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TRIANGULAR PLUS-OPERATORS IN BANACH SPACES: APPLICATIONS TO THE KŒNIGS EMBEDDING PROBLEM

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1. Introduction

The present paper is a continuation of [6, 7, 10]. In [6, 7] the conditions of the so-called Kœnigs Embedding Property (KE-property for brevity) were studied for Linear Fractional Transformations (LFT for brevity) with upper triangular matrix $A = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}$, namely,

$$(I) \qquad F_A(K) = A_{22}K(A_{11} + A_{12}K)^{-1}, \quad A_{ij} \in L(\mathfrak{H}_j, \mathfrak{H}_i),\ i, j = 1, 2,$$

where $\mathfrak{H}_1$ and $\mathfrak{H}_2$ are Hilbert spaces, F_A is a self-mapping of the open unit ball $\mathcal{K}$ of the space $L(\mathfrak{H}_1, \mathfrak{H}_2)$ of all linear bounded operators acting between $\mathfrak{H}_1$ and $\mathfrak{H}_2$. Note that if F_A is well-defined, then A is a plus-operator (see Preliminaries below), A_{11} is invertible, and $\left\|{A_{11}}^{-1}A_{12}\right\| \le 1$; if in addition F_A is not a constant, then $\left\|{A_{11}}^{-1}A_{12}\right\| < 1$, see, for example, [1, 5]. Recall that the problem of embedding of a holomorphic self-mapping F of $\mathcal{K}$ into a continuous one-parameter semigroup $\left\{F^t\right\}_{t\ge 0}$ of holomorphic self-mappings such that $F^1 = F$, is called the Kœnigs Embedding Problem (see [6, 7]). If for a mapping F the Kœnigs Embedding Problem is solvable, i.e., F is embeddable, then we say that F has the KE-property.

The results in [6, 7] were obtained by using biholomorphic linear fractional solutions F_T to Schröder's equation

$$(1.1) \qquad F_T \circ F_A = F_{\widetilde{A}} \circ F_T,$$

where $\widetilde{A} = \operatorname{diag} A = \begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} \end{pmatrix}$ (see also [9]).

In [10] more general Abel–Schröder equations were considered for the general case of LFT F_A, when the operators A_{ij} act between Banach spaces X_j and X_i, $i, j = 1, 2$, and $\mathcal{K}$ is the open unit ball of the space $L(X_1, X_2)$.

The main result of [6, 7] ([6, Theorem 4.7] and [7, Theorem 6.5]) establishes the KE-property for F_A of the form (I) in the case when $A_{11} = I$ in $\mathfrak{H}_1$ and A_{22} is uniformly positive operator in $\mathfrak{H}_2$, that is,

$$(I_u) \qquad F_A(K) = A_{22}K(I + A_{12}K)^{-1}.$$

In the present paper for the general case of complex Banach spaces X_i, $i = 1, 2$, we study both LFT's of the form (I) and the dual affine mappings of the form:

$$(II) \qquad F_B(K) = (B_{22}K + B_{21})B_{11}^{-1}$$

with lower triangular matrices $B = \begin{pmatrix} B_{11} & 0 \\ B_{21} & B_{22} \end{pmatrix}$. We consider LFT of the form (I_u) as well as their dual LFT with respect to the main diagonal of the form

$$(I_\ell) \qquad F_A(K) = K(A_{11} + A_{12}K)^{-1},$$

where $A_{22} = I|_{X_2}$ and $\|A_{11}x_1\| \geq \|x_1\|$ for all $x_1 \in X_1$.

On the other hand, in the class of LFT's F_B of the form (II) one can specify two subclasses of mappings which are dual one to another with respect to the second diagonal:

$$(II_u) \qquad F_B(K) = B_{21} + B_{22}K$$

and

$$(II_\ell) \qquad F_B(K) = (B_{21} + K)B_{11}^{-1}$$

with the matrices $B = \begin{pmatrix} I & 0 \\ B_{21} & B_{22} \end{pmatrix}$ and $B = \begin{pmatrix} B_{11} & 0 \\ B_{21} & I \end{pmatrix}$, respectively.

Note that LFT's defined by (I) and (II) are usually called LFT of type (I) and (II), respectively (see, for example, [7]). It seems to be natural to denote LFT's defined in formulae (I_u), (I_ℓ), (II_u) and (II_ℓ), by I_u, I_l, II_u and II_l respectively. Here indexes 'u' and 'ℓ' mean the upper and lower location respectively of the identity operator I in the main diagonal of the operator block matrix.

We proceed the line of the work [10] and study here the diagonality conditions for upper and lower triangular plus-operators A and B, and on this base we obtain new results on the KE-property, which complete and develop the mentioned above results of [6, 7]. Note that the cases $A = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}$ and $B = \begin{pmatrix} B_{11} & 0 \\ B_{21} & B_{22} \end{pmatrix}$, where operators A_{11} and B_{11} are different from the identity, were not considered previously even in the case of Hilbert spaces.

Along with this, we study conditions to the non-diagonal element B_{12} of the matrix B which provide the LFT's of type II_u and II_l have KE-property. Using the Duality Theorem (see Theorem 6.2 below) we pass these results to the case of LFT's of type I_u and I_l. The main results of the paper are Theorems 4.3, 4.4 and Theorems 6.1–6.3.

2. Preliminaries

In this section we give some auxiliary notions and results which are needed in the sequel.

