is the normalized duality mapping of X^* and P_Λ denotes the Bregman projection onto the set Λ with respect to the function $\theta(x) := \|x\|^2$. It results from (74) that, if the values of J^* are computable (as happens in many usual Banach spaces as, for instance, in Lebesgue spaces L^p , in Sobolev spaces $W^{m,p}$), then computing values of Γ_Λ amounts to determining values of P_Λ . If Λ is a closed hyperplane or a closed half-space, then values of P_Λ can be determined by formulae established in [6] and [23]. If Λ is an arbitrary nonempty closed convex subset of X , then values of P_Λ can be calculated using the algorithms presented in [18].

With these facts in mind, we proceed to the description of the iterative procedure we propose for solving the monotone variational inequality (1).

THE ALGORITHM

Step 0 (Initialization).

(a) Fix three numbers M , M_1 and M_2 such that conditions (61), (63) and (64), respectively, are satisfied, let R_0 be a positive real number such that $\sqrt{R_0} > 2M$ and put

$$(75) \quad K := M + \sqrt{R_0}.$$

(b) Define the functions $\mu, \varkappa, r : [0, \infty) \rightarrow \mathbb{R}$ and $\Upsilon : [0, \infty) \times \mathbb{N} \rightarrow \mathbb{R}$ by

$$(76) \quad \begin{aligned} \mu(t) &= \max \{ \Phi(t) + \bar{\alpha}t + M_2, 1 \}, \\ \varkappa(t) &= \bar{\varepsilon}(\Phi(t) + M_2) + (\bar{\alpha} + 1)t, \\ r(t) &= \sqrt{\mu^2(t) + \varkappa^2(t)}, \end{aligned}$$

$$\Upsilon(t, k) := 2 \left[2Lr^2(t) \rho_{X^*} (4r(t)^{-1} \mu(t) \varepsilon_k) + \varepsilon_k h_k (t^2 + M^2) \right. \\ \left. + (\varepsilon_k \mu(t) + t + M) (G_k + G_{k+1} + \bar{G}_k) \right],$$

where $\bar{\alpha}$ and $\bar{\varepsilon}$ are upper bounds of the sequences $\{\alpha_k\}_{k \in \mathbb{N}}$ and $\{\varepsilon_k\}_{k \in \mathbb{N}}$, respectively.

(c) Let $n_0 \in \mathbb{N}$ be a nonnegative integer such that for any $k \geq n_0$ we have

$$(77) \quad \Upsilon(K, k) \leq R_0 \alpha_k \varepsilon_k,$$

put $p(0) = n_0$ and choose $z^0 \in \Omega_{p(0)}$ such that

$$(78) \quad \sqrt{R_0} - M \geq \|z^0\|.$$

Step 1 (Iteration).

Given $k \in \mathbb{N}$ and $z^k \in \Omega_{p(k)}$, put $p(k+1) = n_0 + k + 1$, choose $\chi^k \in A_{p(k)} z^k$ and compute

$$(79) \quad z^{k+1} = \Gamma_{\Omega_{p(k+1)}} \left[Jz^k - \varepsilon_{p(k)} \left(\chi^k + \alpha_{p(k)} Jz^k - \psi^{p(k)} \right) \right].$$

Step 2 (Loop).

Let $k \rightarrow k + 1$ and go to Step 1.

The following result describes the convergence behavior of our algorithm.

Theorem 4.1. *If the variational inequality (1) has at least one solution and if the assumptions (A1)-(A5) are satisfied, then the sequence $\{z^k\}_{k \in \mathbb{N}}$ generated by the algorithm described above is well defined and is strongly convergent to the minimal norm solution of (1).*

The proof of this theorem is presented below as a succession of lemmas. The basic idea of the proof is to show that the sequence $\{z^k\}_{k \in \mathbb{N}}$ generated by the algorithm and the sequence $\{x^k\}_{k \in \mathbb{N}}$, whose existence and convergence is ensured by Theorem 2.1(ii), have the same limit. One should observe that the algorithm does not require computing the sequence $\{x^k\}_{k \in \mathbb{N}}$, but only to have an evaluation of an upper bound of the sequence $\{\|x^k\|\}_{k \in \mathbb{N}}$. Once such an evaluation is established, one can use Lemma 4.2 below in order to estimate the number M required by the algorithm because, as follows from Lemma 4.1, the sequence $\{G_k\}_{k \in \mathbb{N}}$ occurring in (83) is convergent and, hence, bounded. In order to estimate the number M_1 required by the algorithm one should note that, by virtue of (A4), the number $\Phi(M)$ is an upper bound of the sequence $\{\|\xi^k\|_*\}_{k \in \mathbb{N}}$, where ξ^k is defined by (17). This fact and (A3-2) allow for determining M_1 . These remarks show that implementation of the algorithm does not require computations with the original problem data, but involves computations with the approximative data only.

Now, we start our proof of Theorem 4.1 by establishing well definedness of the algorithm. Note first that, if the number $p(0) = n_0$ is well defined, then the vector $z^0 \in \Omega_{p(0)}$, which is required at (78), exists since at least $x^{p(0)}$ satisfies that condition because $\sqrt{R_0} - M > M \geq \|x^{p(0)}\|$. The following result implies that the number n_0 required in Step 0 of the algorithm (see (77)) exists and, hence, the algorithm is well defined.

Lemma 4.1. *The sequence $\{\gamma_k\}_{k \in \mathbb{N}}$ defined by*

$$(80) \quad \gamma_k = (\alpha_k \varepsilon_k)^{-1} \Upsilon(K, k),$$

where K is the real number given by (75), converges to zero.

