Journal of Nonlinear and Convex Analysis
Volume 6, Number 1, 2005, 165–171

Yokohama Publishers
ISSN 1880-5221 ONLINE JOURNAL

ON A FAMILY OF CONVEX FUNCTIONS ASSOCIATED TO SUBDIFFERENTIALS

R. S. BURACHIK* AND S. FITZPATRICK

ABSTRACT. Associated to every maximal monotone operator T from X to X^*, there is a family of convex functions defined on $X \times X^*$. It is known that, when $T = \partial f$, the element $h(x,v) = f(x) + f^*(v)$ belongs to this family. We further analyze the structure of this family when $T = J = (1/2)\partial\|\cdot\|^2$, the normalized duality mapping of X. We study a particular element of this family, when X is an arbitrary Banach space, and use this element in order to compute explicitly the smallest element of this family when $X = c_0$ and $X = \ell_1$. Also, we prove that, when the maximal monotone operator is the subdifferential of a sublinear function, the associated family of convex functions has only one element.

1. Introduction and Motivation

Let X be a Banach space with dual X^*. Given $f : X \to \mathbb{R} \cup \{+\infty\}$ a convex, proper and lower semicontinuous function, it is well-known [6] that the subdifferential of f is a maximal monotone operator. On the other hand, maximal monotone operators can be represented by convex functions in $X \times X^*$. This is a result due to Fitzpatrick [3], recently rediscovered in [4, 1]. In his work, Fitzpatrick defines for a maximal monotone operator $T : X \rightrightarrows X^*$, the family $\mathcal{F}(T)$ as

$$(1) \qquad \mathcal{F}(T) := \left\{ h : X \times X^* \to \overline{\mathbb{R}} \;\middle|\; \begin{array}{cc} & h \text{ convex, closed;} \\ \forall (x,v) \in X \times X^*, & h(x,v) \geq \langle x, v\rangle, \\ (x,v) \in G(T) \Rightarrow & h(x,v) = \langle x, v\rangle. \end{array} \right\},$$

where $\overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$. In [3], it is proved that the function $\Phi_T : X \times X^* \to \overline{\mathbb{R}}$ defined as

$$\Phi_T(x,v) := \sup\{\langle x - y, u - v\rangle \mid (y,u) \in G(T)\} + \langle x, v\rangle,$$

is the smallest element of $\mathcal{F}(T)$. Our aim is to study the properties of this family when T is the subdifferential of a convex function f and X is a non-reflexive Banach space. It is easy to check that $h(x,v) := f(x) + f^*(v)$ is a member of $\mathcal{F}(\partial f)$. Therefore, this family has an element which is *separable*, i.e, which can be expressed as the sum of a function on the variable x plus a function on the variable v. Reciprocally, it holds [1] that, if the family $\mathcal{F}(T)$ has a separable element $h(x,v) = f(x) + g(v)$, then $T = \partial f$ and $g = f^*$. The element h is in general easier to compute than the Fitzpatrick function. In the particular case in which X is

2000 *Mathematics Subject Classification.* 46N10, 47H05.

Key words and phrases. Convex functions, subdifferentials of convex functions, duality mappings in Banach spaces, maximal monotone operators.

*Research of this author was funded by a post-doctoral fellowship CAPES-BEX664-02/2 and by the School of Mathematics of the University of Western Australia. It was partially completed during her visit to the School of Mathematics of the University of Western Australia.

reflexive, it has been recently noted [5, Example 3] that, when f is sublinear, $\mathcal{F}(\partial f)$ consists of h only. In the first part of this work we prove that this property can be extended to non-reflexive Banach spaces. The second part of this work studies the family $\mathcal{F}((1/2)\partial\|\cdot\|^2)$, i.e., when T is the normalized duality mapping associated to X. We study some elements of this family for an arbitrary Banach space X, and specifically compute the smallest element of the family, when $X = \ell_1$ and $X = c_0$.

The organization of the paper is as follows. In the first section we recall some basic definitions and facts, which are to be used in further sections. The second section proves that for a sublinear function f the family $\mathcal{F}(\partial f)$ has only one element, and hence the smallest function $\Phi_{\partial f}$ is separable. Our last section studies the family $\mathcal{F}(J)$, for the normalized duality mapping J. We identify a particular element in this family, and use this result in order to compute Φ_J for $X = \ell_1$ and $X = c_0$.