Definition 2.1 (see [2])**.** A normed space X is called uniformly convex, if for each ϵ, $0 < \epsilon \leq 2$, there exists $\delta = \delta(\epsilon) > 0$ such that for all $x, y \in X$ with $\|x\| = \|y\| = 1$ and $\|x - y\| > \epsilon$, the following inequality

$$\|x + y\| \leq 2(1 - \delta)$$

holds.

A normed space X is called uniformly smooth, if for each $\eta > 0$ there exists $\epsilon = \epsilon(\eta) > 0$ such that $\|x - y\| \leq \epsilon$ implies that $(1 + \eta)\|x + y\| \geq \|x\| + \|y\|$.

Theorem 2.1 (see [2]). *Uniformly convex space is reflexive. Uniformly smooth space is reflexive.*

Theorem 2.2 (see [2]). *A space X is uniformly convex if and only if X^* is uniformly smooth.*

Let T be a bounded linear operator in a Banach space X. Suppose that the spectrum $\sigma(T)$ does not separate zero and infinity (consequently, this operator is invertible). Then there are a neighborhood of $\sigma(T)$ and a branch of the function $\log z$ analytic in this neighborhood. It is well known that in this case one can define the operator

$$S := \log T$$

using the Riesz–Dunford integral (see [3]). Furthermore, the operator

$$T^t := e^{tS}$$

is well defined for all $t \in \mathbb{R}^+ = [0, \infty)$.

The following fact follows by [4, Lemma 2.1.1].

Proposition 2.1. *Let $X = \mathfrak{H}$ be a Hilbert space, $T \in L(\mathfrak{H})$ be a bounded linear operator such that $\sigma(T)$ does not separate zero and infinity, and $\|T\| \leq 1$. Then $\Re S \leq 0$ and consequently $\|T^t\| \leq 1$ for all $t \in \mathbb{R}^+$.*

In the general case of a Banach space X this fact is no longer true.

Example. Let $X = \mathbb{C}^2$ be endowed with ℓ_p-norm, $1 \leq p \leq \infty$. Define $A \in L(X)$ by

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Then in the case $p = 2$ the operator A^t is unitary for all $t \in \mathbb{R}$, however in the case $p \neq 2$, i.e., X is not a Hilbert space, $\|A^t\| > 1$ for all non-integer t.

Let now X be an indefinite Banach space [9, 10], that is

$$X = X_1 \dot{+} X_2 \tag{2.2}$$

is a topological decomposition (with bounded projections P_1 and P_2 on X_1 and X_2, respectively) of the space X, and the following two sets are defined:

$$\mathfrak{P} = \{x \in X : \ \|x_1\| \geq \|x_2\|\},$$

where $x = x_1 + x_2, \ x_1 \in X_1, \ x_2 \in X_2, \ x_i = P_i x, \ i = 1, 2$, and

$$\mathfrak{N} = \{y \in X : \ \|y_1\| \leq \|y_2\|\},$$

where once again $y = y_1 + y_2, \ y_i \in X_i, \ i = 1, 2$.

The indefinite structure in the conjugate space X^* is defined by the decomposition

$$X^* = X_1^* \dot{+} X_2^*.$$

Let now $X^{(1)}$ and $X^{(2)}$ be two indefinite spaces, and let $T : X^{(1)} \mapsto X^{(2)}$ be a linear operator. The operator T is called a plus-operator if

$$T\mathfrak{P}^{(1)} \subset \mathfrak{P}^{(2)},$$

and a minus-operator if

$$T\mathfrak{N}^{(1)} \subset \mathfrak{N}^{(2)}.$$

The next assertion follows immediately by definitions.

Proposition 2.2. *If T is one-to-one plus-operator, then T^{-1} is a minus-operator.*

Let $\mathcal{L}$ be a subspace of X, $\mathcal{L} \subset \mathfrak{P}^{(1)}$. We say that $\mathcal{L} \in \mathfrak{M}$ if $P_1\mathcal{L} = X_1$.

Proposition 2.3 ([1]). *$\mathcal{L} \in \mathfrak{M}$ if and only if*

$$\mathcal{L} = \{y : \ y = x_1 + Kx_1, \ x_1 \in X_1\}$$

for some $K = K(\mathcal{L}) \in L(X_1, X_2)$ with $\|K\| \le 1$.

We say that $\mathcal{L} \in \mathfrak{M}^0$ if $\mathcal{L} \in \mathfrak{M}$ and the corresponding operator $K(\mathcal{L})$ is a uniform contraction, i.e., $\|K\| < 1$.

Proposition 2.4 ([8]). *Let T be a bounded plus-operator such that $T\mathcal{L} \in \mathfrak{M}$ for all $\mathcal{L} \in \mathfrak{M}^0$. Then $T^* : \ X^{(2)*} \mapsto X^{(1)*}$ is a plus-operator.*

3. Diagonality conditions for triangular plus-operators

In the sequel we consider bounded plus-operators only.

First let us study the case when the upper element of the main diagonal is an isometry.