Proof. According to (76) we have

$$\begin{aligned} \frac{1}{2} \gamma_k &= 2(\alpha_k \varepsilon_k)^{-1} L \bar{r}^2 \rho_{X^*} (4\bar{r}^{-1} \bar{\mu} \varepsilon_k) + \varepsilon_k h_k (K^2 + M^2) \\ &\quad + (\alpha_k \varepsilon_k)^{-1} (\varepsilon_k \bar{\mu} + K + M) (G_k + G_{k+1} + \bar{G}_k), \end{aligned}$$

where

$$(81) \quad \bar{r} = r(K) \text{ and } \bar{\mu} = \mu(K).$$

By (73), the last term of this sum converges to zero as $k \rightarrow \infty$. According to **(A5)**, the sequence $\{\alpha_k \varepsilon_k\}_{k \in \mathbb{N}}$ is bounded from above by 1. Therefore, we have that $\varepsilon_k h_k \leq \alpha_k^{-1} h_k$. Since, by (6), the sequence $\{\alpha_k^{-1} h_k\}_{k \in \mathbb{N}}$ converges to zero, we deduce that the second term of the sum converges to zero as $k \rightarrow \infty$. It remains to show that

$$(82) \quad \lim_{k \rightarrow \infty} (\alpha_k \varepsilon_k)^{-1} \rho_{X^*} (4\bar{r}^{-1} \bar{\mu} \varepsilon_k) = 0.$$

To this end, observe that, according to (76), we have that $\bar{r}^{-1} \bar{\mu} \leq 1$. Since the function ρ_{X^*} is nondecreasing (cf. [28, Lemma 2.7.4]) we deduce that

$$\frac{\rho_{X^*} (4\bar{r}^{-1} \bar{\mu} \varepsilon_k)}{\alpha_k \varepsilon_k} \leq \frac{\rho_{X^*} (4\varepsilon_k)}{\alpha_k \varepsilon_k}.$$

Applying twice Lemma 8 in [26] we deduce

$$\begin{aligned} \frac{\rho_{X^*} (4\varepsilon_k)}{\alpha_k \varepsilon_k} &\leq 4(1 + \varepsilon_k) \frac{\rho_{X^*} (2\varepsilon_k)}{\alpha_k \varepsilon_k} \\ &\leq 16(1 + \varepsilon_k) \left(1 + \frac{1}{2} \varepsilon_k\right) \frac{\rho_{X^*} (\varepsilon_k)}{\alpha_k \varepsilon_k}. \end{aligned}$$

The last two inequalities and (73) imply (82). \square

We are going to use Theorem 3.1 in order to obtain an evaluation of the distance between the vectors x^k and u^k .

Lemma 4.2. *For any $k \in \mathbb{N}$, we have that*

$$(83) \quad \left\| x^k - u^k \right\| \leq G_k.$$

Proof. We apply Theorem 3.1 to the variational inequalities (60) and (4) by taking $T_1 = A$, $T_2 = A_k$, $\varphi^1 = \psi$, $\varphi^2 = \psi^k$, $\omega = \omega_k$, $\tau = \tau_k$, and $\delta = \delta_k$. Condition **(A3-3)** ensures that hypothesis (ii) of Theorem 3.1 is valid in this case. By conditions **(A3-1)** and **(A3-2)** we deduce that hypothesis (i) of Theorem 3.1 is also satisfied. Thus, we deduce the following particular version of (36) with $\alpha = \beta = \alpha_k$:

$$(84) \quad \left\| x^k - u^k \right\| \leq K_0 g_X^\# \left(K_2 \frac{\tau_k + \delta_k}{\alpha_k} + K_3 \sqrt{\frac{\omega_k + h_k}{\alpha_k}} \right),$$

where the numbers K_1, \dots, K_5 are defined by (37)-(40) with α, ω, τ and δ as above. By direct comparison one can see that the numbers K_4 and K_5 , defined by (68) and

(69), are at least equal to their homonymous numbers defined by (39) and (40) in our current circumstances. This implies that the corresponding values of K_0 , K_1 , K_2 and K_3 obtained when K_4 and K_5 are given by (68) and (69) are at least equal to those of their homonymous counterparts obtained when K_4 and K_5 are given by (39) and (40). Since the function g_X^\sharp is nondecreasing, it follows that by replacing on the right hand side of (84) the numbers K_i by their larger counterparts, the inequality still stands and this is exactly (83). \square

The following lemma is a consequence of Theorem 3.1 and of Lemma 4.2. It is the key result for our proof of convergence of the sequences $\{z^k\}_{k \in \mathbb{N}}$ generated by the algorithm described above.

Lemma 4.3. *For each $k \in \mathbb{N}$, we have that*

$$(85) \quad W(Jz^{k+1}, x^{p(k+1)}) \leq (1 - \varepsilon_{p(k)} \alpha_{p(k)}) W(Jz^k, x^{p(k)}) + \Upsilon(\|z^k\|, p(k)).$$