2. Preliminaries

The continuous dual space of X will be denoted by X^*. Denote by $\langle\cdot,\cdot\rangle$ the duality product and by $\|\cdot\|$ the norm both in X and X^*. Denote by J the Gâteaux gradient of the function $g(x) = (1/2)\|x\|^2$. Thus, $J : X \rightrightarrows X^*$ is the *normalized duality mapping*, defined as

$$J(x) := \{x^* \in X^* \mid \langle x, x^*\rangle = \|x\|^2 = \|x^*\|^2\}. \tag{2}$$

Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semi-continuous convex function. Then, the **conjugate** function of f, $f^* : X' \to \mathbb{R} \cup \{+\infty\}$, is defined by

$$f^*(v) = \sup\{\langle v, x\rangle - f(x) \mid x \in \text{ dom } f\},$$

where the domain of f, dom f, is given by

$$\text{dom } f = \{x \in X \mid f(x) < +\infty\}.$$

Given a maximal monotone operator $T : X \rightrightarrows X^*$, the *graph* of T is the set

$$G(T) = \{(x, v) \in X \times X^* \mid v \in T(x)\}.$$

Associated to every maximal monotone operator T, Fitzpatrick [3] defined a convex lower-semicontinuous function, which we call the *Fitzpatrick function* associated to T, $\Phi_T : X \times X^* \to \mathbb{R} \cup \{+\infty\}$. This function is defined as

$$\Phi_T(x, x^*) := \sup_{(y,y^*)\in G(T)} \langle y - x, x^* - y^*\rangle + \langle x, x^*\rangle = \sup_{(y,y^*)\in G(T)} \langle y, x^* - y^*\rangle + \langle x, y^*\rangle. \tag{3}$$

We recall [3, 1] that the family $\mathcal{F}(T)$ given in (1) has a biggest and a smallest element, the latter being the Fitzpatrick function. The biggest element, is given by $\operatorname{cl}\operatorname{conv}(\pi + \delta_{G(T)})$, where $\pi(x, v) := \langle x, v\rangle$ and δ_C represents the indicator function of the set C. In the case $T = \partial f$, for f proper, convex and lower semicontinuous, the function $h(x, x^*) := f(x) + f^*(x^*)$ belongs to $\mathcal{F}(\partial f)$.

Definition 2.1. We say that $\psi : X \to \mathbb{R} \cup \{+\infty\}$ is a *sublinear* function when it is convex and positively homogeneous (i.e., $\psi(\lambda x) = \lambda\psi(x)$, $\forall x \in X, \forall\lambda \geq 0$).

Recall that for a proper lower semi-continuous sublinear ψ it holds that $\psi(0) = 0$, $\partial\psi(0)$ is non-empty and for each $x \in \operatorname{dom}\psi$

$$
\begin{aligned}
&(i)\ \partial\psi(x) = \{v \in \partial\psi(0) \mid \langle v, x\rangle = \psi(x)\} = \partial\psi(\lambda x),\ \text{ for all } \lambda > 0,\\
&(ii)\ \psi^* = \delta_{\partial\psi(0)},\\
&(iii)\ \psi = \sigma_{\partial\psi(0)} := \sup\nolimits_{v\in\partial\psi(0)}\langle v, \cdot\rangle.
\end{aligned} \tag{4}
$$

3. A case in which $\mathcal{F}(\partial f)$ has only one element

Throughout this section we compute $\mathcal{F}(\partial\psi)$, for a sublinear function ψ satisfying the assumptions of the previous section.

Theorem 3.1. *Let X be an arbitrary Banach space and let $\psi : X \to \mathbb{R} \cup \{+\infty\}$ be proper, lower semicontinuous and sublinear. Then the family $\mathcal{F}(\partial\psi)$ has only one element. Hence, $\Phi_{\partial\psi}$ is separable.*

Proof. We pointed out above that the function $h(x, x^*) := \psi(x) + \psi^*(x^*) \in \mathcal{F}(\partial\psi)$. Denoting by $U := \partial\psi(0)$ we get by (4)(ii) that $h(x, v) = \psi(x) + \delta_U(v) \in \mathcal{F}(\partial\psi)$. Hence $\Phi_{\partial\psi} \le h$. For simplicity, call $\Phi := \Phi_{\partial\psi}$ and call η the biggest element of $\mathcal{F}(\partial\psi)$. Since all elements $g \in \mathcal{F}(\partial\psi)$ verify $\Phi \le g \le \eta$, the claim of the theorem will follow from the two steps below.