Theorem 3.1. *Let $B = \begin{pmatrix} B_{11} & 0 \\ B_{21} & B_{22} \end{pmatrix}$ be a lower triangular plus-operator between indefinite spaces $X^{(1)}$ and $X^{(2)}$ such that $B_{11} : X_1^{(1)} \mapsto X_1^{(2)}$ is an isometry and $\|B_{22}\| = 1$. If $X_2^{(2)}$ is uniformly convex, then $B = \operatorname{diag} B$, i.e., $B_{21} = 0$.*

Proof. Suppose the contrary: there exists $x_1 \in X^{(1)}$, $\|x_1\| = 1$, such that $\|B_{21}x_1\| = \epsilon > 0$. Let $x_2 \in X^{(2)}$, $\|x_2\| = 1$. Since B is a plus-operator and B_{11} is an isometry, we have for $\lambda = \pm 1$

$$\|\lambda B_{21}x_1 + B_{22}x_2\| \le \|B_{11}x_1\| = 1.$$

Because

$$\|(B_{21}x_1 + B_{22}x_2) - (B_{22}x_2 - B_{21}x_1)\| = 2\|B_{21}x_1\| > \epsilon,$$

the uniform convexity of $X_2^{(2)}$ implies that

$$2\|B_{22}x_2\| = \|(B_{21}x_1 + B_{22}x_2) + (B_{22}x_2 - B_{21}x_1)\| \le 2\left(1 - \delta(\epsilon)\right),$$

where $\delta(\epsilon) > 0$.

So we have the inequality

$$\|B_{22}\| = \sup_{\|x_2\|=1} \|B_{22}x_2\| \le 1 - \delta(\epsilon) < 1,$$

which contradicts the assumption of the theorem. □

The following statement is a reformulation of the previous result in terms of minus-operators.

Theorem 3.1′. *Let* $C = \begin{pmatrix} C_{11} & C_{12} \\ 0 & C_{22} \end{pmatrix}$ *be an upper triangular minus-operator between indefinite spaces* $X^{(1)}$ *and* $X^{(2)}$ *such that* $C_{22} : X_2^{(1)} \mapsto X_2^{(2)}$ *is an isometry and* $\|C_{11}\| = 1$. *If* $X_1^{(2)}$ *is uniformly convex, then* $C = \operatorname{diag} C$, *i.e.,* $C_{12} = 0$.

Now we establish an assertion which is dual in a certain sense to Theorem 3.1.

Theorem 3.2. *Let* $A = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}$ *be an upper triangular plus-operator between indefinite spaces* $X^{(1)}$ *and* $X^{(2)}$ *such that* A_{11} *is an isometric bijection of* $X_1^{(1)}$ *onto* $X_1^{(2)}$, $\|A_{22}\| = 1$ *and* $\|A_{12}\| \le 1$. *If* $X_2^{(1)}$ *is uniformly smooth, then* $A = \operatorname{diag} A$, *i.e.,* $A_{12} = 0$.

Proof. Let $B = A^*$ be the conjugate operator to A. Evidently, the operator B is lower triangular. Arguing like in the proof of Theorem 2.1 in [8] we get that B is a plus-operator between $\left(X^{(2)}\right)^*$ and $\left(X^{(1)}\right)^*$, and by Theorem 2.2 the subspace $\left(X_2^{(1)}\right)^*$ of the space $\left(X^{(1)}\right)^*$ is uniformly convex. Then the conclusion follows from Theorem 3.1, since B satisfies all its conditions. □

Now we study the case when an isometric operator is in the lower right corner of the block-matrix of a plus-operator. In a certain sense this consideration is dual to the case considered in Theorems 3.1 and 3.2.

Theorem 3.3. *Let* D *be a plus-operator between indefinite spaces* $X^{(1)}$ *and* $X^{(2)}$ *such that* $D_{11} : X_1^{(1)} \mapsto X_1^{(2)}$ *is a bijection with* $\left\|(D_{11})^{-1}\right\| = 1$, *and* D_{22} *is an isometric bijection of* $X_2^{(1)}$ *onto* $X_2^{(2)}$. *Then* $D = \operatorname{diag} D$ *in each of the following two cases:*

(i) D *is upper triangular and* $X_1^{(1)}$ *is uniformly convex;*

(ii) D *is lower triangular and* $X_1^{(2)}$ *is uniformly smooth.*

Proof. First suppose that D is upper triangular. Under the conditions of the theorem the operator D is invertible. By Proposition 2.1 its inverse

$$C := D^{-1} = \begin{pmatrix} (D_{11})^{-1} & -(D_{11})^{-1} D_{12} (D_{22})^{-1} \\ 0 & (D_{22})^{-1} \end{pmatrix}$$

is a minus-operator between the indefinite spaces $X^{(2)}$ and $X^{(1)}$. Moreover, $C_{22} = D_{22}^{-1}$ is an isometry which acts from $X_2^{(2)}$ to $X_2^{(1)}$, and

$$\|C_{11}\| = \left\|D_{11}^{-1}\right\| = 1.$$

Therefore, in the first case the conclusion follows by Theorem 3.1′. Indeed, $C_{12} = 0$ implies that $D_{12} = -D_{11}C_{12}D_{22} = 0$.

Let now D be lower triangular. Like in the proof of Theorem 3.2 we see that the conjugate operator $D^* : \left(X^{(2)}\right)^* \mapsto \left(X^{(1)}\right)^*$ is a plus-operator. Since $X_1^{(2)}$ is uniformly smooth, by Theorem 2.2 the space $\left(X^1\right)^*$ is uniformly convex. Thus the conclusion for the second case follows by the first part of the proof. □

Remark. Example 4 in [10] shows that the conditions of uniform convexity and uniform smoothness in above Theorems can not be omitted.

4. Diagonality conditions for triangular plus-operator acting in an indefinite space

In this section we consider the case of a plus-operator acting in an indefinite space X, that is $X = X^{(1)} = X^{(2)}$.