Proof. Denote

$$(86) \quad \begin{aligned} w^k &:= Jz^k - \varepsilon_{p(k)} \left(\chi^k + \alpha_{p(k)} Jz^k - \psi^{p(k)} \right), \\ \varkappa_k &:= \varkappa(\|z^k\|), \quad r_k := r(\|z^k\|), \quad \mu_k := \mu(\|z^k\|), \end{aligned}$$

and note that, according to (79), we have

$$(87) \quad z^{k+1} = \Gamma_{\Omega_{p(k+1)}} w^k.$$

Observe that, whenever $u \in X$, the function $W(\cdot, u)$ is convex and (Gâteaux) differentiable at any point of X^* because the space X^* is smooth (since X is uniformly convex). Moreover, for any $\zeta \in X^*$ and for any $u \in X$, we have that

$$[W(\cdot, u)]'(\zeta) = 2(J^* \zeta - u),$$

where $J^* = J^{-1}$ is the normalized duality mapping of the space X^* (cf. [5, Lemma 6.1]). Therefore, we obtain that

$$(88) \quad \begin{aligned} &W(Jz^k, x^{p(k+1)}) - W(w^k, x^{p(k+1)}) \\ &\geq \left\langle Jz^k - w^k, [W(\cdot, x^{p(k+1)})]'(w^k) \right\rangle \\ &= 2 \left\langle Jz^k - w^k, J^* w^k - x^{p(k+1)} \right\rangle. \end{aligned}$$

Applying [5, Property 6h] and taking into account (87) and the fact that $x^{p(k+1)} \in \Omega_{p(k+1)}$ we obtain that

$$(89) \quad W(w^k, z^{k+1}) \leq W(w^k, x^{p(k+1)}).$$

Combining this and (88) we deduce that

$$(90) \quad \begin{aligned} &W(Jz^k, x^{p(k+1)}) - W(Jz^{k+1}, x^{p(k+1)}) \\ &\geq 2 \left\langle Jz^k - w^k, J^* w^k - x^{p(k+1)} \right\rangle. \end{aligned}$$

For any $\zeta \in X^*$, the function $W(\zeta, \cdot)$ is also convex and differentiable and we have (cf. [5, Lemma 6.1]) that

$$(91) \quad [W(\zeta, \cdot)]'(u) = 2(Ju - \zeta).$$

Thus, we deduce that

$$\begin{aligned} & W(Jz^k, x^{p(k)}) - W(Jz^k, x^{p(k+1)}) \\ & \geq \left\langle [W(Jz^k, \cdot)]'(x^{p(k+1)}), x^{p(k)} - x^{p(k+1)} \right\rangle \\ & = 2 \left\langle Jx^{p(k+1)} - Jz^k, x^{p(k)} - x^{p(k+1)} \right\rangle. \end{aligned}$$

Adding this and inequality (90) one gets

$$\begin{aligned} & W(Jz^k, x^{p(k)}) - W(Jz^{k+1}, x^{p(k+1)}) \\ & \geq 2 \left\langle Jz^k - w^k, J^*w^k - x^{p(k+1)} \right\rangle \\ & \quad + 2 \left\langle Jx^{p(k+1)} - Jz^k, x^{p(k)} - x^{p(k+1)} \right\rangle. \end{aligned}$$

Thus, we have

$$\begin{aligned} W(Jz^{k+1}, x^{p(k+1)}) & \leq W(Jz^k, x^{p(k)}) \\ & \quad + 2 \left\langle Jx^{p(k+1)} - Jz^k, x^{p(k+1)} - x^{p(k)} \right\rangle \\ & \quad + 2 \left\langle w^k - Jz^k, J^*w^k - x^{p(k+1)} \right\rangle. \end{aligned}$$

By consequence, we obtain

$$(92) \quad \begin{aligned} W(Jz^{k+1}, x^{p(k+1)}) & \leq W(Jz^k, x^{p(k)}) \\ & \quad + 2 \left\langle Jx^{p(k+1)} - Jz^k, x^{p(k+1)} - x^{p(k)} \right\rangle \\ & \quad + 2(U_k + V_k + Z_k). \end{aligned}$$

where

$$(93) \quad \begin{aligned} U_k & := \left\langle w^k - Jz^k, J^*w^k - z^k \right\rangle, \\ V_k & := \left\langle w^k - Jz^k, z^k - x^{p(k)} \right\rangle, \\ Z_k & := \left\langle w^k - Jz^k, x^{p(k)} - x^{p(k+1)} \right\rangle. \end{aligned}$$

Recall (see [5, Theorem 7.5]) that, if $\xi', \xi'' \in X^*$ and $R \geq \sqrt{\frac{1}{2}(\|\xi'\|_*^2 + \|\xi''\|_*^2)}$, then we have that

$$(94) \quad \left\langle \xi' - \xi'', J^*\xi' - J^*\xi'' \right\rangle \leq 2LR^2\rho_{X^*}(4R^{-1}\|\xi' - \xi''\|_*),$$

where ρ_{X^*} is the modulus of smoothness of X^* . Let $\vartheta^k = Jz^k$ and observe that $z^k = J^*(Jz^k) = J^*\vartheta^k$. Note that, according to (76) and **(A4)**, we have that

$$\|w^k\|_* \leq \|z^k\| + \varepsilon_{p(k)} \left(\|\chi^k\|_* + \alpha_{p(k)} \|z^k\| + M_2 \right)$$

$$\leq (1 + \bar{\alpha}) \|z^k\| + \varepsilon_{p(k)} \left(\Phi(\|z^k\|) + M_2 \right) \leq \varkappa_k.$$

Thus, we deduce that

$$\begin{aligned} r_k &\geq \sqrt{\varkappa_k^2 + \|z^k\|^2} \geq \sqrt{\|w^k\|_*^2 + \|z^k\|^2} \\ &\geq \sqrt{\frac{1}{2} \left(\|w^k\|_*^2 + \|z^k\|^2 \right)} = \sqrt{\frac{1}{2} \left(\|w^k\|_*^2 + \|\vartheta^k\|_*^2 \right)}, \end{aligned}$$

because of (76). Then, by (94), we obtain that

$$(95) \quad \begin{aligned} U_k &= \left\langle w^k - \vartheta^k, J^* w^k - J^* \vartheta^k \right\rangle \\ &\leq 2Lr_k^2 \rho_{X^*} \left(4r_k^{-1} \|w^k - Jz^k\|_* \right). \end{aligned}$$