(Step 1.) $\Phi = h$.

(Step 2.) $h = \eta$.

Proof of Step 1. Note that $\Phi(x, v) \ge \psi(x)$ for all $x \in X$. Indeed, taking $y = 0$ in (3), we obtain

$$\Phi(x, v) \ge \sup_{w\in U}\langle w, x\rangle = \psi(x), \tag{5}$$

where we use (4)(iii). Now suppose $v \notin U$. The definition of U yields the existence of some $z \in X$ for which

$$\psi(z) < \langle v, z\rangle.$$

Using also the rightmost equality in (4)(i) we can write

$$
\begin{aligned}
\Phi(x, v) &\ge \sup_{\lambda>0, u\in\partial\psi(z)} \{\langle u, x\rangle - \lambda\langle u, z\rangle + \lambda\langle v, z\rangle\}\\
&= \sup_{\lambda>0, u\in\partial\psi(z)} \{\langle u, x\rangle + \lambda(\langle v, z\rangle - \psi(z))\}\\
&= \sup_{u\in\partial\psi(z)} \langle u, x\rangle + \sup_{\lambda>0} \{\lambda(\langle v, z\rangle - \psi(z))\}.
\end{aligned}
$$

Since $z \in \operatorname{dom}\psi$, the first supremum on the right-hand side is not $-\infty$, and therefore the assumption on z yields $\Phi(x, v) = +\infty = \psi(x) + \delta_U(v)$ whenever $v \notin U$. On the other hand, we get for $v \in U$, $x \in X$

$$\psi(x) = \psi(x) + \delta_U(v) \ge \Phi(x, v) \ge \psi(x),$$

where we use the fact that $\Phi \le h$ in the first inequality and (5) in the second one. Hence $\psi(x) + \delta_U(v) = \Phi(x, v)$ also in this case. The proof of Step 1 is complete.

Proof of Step 2. By definition of η, $\Phi \le \eta$ and it has been proved in [2] that $\operatorname{Epi}(\eta) = \operatorname{cl}\operatorname{conv}(\operatorname{Epi}(\pi + \delta_{G(\partial\psi)}) =: E_1$, where $\pi(x, v) := \langle x, v\rangle$. Call $E_2 := \operatorname{Epi}(\Phi) = \{(x, v, a) \mid x \in X,\ v \in U,\ \psi(x) \le a\}$. We know that $E_1 \subset E_2$. Let us prove that $E_2 \subset E_1$. Let $(x, v, a) \in E_2$. There are to cases:

- Case 1. $v \in \partial\psi(x)$.
- Case 2. $v \notin \partial\psi(x)$.

In case 1, $(\pi + \delta_{G(\partial\psi)})(x, v) = \langle x, v\rangle = \psi(x) \le a$ and hence $(x, v, a) \in E_1$.

In case 2, fix $w \in \partial\psi(x)$. Define the sequence $\{z_k\}$ as $z_k := (kx, w, ka)$. We claim that $\{z_k\} \subset \operatorname{Epi}(\pi + \delta_{G(\partial\psi)})$. Indeed; since $\partial\psi(x) = \partial\psi(kx)$ we have that $(\pi + \delta_{G(\partial\psi)})(kx, w) = k\psi(x) \le ka$ and hence the claim is true. Fix $z' := (0, v, 0)$, since $v \in U = \partial\psi(0)$, it also holds that $z' \in \operatorname{Epi}(\pi + \delta_{G(\partial\psi)})$. Define $y_k := (1/k)z_k + (1-(1/k))z' = (x, w/k + (1-(1/k))v, a)$. Then $\{y_k\} \subset \operatorname{convEpi}(\pi + \delta_{G(\partial\psi)})$. So $(x, v, a) = \lim_{k\to\infty} y_k \in \operatorname{cl}\operatorname{convEpi}(\pi + \delta_{G(\partial\psi)}) = E_1$. This completes step 2 and hence $\mathcal{F}(\partial\psi)$ has only one element. □

Remark 3.1. In the case in which $f = \|\cdot\|$, the norm of the Banach space X, then the above result yields $\mathcal{F}(\partial f) = \{\|\cdot\| + \delta_{B^*}\}$, where B^* is the unit ball in X^*.