Theorem 4.1. *Let D be a triangular plus-operator in X such that D_{11} is an isometry and $\|D_{22}\| = 1$. Then $D = \operatorname{diag} D$ in each of the following two cases:*

(i) *D is lower triangular and X_2 is uniformly convex;*
(ii) *D is upper triangular, $\Im D_{12} \subset \Im D_{11}$, and X_2 is uniformly smooth.*

Proof. In the first case the conclusion follows by Theorem 3.1.

Now suppose that D is an upper triangular plus-operator. Since $\|D_{22}\| = 1$, it follows by [10, Proposition 3] that $\|D_{12}\| \leq 1$.

Set $X^{(1)} = X$ and $X^{(2)} = \Im D_{11} \dot{+} X_2$. Then $D : X^{(1)} \mapsto X^{(2)}$, and the result follows by Theorem 3.2. □

Remark. Note that the second part of Theorem 4.1 generalizes the part b) of Theorem 10 in [10].

The dual result is presented by

Theorem 4.2. *Let D be a triangular plus-operator in X such that D_{22} is an isometry, D_{11} is a bijection of X_1 onto X_1 and $\left\|(D_{11})^{-1}\right\| = 1$. Then $D = \operatorname{diag} D$ in each of the following two cases:*

(i) *D is upper triangular and X_1 is uniformly convex;*
(ii) *D is lower triangular, $\Im D_{21} \subset \Im D_{22}$, and X_1 is uniformly smooth.*

Proof. In the first case the conclusion follows by Theorem 3.3.

Let D be a lower triangular plus-operator. Setting $X^{(1)} = X$ and $X^{(2)} = X_1 \dot{+} \Im D_{22}$, we see that $D : X^{(1)} \mapsto X^{(2)}$. Then the result follows by Theorem 3.3. □

The following examples show that the conditions $\Im D_{12} \subset \Im D_{11}$ and $\Im D_{21} \subset \Im D_{22}$ in Theorems 4.1 and 4.2 can not be omitted.

Example 4.1. Let $\mathfrak{H} = \mathfrak{H}_1 \oplus \mathfrak{H}_2$ be a separable Krein space, where $\mathfrak{H}_1 = \overline{\operatorname{Span}\{e_1^n\}_{n=1}^{\infty}}$ with an orthonormal system $\{e_1^n\}_{n=1}^{\infty}$, and $\mathfrak{H}_2$ is one-dimensional, $\mathfrak{H}_2 = \operatorname{Span}\{e_2\}$, $\|e_2\| = 1$. Define $A \in L(\mathfrak{H})$ by

$$A_{11} e_1^n = e_1^{n+1},\ n \in \mathbb{N},\ A_{12} e_2 = e_1^1,\ A_{21} = 0,\ A_{22} = I.$$

Then A is a plus-operator (since $\|P_1 A x\| = \|x\| \geq \|x_2\| = \|P_2 A x\|$) and $\|A_{11} x_1\| = \|x_1\|$ for all $x_1 \in \mathfrak{H}_1$. At the same time, $\Im A_{12} \not\subset \Im A_{11}$, that is, Theorem 4.1 is not applicable.

Example 4.2. Let $\mathfrak{H} = \mathfrak{H}_1 \oplus \mathfrak{H}_2$ be once again a separable Krein space, where $\mathfrak{H}_1$ is two-dimensional, $\mathfrak{H}_1 = \operatorname{Span}\left\{e_1^1, e_1^2\right\}$, $(e_1^i, e_1^j) = \delta_{ij}$, and $\mathfrak{H}_2 = \overline{\operatorname{Span}\{e_2^n\}_{n=1}^{\infty}}$ with an orthonormal system $\{e_2^n\}_{n=1}^{\infty}$. Define $B \in L(\mathfrak{H})$ as follows

$$B_{11} = \begin{pmatrix} \mu & 0 \\ 0 & 1 \end{pmatrix},\ |\mu| \geq \sqrt{2},\ B_{12} = 0,$$

$$B_{21}e_1^1 = e_2^1,\ B_{21}e_1^2 = 0,\ B_{22}e_2^n = e_2^{n+1},\ n \in \mathbb{N}.$$

Then B is a plus-operator since for each $x = (x_1^1, x_1^2, x_2)$ with

$$\|x_1\| \left(= \sqrt{|x_1^1|^2 + |x_1^2|^2}\right) \geq \|x_2\|$$

we have

$$\|P_2Bx\|^2 = |x_1^1|^2 + \|x_2\|^2 \leq (|\mu|^2 - 1)|x_1^1|^2 + \|x_1\|^2 = \|P_1Bx\|^2.$$

So, B is a lower triangular non-diagonal plus-operator. The reason for this is the following: $\Im B_{21} \not\subset \Im B_{22}$.

Further, Theorems 4.1 and 4.2 enable us to establish new conditions for the KE-property for LFT.

In both Theorems 4.3 and 4.4 below we assume that *the spectra* $\sigma(D_{11})$ *and* $\sigma(D_{22})$ *of diagonal entries* D_{11} *and* D_{22}*, respectively, do not separate zero and infinity.* Then D_{11} and D_{22} are bijections of X_1 and X_2, respectively, and, as we noted above, the fractional powers $(D_{11})^t$ and $(D_{22})^t$ are well-defined for all $t \in \mathbb{R}^+$. In light of Example in Section 2 *we also assume that* $\left\|(D_{11})^{-t}\right\| \leq 1$ *and* $\left\|(D_{22})^t\right\| \leq 1$ *for all* $t \in \mathbb{R}^+$.