Note that, by (86), we have

$$(96) \quad \|w^k - Jz^k\|_* = \varepsilon_{p(k)} \left\| \chi^k + \alpha_{p(k)} Jz^k - \psi^{p(k)} \right\|_*,$$

where, according to (59), $\|\chi^k\|_* \leq \Phi(\|z^k\|)$. Thus, we deduce that

$$(97) \quad \begin{aligned} \|w^k - Jz^k\|_* &\leq \varepsilon_{p(k)} \left(\|\chi^k\|_* + \alpha_{p(k)} \|z^k\| + \|\psi^{p(k)}\|_* \right) \\ &\leq \varepsilon_{p(k)} \left(\Phi(\|z^k\|) + \alpha_{p(k)} \|z^k\| + M_2 \right) \leq \varepsilon_{p(k)} \mu_k, \end{aligned}$$

where μ_k is given by (86). This and (95) imply

$$(98) \quad U_k \leq 2Lr_k^2 \rho_{X^*} \left(4r_k^{-1} \mu_k \varepsilon_{p(k)} \right).$$

Now we are going to estimate V_k . Observe that, since $x^{p(k)}$ is a solution of (4) with $p(k)$ instead of k , there exists $\phi^{p(k)} \in A_{p(k)} x^{p(k)}$ such that

$$(99) \quad \left\langle \phi^{p(k)} + \alpha_{p(k)} Jx^{p(k)} - \psi^{p(k)}, y - x^{p(k)} \right\rangle \geq 0, \quad \forall y \in \Omega_{p(k)}.$$

According to (86) and (99), we have

$$(100) \quad \begin{aligned} V_k &= -\varepsilon_{p(k)} \left\langle \chi^k + \alpha_{p(k)} Jz^k - \psi^{p(k)}, z^k - x^{p(k)} \right\rangle \\ &= -\varepsilon_{p(k)} \left[\left\langle \chi^k - \phi^{p(k)}, z^k - x^{p(k)} \right\rangle \right. \\ &\quad \left. + \alpha_{p(k)} \left\langle Jz^k - Jx^{p(k)}, z^k - x^{p(k)} \right\rangle \right. \\ &\quad \left. + \left\langle \phi^{p(k)} + \alpha_{p(k)} Jx^{p(k)} - \psi^{p(k)}, z^k - x^{p(k)} \right\rangle \right] \\ &\leq -\varepsilon_{p(k)} \left[\left\langle \chi^k - \phi^{p(k)}, z^k - x^{p(k)} \right\rangle \right. \\ &\quad \left. + \alpha_{p(k)} \left\langle Jz^k - Jx^{p(k)}, z^k - x^{p(k)} \right\rangle \right]. \end{aligned}$$

The operator $A_{p(k)}$ is strongly $h_{p(k)}$ -hypomonotone on $\Omega_{p(k)}$, and this implies

$$\left\langle \chi^k - \phi^{p(k)}, z^k - x^{p(k)} \right\rangle \geq -h_{p(k)} \left(\|z^k\| - \|x^{p(k)}\| \right)^2.$$

Using (91) and the convexity of $W(Jz^k, \cdot)$, we deduce that

$$W(Jz^k, x^{p(k)}) - W(Jz^k, z^k) \leq 2 \left\langle Jz^k - Jx^{p(k)}, z^k - x^{p(k)} \right\rangle,$$

where $W(Jz^k, z^k) = 0$. Thus, by (100), we get

$$(101) \quad V_k \leq \varepsilon_{p(k)} h_{p(k)} \left(\left\| z^k \right\| - \left\| x^{p(k)} \right\| \right)^2 - \frac{\varepsilon_{p(k)} \alpha_{p(k)}}{2} W(Jz^k, x^{p(k)}).$$

For evaluating Z_k we take into account that

$$\begin{aligned} Z_k &\leq \left\| w^k - Jz^k \right\|_* \left\| x^{p(k)} - x^{p(k+1)} \right\| \\ &\leq \varepsilon_{p(k)} \mu_k \left\| x^{p(k)} - x^{p(k+1)} \right\|, \end{aligned}$$

where the last inequality follows from (97). Consequently,

$$(102) \quad \begin{aligned} Z_k &\leq \varepsilon_{p(k)} \mu_k \left(\left\| x^{p(k)} - u^{p(k)} \right\| + \left\| u^{p(k+1)} - x^{p(k+1)} \right\| \right. \\ &\quad \left. + \left\| u^{p(k)} - u^{p(k+1)} \right\| \right). \end{aligned}$$

Applying Lemma 4.2 successively, first for $p(k)$ instead of k and next for $p(k+1)$ instead of k , we obtain

$$(103) \quad \left\| x^{p(k)} - u^{p(k)} \right\| + \left\| u^{p(k+1)} - x^{p(k+1)} \right\| \leq G_{p(k)} + G_{p(k+1)}.$$

Consider the variational inequalities

$$\langle (A + \alpha_{p(k)} J)x - \psi, y - x \rangle \geq 0, \quad \forall y \in \Omega,$$

and

$$\langle (A + \alpha_{p(k+1)} J)x - \psi, y - x \rangle \geq 0, \quad \forall y \in \Omega,$$

which have the solutions $u^{p(k)}$ and $u^{p(k+1)}$, respectively. These variational inequalities satisfy the requirements of Theorem 3.1 when \bar{a} , \bar{b} , \bar{c} , h , δ , τ and ω are all zero. Therefore, application of Theorem 3.1 in this particular case, leads to the inequality

$$(104) \quad \left\| u^{p(k)} - u^{p(k+1)} \right\| \leq K_0 g_X^\sharp \left(K_1 \frac{|\alpha_{p(k)} - \alpha_{p(k+1)}|}{\alpha_{p(k)}} \right),$$

which, in conjunction with (103), implies

$$(105) \quad \left\| x^{p(k)} - x^{p(k+1)} \right\| \leq G_{p(k)} + G_{p(k+1)} + \bar{G}_{p(k)}.$$