4. The family $\mathcal{F}(J)$

For an arbitrary Banach space X, consider the normalized duality mapping J defined in (2). We study in this section some elements of the family $\mathcal{F}(J)$.

Proposition 4.1. *Let J be the normalized duality mapping on a Banach space X. Then the Fitzpatrick function Φ_J satisfies*

$$\Phi_J(x, x^*) \le (\|x\| + \|x^*\|)^2/4 \le 1/2(\|x\|^2 + \|x^*\|^2).$$

Proof. Since J is the subdifferential of the function $(1/2)\|\cdot\|^2$, we know that h defined as $h(x, x^*) := (1/2)(\|x\|^2 + \|x^*\|^2)$ belongs to $\mathcal{F}(J)$. Define also $f(x, x^*) := (\|x\|+\|x^*\|)^2/4$. The rightmost inequality follows easily from the fact that $h(x, x^*) - f(x, x^*) = (1/4)(\|x\| - \|x^*\|)^2$. The leftmost inequality follows from the fact that $f \in \mathcal{F}(J)$. Indeed, it is continuous and convex and for $x^* \in Jx$ we have $\langle x, x^*\rangle = \|x\|^2 = f(x, x^*)$. For checking the remaining condition in (1), note that for all $x \in E$ and $x^* \in E^*$ we have $\langle x, x^*\rangle \le \|x\|\,\|x^*\| \le f(x, x^*)$. Thus $f \in \mathcal{F}(J)$ and hence $\Phi_J \le f$. □

We do not know of a Banach space where $\Phi_J(x, x^*) = (\|x\| + \|x^*\|)^2/4$ for all $x \in X$ and $x^* \in X^*$ but the following example shows that equality can occur for many x and x^*.

Theorem 4.1. *For the duality mapping $J : c_0 \to c_0^* = \ell_1$ we have*

$$\Phi_J(x, u) = \sup_j \left(\|u\|_1 + |x_j + u_j| - |u_j|\right)^2/4.$$

Proof. Define the function $g : c_0 \times \ell_1 \to \overline{\mathbb{R}}$ as $g(x, u) := \sup_j(\|u\|_1 + |x_j + u_j| - |u_j|)^2/4$. This function is convex and continuous (as it is the supremum of the family of convex functions $\sum_{k\ne j}|u_k| + |x_j + u_j|$). Note that, when $u \in Jx$ then $|x_j + u_j| - |u_j| = |x_j|$. Indeed, this equality holds for j such that $u_j x_j \ge 0$. We claim that there is no j for which the latter inequality doesn't hold. Otherwise, suppose that there is some j for which $u \in Jx$ and $x_j u_j < 0$. Take y as follows

$$y_n := \begin{cases} -x_n & n = j \\ x_n & n \ne j. \end{cases}$$

Then $y \in c_0$. Since $u \in Jx$ we have also that

$$(1/2)\|y\|_\infty^2 \geq (1/2)\|x\|_\infty^2 + \langle u, y - x\rangle.$$

Using that $\|y\|_\infty = \|x\|_\infty$ we get $0 \geq u_j(-2x_j)$, a contradiction. So for all $x \in c_0$ and $u \in Jx$ we have that $g(x,u) = (\|u\|_1 + \|x\|_\infty)^2/4 = \langle x, u\rangle$. Let us prove now that $g \geq \langle \cdot, \cdot \rangle$ on $c_0 \times \ell_1$. Fix $x \in c_0$ and $u \in \ell_1$. Define $J_+ := \{j \in \mathbb{N} \mid u_j x_j \geq 0\}$. Take the element $x_+ \in c_0$ such that $(x_+)_j = \max\{0, x_j\}$. By the previous proposition we have that

$$\begin{aligned}\langle x, u\rangle \leq \langle x_+, u\rangle &\leq (1/4)(\|x_+\|_\infty + \|u\|_1)^2 \\ &\leq (1/4)(\|x\|_\infty + \|u\|_1)^2 = g(x,u).\end{aligned}$$