Theorem 4.3. *Let D be a triangular plus-operator in X such that D_{11} is an isometry on X_1 and $\|D_{22}\| = 1$. Then F_D has the KE-property in each of the following two cases:*

(i) *D is lower triangular and X_2 is uniformly convex;*
(ii) *D is upper triangular and X_2 is uniformly smooth.*

Proof. By Theorem 4.1 the operator D is diagonal, that is, the LFT F_D is linear, $F_D(K) = D_{22}K\,(D_{11})^{-1}$. According to our assumption and the conditions of the theorem $\left\|(D_{22})^t\right\| \leq 1$ and $\left\|(D_{11})^{-t}\right\| \leq 1$ for all $t \in \mathbb{R}^+$. Therefore, $F_{D^t}(K) \in \mathcal{K}$ for all $K \in \mathcal{K}$ and $t \in \mathbb{R}^+$. □

The following dual statement can be proved analogously.

Theorem 4.4. *Let D be a triangular plus-operator in X such that $\left\|(D_{11})^{-1}\right\| = 1$ and D_{22} is an isometry on X_2. Then F_D has the KE-property in each of the following two cases:*

(i) *D is upper triangular and X_1 is uniformly convex;*
(ii) *D is lower triangular and X_1 is uniformly smooth.*

Remark. In the particular case when both X_1 and X_2 are Hilbert spaces, one can omit the inequalities $\left\|(D_{11})^{-t}\right\| \leq 1$ and $\left\|(D_{22})^t\right\| \leq 1$ (see Proposition 2.1). On the other hand, in this case the conditions of the uniform convexity and uniform smoothness are fulfilled automatically.

5. Fixed points of affine mappings and of their dual LFT's

In this Section we establish the KE-property for LFT's of types (II), i.e., affine mappings, and for their dual LFT's of type (I) based on the existence of sufficiently small (with respect to the norm) fixed points to above affine mappings. Thereby we need the reflexivity of the subspace X_2 (there are well known examples of affine mappings of the closed unit ball of a non reflexive Banach space which do not have fixed points). In the previous Sections we imposed on X_2 the conditions of uniform convexity or uniform smoothness. By Theorem 2.1 both these conditions imply the reflexivity of X_2. Moreover, if X_2 is uniformly convex, then it follows by Theorems 4.1 and 4.2 that a LFT F_B of type II_u or of type $II_l$with non diagonal block-matrix B satisfies $\|B_{22}\| < 1$ or $\left\|B_{11}{}^{-1}\right\| < 1$, respectively. Hence F_B is a uniform contraction, that is $\|F_B(K_1) - F_B(K_2)\| \le q\|K_1 - K_2\|$ with $q = \|B_{22}\|$ or $q = \left\|B_{11}^{-1}\right\|$, respectively. Consequently, F_B has a unique fixed point $S_0 \in \overline{\mathcal{K}}$.

Now let us return to the general case of LFT's of type (II). Let X_2 be reflexive. Then $\overline{\mathcal{K}}$ is compact in the weak operator topology [3]. The mapping (II) is evidently continuous in this topology, hence it has a (not necessarily unique) fixed point $S_0 \in \overline{\mathcal{K}}$. Using this fixed point we can rewrite (II) in the following manner:

$$F_B(K) = B_{22}(K - S_0)\,(B_{11})^{-1} + S_0. \tag{5.3}$$

So, the following assertion holds.

Proposition 5.1. *If X_2 is reflexive, then a LFT F_B of type (II) can be rewritten in the form (5.3), where $S_0 \in \overline{\mathcal{K}}$ is a fixed point of F_B satisfying*

$$S_0 B_{11} - B_{22} S_0 = B_{21}.$$

.

Now consider the dual mapping F_A defined by the matrix

$$A := B^* = \begin{pmatrix} (B_{11})^* & (B_{11})^*\, S_0^* - S_0^*\,(B_{22})^* \\ 0 & (B_{22})^* \end{pmatrix}.$$

Theorem 5.1. *Let S_0 be an invertible operator such that $\|S_0\| = \left\|S_0{}^{-1}\right\| = 1$. Then S_0 is a fixed point to an LFT F_B of type (II) if and only if $-\left(S_0^*\right)^{-1}$ is a fixed point to an LFT F_{B^*} of type (I).*

Proof. Let S_0 be a fixed point of an LFT F_B of type (II). By Proposition 5.1

$$B_{21} = S_0 B_{11} - B_{22} S_0.$$

By the assumption of the theorem S_0 is a bijection, i.e., the bounded linear operator $(S_0)^{-1}$ exists. Then we have

$$\left(-S_0^*\right)^{-1}\left[B_{11}^* + (B_{11}^* S_0^* - S_0^* B_{22}^*)\left(-S_0^*\right)^{-1}\right] = B_{22}^*\left(-S_0^*\right)^{-1},$$

i.e., $\left(-S_0^*\right)^{-1}$ is a fixed point of F_A with $A = B^*$.

The second part of the proof can be performed analogously. □

Remark. In the formulation of this theorem we require that $\|S_0\| = \left\|S_0{}^{-1}\right\| = 1$ only to provide the invertibility of the operator $B_{11}^* + B_{21}\left(-S_0^*\right)^{-1}$. One can extend the assertion in some directions. For example, considering LFT defining not only on the unit ball $\mathcal{K}$, but on their natural domains.