So, from (102) and (105), we get

$$(106) \quad Z_k \leq \varepsilon_{p(k)} \mu_k (G_{p(k)} + G_{p(k+1)} + \bar{G}_{p(k)}),$$

where $\bar{G}_{p(k)}$ is defined by (71). Combining (92), (93), (98), (101) and (106) we deduce that

$$(107) \quad \begin{aligned} W(Jz^{k+1}, x^{p(k+1)}) &\leq W(Jz^k, x^{p(k)}) - \varepsilon_{p(k)} \alpha_{p(k)} W(Jz^k, x^{p(k)}) \\ &\quad + 2 \left\langle Jx^{p(k+1)} - Jz^k, x^{p(k+1)} - x^{p(k)} \right\rangle \\ &\quad + 2 [2Lr_k^2 \rho_{X^*} (4r_k^{-1} \varepsilon_{p(k)} \mu_k)] \end{aligned}$$

$$\begin{aligned}
& + \varepsilon_{p(k)} h_{p(k)} \left(\left\| z^k \right\| - \left\| x^{p(k)} \right\| \right)^2 \\
& + \varepsilon_{p(k)} \mu_k \left(G_{p(k)} + G_{p(k+1)} + \bar{G}_{p(k)} \right)].
\end{aligned}$$

Note that

$$\begin{aligned}
& \left\langle Jx^{p(k+1)} - Jz^k, x^{p(k+1)} - x^{p(k)} \right\rangle \\
& \leq \left\| Jx^{p(k+1)} - Jz^k \right\|_* \left\| x^{p(k+1)} - x^{p(k)} \right\| \\
& \leq \left(\left\| Jx^{p(k+1)} \right\|_* + \left\| Jz^k \right\|_* \right) \left\| x^{p(k+1)} - x^{p(k)} \right\| \\
& = \left(\left\| x^{p(k+1)} \right\| + \left\| z^k \right\| \right) \left\| x^{p(k+1)} - x^{p(k)} \right\| \\
& \leq \left(M + \left\| z^k \right\| \right) \left\| x^{p(k+1)} - x^{p(k)} \right\| \\
& \leq \left(M + \left\| z^k \right\| \right) \left(G_{p(k)} + G_{p(k+1)} + \bar{G}_{p(k)} \right),
\end{aligned}$$

where the last two inequalities result from (61) and (105), respectively. This and (107) imply

$$\begin{aligned}
(108) \quad & W(Jz^{k+1}, x^{p(k+1)}) \\
& \leq W(Jz^k, x^{p(k)}) - \varepsilon_{p(k)} \alpha_{p(k)} W(Jz^k, x^{p(k)}) \\
& + 2 \left[2Lr_k^2 \rho_{X^*} \left(4r_k^{-1} \mu_k \varepsilon_{p(k)} \right) \right. \\
& + \varepsilon_{p(k)} h_{p(k)} \left(\left\| z^k \right\| - \left\| x^{p(k)} \right\| \right)^2 \\
& \left. + \left(\varepsilon_{p(k)} \mu_k + M + \left\| z^k \right\| \right) \left(G_{p(k)} + G_{p(k+1)} + \bar{G}_{p(k)} \right) \right].
\end{aligned}$$

According to (61) we have that

$$\left(\left\| z^k \right\| - \left\| x^{p(k)} \right\| \right)^2 \leq \left\| z^k \right\|^2 + M^2.$$

Consequently, we deduce that

$$\begin{aligned}
& 2Lr_k^2 \rho_{X^*} \left(4r_k^{-1} \mu_k \varepsilon_{p(k)} \right) + \varepsilon_{p(k)} h_{p(k)} \left(\left\| z^k \right\| - \left\| x^{p(k)} \right\| \right)^2 \\
& + \left(\varepsilon_{p(k)} \mu_k + M + \left\| z^k \right\| \right) \left(G_{p(k)} + G_{p(k+1)} + \bar{G}_{p(k)} \right) \\
& \leq 2Lr_k^2 \rho_{X^*} \left(4r_k^{-1} \mu_k \varepsilon_{p(k)} \right) + \varepsilon_{p(k)} h_{p(k)} \left(\left\| z^k \right\|^2 + M^2 \right) \\
& + \left(\varepsilon_{p(k)} \mu_k + M + \left\| z^k \right\| \right) \left(G_{p(k)} + G_{p(k+1)} + \bar{G}_{p(k)} \right) \\
& = \frac{1}{2} \Upsilon \left(\left\| z^k \right\|, p(k) \right).
\end{aligned}$$

This and (108) implies (85). \square

The following result ensures boundedness of the sequence $\{z^k\}_{k \in \mathbb{N}}$ generated by the algorithm.

Lemma 4.4. *For any $k \in \mathbb{N}$ we have that*

$$\|z^k\| \leq K \quad \text{and} \quad W(Jz^k, x^{p(k)}) \leq R_0,$$

where R_0 and K are defined in Step 0 of the algorithm (see (75)).