Thus $g \in \mathcal{F}(J)$ and hence $\Phi_J(x,u) \leq \sup_j \left(\|u\|_1 + |x_j + u_j| - |u_j|\right)^2/4$. Let us prove now that $\Phi_J \geq g$. Fix $x \in c_0$ and $u \in \ell_1$. We will prove that, for arbitrary $\varepsilon > 0$, $\Phi_J(x,u) \geq -\varepsilon + g(x,u)$. Fix an index j, and define the number

$$\alpha_j(x,u) := \left(x_j + u_j + \sum_{k \neq j} |u_k| \ sgn(x_j + u_j)\right)/2.$$

Since $u \in \ell_1$, there exists $k_0 \geq j$ such that $\sum_{k \geq k_0} |u_k| < \varepsilon/|\alpha_j(x,u)|$. Take now $y \in c_0$ in the following way.

$$y_k := \begin{cases} \alpha_j(x,u) & k = j \\ \alpha_j(x,u) sgn(x_j + u_j) sgn(u_k) & k < k_0,\, k \neq j \\ 0 & k \geq k_0. \end{cases}$$

Note that $\|y\|_\infty = |y_j| = |\alpha_j(x,u)|$ and

$$\left| \sum_{k \neq j} u_k y_k - y_j sgn(x_j + u_j) \sum_{k \neq j} |u_k| \right| = |y_j| \sum_{k \geq k_0} |u_k| < \varepsilon.$$

The element $v \in \ell_1$ defined as $v_j = y_j$ and $v_k = 0$ for all $k \neq j$ verifies that $\|v\|_1^2 = |y_j|^2 = \|y\|_\infty^2 = \langle v, y\rangle$, and hence $v \in Jy$. So that

$$\begin{aligned}\Phi_J(x,u) &= \sup_{(y',v') \in G(J)} \langle x, v'\rangle + \langle y', u - v'\rangle \geq x_j y_j + \sum_k y_k u_k - y_j^2 \\ &= (x_j + u_j) y_j + \sum_{k \neq j} y_k u_k - y_j^2 = (x_j + u_j) y_j - y_j^2 \\ &\quad + \left(\sum_{k \neq j} y_k u_k - y_j sgn(x_j + u_j) \sum_{k \neq j} |u_k|\right) + y_j sgn(x_j + u_j) \sum_{k \neq j} |u_k| \\ &> -\varepsilon + y_j \left((x_j + u_j) + sgn(x_j + u_j) \sum_{k \neq j} |u_k|\right) - y_j^2 \\ &= -\varepsilon + (1/2) \left((x_j + u_j) + sgn(x_j + u_j) \sum_{k \neq j} |u_k|\right)^2 - y_j^2\end{aligned}$$

$$= -\varepsilon + (1/4)\left((x_j + u_j) + sgn(x_j + u_j)\sum_{k\neq j}|u_k|\right)^2$$
$$= -\varepsilon + (1/4)\left(\|u\|_1 + |x_j + u_j| - |u_j|\right)^2,$$

where we also use the definition of y_j. Taking the sup over j we see that $\Phi_J(x,u) \geq -\varepsilon + g(x,u)$ as required. □

The result for the duality mapping on c_0 can be extended to the duality mapping on ℓ_1.

Corollary 4.1. *Let $J : \ell_1 \to \ell_\infty$ be the duality mapping on ℓ_1. Then*

$$\Phi_J(u,z) = \sup_j \left(\|u\|_1 + |z_j + u_j| - |u_j|\right)^2/4.$$

Proof. By the theorem this holds if $z \in c_0$. Since c_0 is weak* dense in ℓ_∞ we see that it holds for all $u \in \ell_1$ and $z \in \ell_\infty$. □

In view of the Proposition it is of interest to calculate the dual of $f(x,x^*) = (\|x\| + \|x^*\|)^2/4$.