Theorem 5.2. *An invertible operator S_0 is a fixed point to a LFT of type (II) with invertible A_{22}, or of type I_l, or of type II_u with invertible B_{22}, or of type II_l if and only if $S_0{}^{-1}$ is a fixed point to $G_{A^{-1}}$ ($G_{B^{-1}}$, respectively).*

Proof. First we deal with a mapping of type II_u with an invertible entry B_{22} and $\|B_{21}\| \le 1$. Thus the matrix $B = \begin{pmatrix} I & 0 \\ B_{21} & B_{22} \end{pmatrix}$ is invertible too, and $C := B^{-1} = \begin{pmatrix} I & 0 \\ -{B_{22}}^{-1}B_{21} & {B_{22}}^{-1} \end{pmatrix}$ is a minus-operator in X by Proposition 2.2.

Recall that in the general case of a minus-operator with invertible C_{22} and $\|{C_{22}}^{-1}C_{21}\| \le 1$, the mapping G_C of the open unit ball of the space $L(X^{(2)}, X^{(1)})$ is defined as follows:

$$G_C(Z) = (C_{12} + C_{11}Z)\,(C_{22} + C_{21}Z)^{-1},$$

where $Z \in L\left(X^{(2)}, X^{(1)}\right)$ with $\|Z\| < 1$. In particular, for the matrix $C = B^{-1}$ defined above we have $\left\|{C_{22}}^{-1}C_{21}\right\| = \|-B_{21}\| \le 1$ and

$$G_C(Z) = Z\left({B_{22}}^{-1} - {B_{22}}^{-1}B_{21}Z\right)^{-1}.$$

Let S_0 be a fixed point to F_B, i.e., $B_{21} + B_{22}S_0 = S_0$. Then

$$-{B_{22}}^{-1}B_{21}{S_0}^{-1} + {B_{22}}^{-1} = {B_{22}}^{-1}\left[-\left(I - B_{22}\right)S_0{S_0}^{-1} + I\right] = {B_{22}}^{-1}B_{22} = I.$$

Consequently, ${S_0}^{-1}\left({B_{22}}^{-1} - {B_{22}}^{-1}B_{21}{S_0}^{-1}\right) = {S_0}^{-1}$, i.e., ${S_0}^{-1}$ is a fixed point to the mapping $G_C(Z) = Z\left(-{B_{22}}^{-1}B_{21}Z + {B_{22}}^{-1}\right)^{-1}$.

It is easy to show that the inverse statement is also true: if S_0 is a fixed point to LFT of type I_u such that both A_{22} and S_0 are invertible operators, then ${S_0}^{-1}$ is a fixed point to $G_{A^{-1}}$.

In the case of LFT F_B of type II_l the matrix $B = \begin{pmatrix} B_{11} & 0 \\ B_{21} & I \end{pmatrix}$ is evidently invertible, and one can perform a consideration analogous to the above one. □

Now we turn to the KE-property for affine mappings. First for given F_T of type (II) we find a general affine solution F_T to the following Schröder equation

$$F_T \circ F_B = F_{\widetilde{B}} \circ F_T, \quad \text{where } \widetilde{B} = \operatorname{diag} B. \tag{5.4}$$

Theorem 5.3. *Let X_2 be reflexive. Let F_B be an LFT of type (II) with an invertible entry B_{22}, and let $S_0 \in \overline{\mathcal{K}}$ be its fixed point. Then for any invertible operators T_{11} and T_{22} commuting with B_{11} and B_{22}, respectively, the affine mapping*

$$F_T(K) := T_{22}(K - S_0){T_{11}}^{-1} \tag{5.5}$$

is a solution to the Schroeder's equation (5.4). Conversely, any affine mapping satisfying (5.4) has the form (5.5), where the invertible operators T_{11} and T_{22} scalar commute with B_{11} and B_{22}, respectively, i.e., there is $\lambda \ne 0$ such that $T_{ii}B_{ii} = \lambda B_{ii}T_{ii}$.

Proof. Sufficiency. Consider the operator $T = \begin{pmatrix} T_{11} & 0 \\ -T_{22}S_0 & T_{22} \end{pmatrix}$, where T_{11}, T_{22} and S_0 satisfy the conditions of the theorem. Consider the corresponding affine mapping F_T, that is, LFT of type (II). According to the "chain rule", to prove the equality (5.4), it is sufficient to show that

$$TB = \widetilde{B}T. \tag{5.6}$$

The latter equality can be checked directly by using Proposition 5.1.

Necessity. By [11, Theorem 3.1] it follows by (5.4) that there exists $\lambda \in \mathbb{C}$ such that

$$TB = \lambda \widetilde{B} T. \tag{5.7}$$

The latter means that the operators T_{11} and T_{22} scalar commute with B_{11} and B_{22}, respectively. The proof is complete. □

6. The Duality Theorem and the KE-property

In this Section as in Theorems 4.3 and 4.4 we assume that the spectrum $\sigma(\widetilde{B})$ of the operator $\widetilde{B} = \operatorname{diag} B$ does not separate zero and infinity. It follows by Theorem 5.3 that the family $\{F^t\}_{t\geq 0}$, where

$$F^t = F_T^{-1} \circ F_{\widetilde{B}^t} \circ F_T,$$

is a semigroup of affine mappings acting on the whole space $L(X_1, X_2)$ which is a solution to the KE-problem in this space. At the same time, these affine mappings can be not self-mappings of the unit ball $\mathcal{K}$.