Proof. It is sufficient to show that

$$(109) \quad W(Jz^k, x^{p(k)}) \leq R_0, \quad \forall k \in \mathbb{N}.$$

Indeed, if (109) is true, then we have

$$\left(\|z^k\| - \|x^{p(k)}\| \right)^2 \leq W(Jz^k, x^{p(k)}) \leq R_0,$$

where the first inequality follows from the definition of W . This implies

$$\|z^k\| \leq \sqrt{R_0} + \|x^{p(k)}\| \leq \sqrt{R_0} + M = K.$$

We prove (109) by induction upon k . If $k = 0$, then (109) holds because, according to (78), we have that

$$W(Jz^0, x^{p(0)}) \leq \left(\|z^0\| + \|x^{p(0)}\| \right)^2 \leq \left(\sqrt{R_0} - M + M \right)^2 = R_0.$$

Now, assume that (109) holds for some nonnegative integer k . Suppose by contradiction that

$$(110) \quad W(Jz^{k+1}, x^{p(k+1)}) > R_0.$$

Then, according to (77), we deduce that

$$W(Jz^{k+1}, x^{p(k+1)}) > R_0 \geq \Upsilon(K, p(k)) (\alpha_{p(k)} \varepsilon_{p(k)})^{-1}$$

because $p(k) = k + n_0 \geq n_0$. As noted above, the assumption that (109) holds for k implies that $\|z^k\| \leq K$. The function $\Upsilon(\cdot, p(k))$, defined at (76), is nondecreasing. Thus, we have

$$\begin{aligned} W(Jz^{k+1}, x^{p(k+1)}) &\geq \Upsilon(K, p(k)) (\alpha_{p(k)} \varepsilon_{p(k)})^{-1} \\ &\geq \Upsilon(\|z^k\|, p(k)) (\alpha_{p(k)} \varepsilon_{p(k)})^{-1}. \end{aligned}$$

This and Lemma 4.3 imply that

$$\begin{aligned} &(1 - \alpha_{p(k)} \varepsilon_{p(k)}) W(Jz^{k+1}, x^{p(k+1)}) \\ &\leq W(Jz^{k+1}, x^{p(k+1)}) - \Upsilon(\|z^k\|, p(k)) \\ &\leq (1 - \varepsilon_{p(k)} \alpha_{p(k)}) W(Jz^k, x^{p(k)}). \end{aligned}$$

According to (A5), we have that $0 < \varepsilon_{p(k)} \alpha_{p(k)} < 1$. Hence, we obtain that

$$W(Jz^{k+1}, x^{p(k+1)}) \leq W(Jz^k, x^{p(k)}) \leq R_0,$$

and this contradicts (110). \square

The following result completes the proof of Theorem 4.1. Recall that, as already shown in Section 2, in the current circumstances, the solution set $\mathcal{S}(A, \psi, \Omega)$ is convex and has a minimal norm element.

Lemma 4.5. *The sequence $\{z^k\}_{k \in \mathbb{N}}$ generated by the algorithm converges strongly to the minimal norm solution of the variational inequality (1).*

Proof. Lemma 4.4 ensures that the sequence of nonnegative real numbers $\{W(Jz^k, x^{p(k)})\}_{k \in \mathbb{N}}$ is bounded. Lemma 4.3 and Lemma 4.4 show that the numbers $\lambda_k = W(Jz^k, x^{p(k)})$ satisfy the inequality

$$\lambda_{k+1} \leq \lambda_k - \alpha_{p(k)} \varepsilon_{p(k)} \lambda_k + \Upsilon(K, p(k)),$$

where, according to (A5) and Lemma 4.1, we have

$$\lim_{k \rightarrow \infty} \alpha_{p(k)} \varepsilon_{p(k)} = 0 = \lim_{k \rightarrow \infty} \Upsilon(K, p(k)) (\alpha_{p(k)} \varepsilon_{p(k)})^{-1}.$$

These allow us to apply Lemma 1 in [3] in order to deduce that

$$(111) \quad \lim_{k \rightarrow \infty} W(Jz^k, x^{p(k)}) = 0.$$

The space X being uniformly convex and smooth, the function $\theta(x) = \|x\|^2$ is uniformly convex on bounded sets and differentiable. According to [24, Proposition 4.2], in these circumstances, the modulus of total convexity of the function $\theta(x)$ on the set E consisting of all the terms of the bounded sequence $\{z^k\}_{k \in \mathbb{N}}$, denoted $\nu_\theta(E, \cdot)$, has $\nu_\theta(E, t) > 0$ when $t > 0$, and also satisfies

$$\nu_\theta \left(E, \left\| z^k - x^{p(k)} \right\| \right) \leq W(Jz^k, x^{p(k)}), \quad \forall k \in \mathbb{N}.$$

This and (111) imply that

$$\lim_{k \rightarrow \infty} \nu_\theta \left(E, \left\| z^k - x^{p(k)} \right\| \right) = 0.$$

This cannot hold unless $\lim_{k \rightarrow \infty} \|z^k - x^{p(k)}\| = 0$ because the function $\nu_\theta(E, \cdot)$ is (strictly) increasing on $[0, \infty)$ – cf. [25, Lemma 2.4]. By Theorem 2.1(ii), the sequence $\{x^{p(k)}\}_{k \in \mathbb{N}}$ converges strongly to the minimal norm solution of the variational inequality (1). Consequently, the sequence $\{z^k\}_{k \in \mathbb{N}}$ has the same limit and the proof is complete. \square

Acknowledgements. The work of Yakov Alber was supported in part by the KAMEA Program of the Israeli Ministry of Absorption. Dan Butnariu gratefully acknowledges the support of the Israel Science Foundation founded by the Israel Academy of Sciences and Humanities (Grant No. 592/00). All three authors wish to thank a referee for his careful reading of the manuscript and for suggestions which helped them improve an earlier version of this work.