Theorem 4.2. *The dual of $f(x,x^*) = (\|x\| + \|x^*\|)^2/4$ is*

$$f^*(y^*, y^{**}) = \max\left(\|y^*\|, \|y^{**}\|\right)^2.$$

Proof. We may choose either $x = 0$ or $x^* = 0$ to get $f^*(y^*, y^{**}) \geq \|y^*\|^2$ and $f^*(y^*, y^{**}) \geq \|y^{**}\|^2$. Indeed, assume for instance that $x^* = 0$, then

$$f^*(y^*, y^{**}) \geq \sup_{x\in X} \langle y^*, x\rangle - \|x\|^2/4$$
$$= 1/2 \sup_{x\in X} \langle 2\,y^*, x\rangle - \|x\|^2/2 = 1/2\ \varphi^*(2y^*),$$

where φ^* is the Fenchel-Moreau conjugate of $\varphi = (1/2)\|\cdot\|^2$. Using the fact that $\varphi^* = (1/2)\|\cdot\|^2$, we get

$$f^*(y^*, y^{**}) \geq 1/2\ \varphi^*(2y^*) = 1/2\ \|2y^*\|^2/2 = \|y^*\|^2.$$

On the other hand $f^*(y^*, y^{**}) \leq \max\left(\|y^*\|, \|y^{**}\|\right)^2$ because $\|x\|\,\|y^*\| + \|x^*\|\,\|y^{**}\| - (\|x\| + \|x^*\|)^2/4 \leq \max\left(\|y^*\|, \|y^{**}\|\right)^2$. □

We say that a convex function $\varphi : X \times X^* \to \overline{\mathbb{R}}$ *generates* the duality mapping J when

$$Jx = \{x^* \mid \langle x, x^*\rangle = \varphi(x,x^*)\} \text{ for all } x \in X.$$

Corollary 4.2. *All convex functions $h(x,x^*)$ with $(\|x\| + \|x^*\|)^2/4 \leq h(x,x^*) \leq \max(\|x\|, \|x^*\|)^2$ generate the duality mapping J.*

Proof. We have that $x^* \in Jx$ if and only if $\langle x, x^*\rangle = \|x\|^2 = \|x^*\|^2$. In this case, it holds that

$$\langle x, x^*\rangle = (\|x\| + \|x^*\|)^2/4 = \max(\|x\|, \|x^*\|)^2,$$

This forces h to satisfy $\langle x, x^*\rangle = h(x,x^*)$ for all $(x,x^*) \in G(J)$. □

References

[1] Burachik, R.S. and Svaiter, B.F.: Maximal monotone operators, convex functions and a special family of enlargements, *Set Valued Analysis,* **10**(4), 2002, pp. 297–316.

[2] Burachik, R.S. and Svaiter, B.F.: ε-Enlargements of Maximal Monotone Operators in Banach spaces, *Set Valued Analysis*, **7**, 1999, pp. 117–132.

[3] Fitzpatrick, S.: Representing monotone operators by convex functions, *Workshop/Miniconference on Functional Analysis and Optimization (Canberra, 1988)* pp. 59–65, *Proc. Centre Math. Anal. Austral. Nat. Univ.*,20 *Austral. Nat. Univ., Canberra,* 1988.

[4] Martínez-Legaz, J.-E. and Théra, M.: A convex representation of maximal monotone operators, *Journal of Nonlinear and Convex Analysis*, **2**, 2001, pp. 243–247.

[5] Penot, J.-P.: A representation of maximal monotone operators by closed convex functions and its impact on calculus rules, *C. R. Acad. Sci. Paris*, Scr. **I** 338, 2004, pp. 853–858.

[6] Rockafellar, R. T.: On the maximal monotonicity of subdifferential mappings, *Pacific Journal of Mathematics* **33**, 1970, pp. 209–216.

Manuscript received AUgust 10, 2004
revised November 5, 2004

Regina Sandra burachik
Engenharia de Sistemas e Computação, COPPE-Universidade Federal do Rio de Janeiro, CP 68511, Rio de Janeiro-RJ, CEP 21945-970, Brazil
E-mail address: regi@cos.ufrj.br

Simon Fitzpatrick
School of Mathematics, The University of Western Australia, Nedlands, W.A. 6907 Australia
E-mail address: fitzpatrick@maths.uwa.edu.au