Example. Let $F(z) = \frac{i}{4}\left(z - \frac{1}{\sqrt{2}}\right) + \frac{1}{\sqrt{2}}$ be an affine mapping of the complex plane $\mathbb{C}^1$ related to the operator $B = \begin{pmatrix} 1 & 0 \\ \frac{4-i}{4\sqrt{2}} & \frac{i}{4} \end{pmatrix}$. There are infinitely many semigroups $\{F_k^t\}_{t\geq 0}$, $k \in \mathbb{Z}$, acting on $\mathbb{C}^1$ such that F is embedded into any one of them. Namely,

$$F_k^t(z) = \frac{e^{it\left(\frac{\pi}{2}+2\pi k\right)}}{4^t}\left(z - \frac{1}{\sqrt{2}}\right) + \frac{1}{\sqrt{2}}.$$

At the same time, it is easy to check that the given function F is a self-mapping of the open unit disk Δ, but no semigroup $\{F_k^t\}_{t\geq 0}$ is a semigroup of self-mappings of Δ. For instance, one can check that for $t = 0.5$ no function

$$F_k^{0.5}(z) = \frac{e^{i\left(\frac{\pi}{4}+\pi k\right)}}{2}\left(z - \frac{1}{\sqrt{2}}\right) + \frac{1}{\sqrt{2}}, \quad k \in \mathbb{Z},$$

maps Δ into Δ, that is $F_k^{0.5}$ is not a self-mapping of Δ.

Below in Theorems 6.1 and 6.3 we impose some additional restrictions on the block-matrices A and B providing LFT's F_A of type (I) and F_B of type (II) to have the KE-property.

In the sequel we use the following notation:

$$\gamma := \inf_{t\geq 0} \frac{1 - \|B_{22}^t\|}{\|I - B_{22}^t\|} \tag{6.8}$$

and

$$\delta := \inf_{t\geq 0} \frac{1 - \left\|B_{11}{}^{-t}\right\|}{\left\|I - B_{11}{}^{-t}\right\|}. \tag{6.9}$$

Theorem 6.1. *Let $B_{11} \in L(X_1)$ and $B_{22} \in L(X_2)$. Assume that both their spectra $\sigma(B_{11})$ and $\sigma(B_{22})$ do not separate zero and infinity, and $\|B_{22}^t\| \le 1$ (or $\|{B_{11}}^{-t}\| \le 1$) for all $t \in \mathbb{R}^+$. Then*

a) the block-matrix

$$B = \begin{pmatrix} I & 0 \\ (I - B_{22})S_0 & B_{22} \end{pmatrix}$$

of type II_u (or

$$B = \begin{pmatrix} B_{11} & 0 \\ S_0(B_{11} - I) & I \end{pmatrix}$$

of type II_l) is a plus-operator for all S_0 with $\|S_0\| \le \gamma$ (or $\|S_0\| \le \delta$, respectively);

b) the corresponding LFT F_B of type (II) has the KE-property.

Moreover, in this case the point S_0 is a common fixed point for the semigroup $\{F^t\}_{t\in\mathbb{R}^+}$ of self-mappings of the ball $\mathcal{K}$ corresponding to the semigroup of linear operators B^t.

Proof. We will prove this theorem for the case of LFT's of type II_u. To prove the first assertion we have to show that for any positive vector $x = (x_1, x_2) \in X_1 \oplus X_2$, $x_i \in X_i$, the vector Bx is also positive, i.e., the inequality $\|x_1\| \ge \|x_2\|$ implies $\|x_1\| \ge \|(I - B_{22})S_0x_1 + B_{22}x_2\|$. Indeed, by (6.8) we have $\|S_0\| \le \frac{1-\|B_{22}\|}{\|I-B_{22}\|}$ and consequently

$$\begin{aligned}\|(I - B_{22})S_0x_1 + B_{22}x_2\| &\le \|(I - B_{22})S_0x_1\| + \|B_{22}x_2\| \\ &\le (1 - \|B_{22}\|)\|x_1\| + \|B_{22}\|\|x_2\| \le \|x_1\|.\end{aligned}$$

Further, to prove the second assertion we note that in our assumption one can calculate

$${B_u}^t = \begin{pmatrix} I & 0 \\ (I - B_{22})S_0 & B_{22} \end{pmatrix}^t = \begin{pmatrix} I & 0 \\ (I - B_{22}^t)S_0 & B_{22}^t \end{pmatrix}.$$

Thus the LFM $F_{{B_u}^t}$ generated by ${B_u}^t$ is actually affine and has the form

$$F_{{B_u}^t}(K) = (I - B_{22}^t)S_0 + B_{22}^t K.$$

It is clear: $F_{{B_u}^t}(S_0) = S_0$. Moreover, $F_{{B_u}^t} \circ F_{{B_u}^s} = F_{{B_u}^{t+s}}$. To complete the proof we just estimate $F_{B^t}(K)$ for any $K \in \mathcal{K}$:

$$\|F_{{B_u}^t}(K)\| \le \|(I - B_{22}^t)S_0\| + \|B_{22}^t K\| \le (1 - \|B_{22}^t\|) + \|B_{22}^t\| \cdot \|K\| < 1.$$

So, F_{B_u} is embeddable into $\{F_{{B_u}^t}\}$.

The case of LFT's of type II_l can be considered analogously. □

The following assertion follows immediately from Theorem 6.1 and, in fact, gives examples for explicit estimation of γ and δ.