REFERENCES

- [1] Alber Y.I.: A continuous regularization of linear operator equations in Hilbert spaces (Russian), *Math. Notes* **9** (1968), 42–54.
- [2] Alber Y.I.: On a solution of nonlinear equations with monotone operators in Banach spaces (Russian), *Sibir. Math. J.* **16** (1975), 1–8.
- [3] Alber Y.I.: Recurrence relations and variational inequalities, *Soviet Math. Dokl.* **27** (1983), 511–517.

- [4] Alber Y.I.: The regularization method for variational inequalities with nonsmooth unbounded operators in Banach space, *Appl. Math. Lett.* **6** (1993), 63-68.
- [5] Alber Y. I.: Metric and generalized projection operators in Banach spaces: properties and applications, in "Theory and Applications of Nonlinear Operators of Accretive and Monotone Type", pp. 15–50, edited by A. G. Kartsatos, Lecture Notes in Pure and Appl. Math., 178, *Marcel Dekker*, New York, 1996.
- [6] Alber Y.I. and Butnariu D.: Convergence of Bregman-Projection Methods for Solving Convex Feasibility Problems in Reflexive Banach Spaces, *J. Optim. Theory Appl.* **92** (1997), 33-61.
- [7] Alber Y., Butnariu D., and Kassay G.: Convergence and stability of a regularization method for maximal monotone inclusions and its applications to convex optimization, in: "Variational Analysis and Applications" edited by F. Giannessi and A. Maugeri, *Kluwer Academic Publishers*, Dordrecht, 2004, to appear. [available at <http://math2/haifa.ac.il/dbutnaru/publications>]
- [8] Alber Y.I., Butnariu D., and Ryazantseva I.: Regularization methods for ill-posed inclusions and variational inequalities with domain perturbations, *J. Nonlin. Convex Anal.* **2** (2001), 53-79.
- [9] Alber Y.I., Butnariu D., and Ryazantseva I.: Regularization of Monotone Variational Inequalities with Mosco Approximations of the Constraint Sets, *Set-Valued Analysis* **13** (2005), to appear. [available at <http://math2.haifa.ac.il/dbutnaru/>]
- [10] Alber Y.I., Kartsatos A.G., and Litsyn E.: Iterative solutions of unstable variational inequalities on approximately given sets, *Abstr. Appl. Anal.* **1** (1996), 45-54.
- [11] Alber Y.I. and Nashed Z.M.: Iterative-projection regularization of ill-posed variational inequalities, *Analysis*, to appear.
- [12] Alber Y.I. and Notik A. I.: Geometric properties of Banach spaces and approximate methods for solving nonlinear operator equations, *Soviet Math. Dokl.* **29** (1984), 611-615.
- [13] Alber Y.I. and Notik A. I.: Perturbed unstable variational inequalities with unbounded operators on approximately given sets, *Set-Valued Anal.* **1** (1993), 393-402.
- [14] Alber Y.I. and Reich S.: An iterative method for solving a class of nonlinear operator equations in Banach spaces, *Panamerican Math. J.* **4** (1994), 39-54.
- [15] Bakushinskii A.B.: On the principle of iterative regularization, *USSR Comput. Math. Math. Phys.* **19** (1979), 256-260.
- [16] Bakushinskii A.B. and Gontscharskii A.V.: Ill-posed Problems - Numerical Methods and Applications (Russian), *Nauka*, Moscow, 1989.
- [17] Bakushinskii A.B. and Poljak B.T.: On the solution of variational inequalities, *Soviet Math. Dokl.* **15** (1974), 1705-1710.
- [18] Bauschke H. and Combettes P.L.: Construction of best Bregman approximations in reflexive Banach spaces, *Proc. Amer. Math. Soc.* **131** (2003), 3757–3766.
- [19] Browder F.E.: Multivalued monotone nonlinear mappings and duality mappings in Banach spaces, *Trans. Amer. Math. Soc.* **118** (1965), 338-351.
- [20] Browder F.E.: Existence and approximation of solutions of nonlinear variational inequalities, *Proc. Nat. Acad. Sci. U.S.A.* **56** (1966), 1080–1086.
- [21] Bruck R.E.: A strongly convergent iterative solution of $0 \in U(x)$ for a maximal monotone operator U in Hilbert space. *J. Math. Anal. Appl.* **48** (1974), 114–126.
- [22] Bruck R.E.: An iterative solution for a variational inequality for certain monotone operators in Hilbert space, *Bull. Amer. Math. Soc.* **81** (1975), 890-892.
- [23] Butnariu D., Iusem A.N., and E. Resmerita E.: Total convexity of the powers of the norm in uniformly convex Banach spaces, *J. Convex Anal.* **7** (2000), 319-334.
- [24] Butnariu D., Iusem A.N., and Zalinescu C.: Uniform convexity, total convexity and the convergence of the outer Bregman projection and proximal point algorithms in Banach spaces, *J. Convex Anal.*, **10** (2003), 35-61.
- [25] Butnariu D. and Resmerita E.: The outer Bregman projection method for stochastic feasibility problems in Banach spaces, in: D. Butnariu, Y. Censor and S. Reich (Eds.): "Inherently Parallel Algorithms in Feasibility and Optimization and their Applications", pp. 69-87, *Elsevier*, Amsterdam, 2001.
- [26] Figiel T.: On the moduli of convexity and smoothness, *Studia Math.* **56** (1976), 121-155.