Corollary 6.1. *Let $X = \mathfrak{H}$ be a Hilbert space, and let $B_{22} \in L(\mathfrak{H}_2)$ be such that*

$$cI \le B_{22} \le dI \tag{6.10}$$

(or let $B_{11} \in L(\mathfrak{H}_1)$ be invertible with $cI \le B_{11}^{-1} \le dI$), where $0 < c \le d < 1$. Suppose that $S_0 \in L(\mathfrak{H}_1, \mathfrak{H}_2)$ satisfies $\|S_0\| \le \dfrac{\ln d}{\ln c}$. Then $\sigma(B_{22})$ (or $\sigma(B_{11})$) does

not separate zero and infinity and the LFT F_B of type (II) with

$$B = \begin{pmatrix} I & 0 \\ (I - B_{22})S_0 & B_{22} \end{pmatrix} \quad \left(or\ B = \begin{pmatrix} B_{11} & 0 \\ S_0(I - B_{11}^{-1}) & I \end{pmatrix} \right)$$

has the KE-property.

Proof. We just calculate:

$$\gamma = \inf_{t\geq 0} \frac{1 - \|B_{22}^t\|}{\|I - B_{22}^t\|} \geq \inf_{t\geq 0} \frac{1 - d^t}{1 - c^t} = \frac{\ln d}{\ln c}.$$

By the same way one can estimate δ. □

To throw over these results to LFT's of type (I) we need the following duality theorem.

Theorem 6.2. *Let F_D be a LFT with a triangular block-matrix D such that D_{22} is a bijection of X_2. Then the following two assertions are equivalent:*

(i) *F_D has the KE-property;*
(ii) *F_{D^*} has the KE-property.*

Proof. Let $D = A$ be upper triangular, that is $A_{21} = 0$. In terms of the block-matrix A the KE-property for F_A means that A^t is a plus-operator for all $t \in \mathbb{R}^+$. By the definition of A^t (see Section 2) it follows that $(A^*)^t = \left(A^t\right)^*$. So, it is enough to prove that $\left(A^t\right)^*$ is a plus-operator on X^* for all $t \in \mathbb{R}^+$. Since A_{22} is a bijection of X_2, then ${A_{22}}^t = \left(A^t\right)_{22}$ is also a bijection of X_2 (see Section 2). Hence by Proposition 3 [10] it follows that $\left\|\left(A^t\right)_{12}\right\| \leq 1$, and by Theorem 2.1 [8] we obtain that $\left(A^t\right)^*$ is a plus-operator.

Now suppose that $D = B$ is a lower triangular plus-operator, that is, $B_{12} = 0$. Arguing as above we conclude that $\left(B^t\right)^*$ is a plus-operator on X^*. This completes the proof. □

As a result of application of Theorem 6.2 (the duality theorem), we obtain the following

Theorem 6.3. *Let $A_{11} \in L(X_1)$ and $A_{22} \in L(X_2)$. Suppose that both their spectra $\sigma(A_{11})$ and $\sigma(A_{22})$ do not separate zero and infinity, and $\|A_{22}^t\| \leq 1$ (or $\left\|{A_{11}}^{-t}\right\| \leq 1$) for all $t \in \mathbb{R}^+$. Denote*

$$\widetilde{\gamma} := \inf_{t\geq 0} \frac{1 - \|A_{22}^t\|}{\|I - A_{22}^t\|}$$

and

$$\widetilde{\delta} := \inf_{t\geq 0} \frac{1 - \left\|{A_{11}}^{-t}\right\|}{\left\|I - {A_{11}}^{-t}\right\|}.$$

Then

a) the block-matrix

$$A = \begin{pmatrix} I & {S_0}^*(I - A_{22}) \\ 0 & A_{22} \end{pmatrix}$$

of type I_u (or

$$A = \begin{pmatrix} A_{11} & (I - A_{11}^{-1}){S_0}^* \\ 0 & I \end{pmatrix}$$

of type I_l) is a plus-operator for all S_0 with $\|S_0\| \le \widetilde{\gamma}$ (or $\|S_0\| \le \widetilde{\delta}$, respectively);
b) the corresponding LFT F_A of type (I) has the KE-property.

Remark. In the case when $X = \mathfrak{H}$ is a Hilbert space using Theorem 6.2 we get the following assertion which is dual to Corollary 6.1:

Let $A_{22} \in L(\mathfrak{H}_2)$ be such that $cI \le A_{22} \le dI$ (or let $A_{11} \in L(\mathfrak{H}_1)$ be invertible with $cI \le A_{11}^{-1} \le dI$), where $0 < c \le d < 1$. Suppose that $S_0 \in L(\mathfrak{H}_1, \mathfrak{H}_2)$ satisfies $\|S_0\| \le \dfrac{\ln d}{\ln c}$. Then $\sigma(A_{22})$ (or $\sigma(A_{11})$) does not separate zero and infinity and the LFT F_A of type (I) with

$$A = \begin{pmatrix} I & S_0^*(I - A_{22}) \\ 0 & A_{22} \end{pmatrix} \quad \left(or\ A = \begin{pmatrix} A_{11} & (I - A_{11}^{-1})S_0^* \\ 0 & I \end{pmatrix} \right)$$

has the KE-property.

Note that in the case when A is of type I_u this assertion is slightly weaker then Theorem 4.7 [6] or Theorem 6.5 [7].

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