- [27] Holmes P.R.: Geometric Functional Analysis, *Springer Verlag*, New York, 1975.
- [28] Istratescu V.: Strict Convexity and Complex Strict Convexity: Theory and Applications, *Marcel Dekker*, New York, 1984.
- [29] Kaplan A. and Tichatschke, R.: Stable Methods for Ill-posed Variational Problems: Prox-Regularization of Elliptic Variational Inequalities and Semi-Infinite Problems, *Akademie Verlag*, Berlin, 1994.
- [30] Lattes R. and Lions J.-L.: Méthode de Quasi-réversibilité et Applications, *Dunod*, Paris, 1967.
- [31] Lions J.-L.: Quelques Méthodes de Résolution des Problèmes aux Limites Non Linéaires, *Dunod*, Paris, 1969.
- [32] Liskovets O. A.: Regularization of problems with discontinuous monotone arbitrarily perturbed operators, *Soviet Math. Dokl.* **28** (1983), 324-327.
- [33] Liskovets O. A.: Discrete convergence of elements and operators for ill-posed problems with a monotone operator, *Soviet Math. Dokl.* **31** (1985), 202-206.
- [34] Liskovets O. A.: External approximations for the regularization of monotone variational inequalities, *Soviet Math. Dokl.* **36** (1988), 220-224.
- [35] Liskovets O. A.: Regularized variational inequalities with pseudo-monotone operators on approximately given sets (Russian), *Differential Equations* **11** (1989), 1970-1977.
- [36] Liskovets O. A.: Regularization of ill-posed variational inequalities (Russian), private communication, 1990.
- [37] Mosco U.: Convergence of convex sets and solutions of variational inequalities, *Advances in Math.* **3** (1969), 510-585.
- [38] Mosco U.: Perturbations of variational inequalities, in: "Nonlinear Functional Analysis", Proceedings of the Symposium in Pure Mathematics, 18, Part 1, pp. 182-194, *American Mathematical Society*, Providence, R.I., 1970.
- [39] Pascali D. and Sburlan S.: Nonlinear Mappings of Monotone Type, *Stijthoff&Noordhoff International Publishers*, Alphen aan den Rijn, The Netherlands, 1978.
- [40] Penot J.-P. and Volle M.: Inversion of real-valued functions and applications, *Z. Oper. Res.* **34** (1990), 117-141.
- [41] Polyak B.T.: Minimization of nonsmooth functionals, *USSR Comput. Math. Math. Phys.* **9** (1969), 509-521.
- [42] Reich S.: Extension problems for accretive sets in Banach spaces, *J. Funct. Anal.* **26** (1977), 378-395.
- [43] Reich S.: Strong convergence theorems for resolvents of accretive operators in Banach spaces, *J. Math. Anal. Appl.* **75** (1980), 287-292.
- [44] Rockafellar R.T.: On the maximality of sums of nonlinear monotone operators, *Trans. Amer. Math. Soc.* **149** (1970), 75-88.
- [45] Rockafellar R.T.: On the maximal monotonicity of subdifferential mappings, *Pacific J. Math.* **33** (1970), 209-216.
- [46] Rockafellar R.T. and Wets R. J.-B.: Variational Analysis, *Springer Verlag*, Berlin, Heidelberg, 1998.
- [47] Rockafellar R.T.: Favorable classes of Lipschitz continuous functions in subgradient optimization, in: "Progress in Nondifferentiable Optimization" edited by E. Numinski, pp. 125-143, *IIASA*, Laxenburg, 1982.
- [48] Ryazantseva I.P.: Solution of variational inequalities with monotone operators by the regularization method. *USSR Comput. Math. Math. Phys.* **23** (1983), 145-148.
- [49] Ryazantseva I.P.: Variational inequalities with monotone operators on sets which are specified approximately, *USSR Comput. Math. Math. Phys.* **24** (1984), 194-197.
- [50] Ryazantseva I.P.: Stable methods for solving nonlinear monotone ill-posed problems, *D.Sc. Thesis*, Nizhnii Novgorod, 1996.
- [51] Ryazantseva I.P.: A regularized proximal algorithm for nonlinear equations of monotone type in a Banach space, *Comput. Math. Math. Phys.* **42** (2002), 1247-1255.
- [52] Serb I.: Estimates of the modulus of convexity and smoothness of a Banach space, *Mathematica* **34** (1992), 61-70.

- [53] Tikhonov A.N.: Regularization of incorrectly posed problems, *Soviet Math. Dokl.* **4** (1963), 1035-1038.
- [54] Tikhonov A.N. and Arsenin V.Y.: Solutions of Ill-Posed Problems, *V.H. Winston & Sons*, New York, 1977.
- [55] Vainikko G.M.: The residual principle for a class of regularization methods, *Comput. Math. Math. Phys.* **22** (1982), 1-19.
- [56] Zeidler E.: Nonlinear Functional Analysis and Applications IIB: Nonlinear Monotone Operators, *Springer Verlag*, New York, 1990.
- [57] Xu Zong Ben and Roach G.F.: Characteristic inequalities of uniformly convex and uniformly smooth Banach spaces, *J. Math. Anal. Appl.* **157** (1991), 189-210.

Manuscript received March 17, 2004

revised July 14, 2004

YAKOV ALBER

Faculty of Mathematics, The Technion-Israel Institute of Technology, 32000 Haifa, Israel

E-mail address: `alberya@tx.technion.ac.il`

DAN BUTNARIU

Department of Mathematics, University of Haifa, 31905 Haifa, Israel

E-mail address: `dbutnaru@math.haifa.ac.il`

IRINA RYAZANTSEVA

Department of Applied Mathematics, Nizhnii Novgorod State Technical University, 603155 Nizhnii Novgorod, Russia

E-mail address: `ntu@adm.ntu.sci-nnov.ru`