Journal of Nonlinear and Convex Analysis
Volume 6, Number 1, 2005, 153–163

Yokohama Publishers
ISSN 1880-5221 ONLINE JOURNAL

RESISTIVE NETWORKS WITH MONOTONE N-PORTS

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Abstract. This paper presents an existence result on nonlinear networks constructed from N-ports with $i-v$ characteristics which are monotone, but not necessarily surjective. Our result extends the work of Desoer and Wu, who gave conditions for the existence of solutions for networks of two terminal elements. We also extend the two terminal theory by including ideal diodes, in fact all monotone $i-v$ characteristics.

1. Introduction

As early as 1947 R.Duffin [10] produced existence theorems for nonlinear resistive networks, yet there are open questions in this field, see e.g. [17] on transistor networks. Given a resistive network, consisting of nonlinear resistors with a specified relation between the current and voltage in each resistor, and current and voltage sources, the basic question is whether there is a unique set of resistor currents and voltages which satisfy these relations, and Kirchhoff's laws.

One strand of the literature deals with monotone networks, in which each of these branch relations is assumed to be given by a maximal monotone relation in the real line $\mathbb{R}$, usually required to be the graph of a function from $\mathbb{R}$ to $\mathbb{R}$, and in this paper we focus on monotone networks only. Minty [12] gave key results on solutions to a given monotone network. Others studied the class of networks formed by adding arbitrary sources to a given monotone network. The first results naturally assumed the branch relations were homeomorphisms of $\mathbb{R}$, but the work of Desoer and Katzenelson [8] shows it is enough to assume the resistors which are only current controlled form a loop free set and resistors which are only voltage controlled form a cut free set. Desoer and Wu [9], stimulated by Sandberg and Willson's paper [15], took this further and gave necessary and sufficient conditions for existence and uniqueness, by introducing type H resistors. There has been no extension of their theorem until now.

This article generalises Desoer and Wu's sufficiency result [9] to a network in which each resistor is an N-port, having N pairs of terminals, rather than a 1-port, having two terminals. We carefully discuss N-ports in the next section. This paper also extends [9] in the two terminal case, to include all maximal monotone relations. Ohtsuki et al. [13] and [14] gave sufficient conditions for a network of N-port resistors to have a solution; their argument works if we assume each to be coercive, continuous and strictly monotone. Further work under these assumptions appears in [5]. In this paper we introduce N-port resistors of type H, which do not assume the current voltage relation is surjective, to parallel the hypotheses of [9].

2000 *Mathematics Subject Classification.* Primary 94C05; Secondary 47N10, 47H05, 31C20.
Key words and phrases. network, N-port, multiterminal, resistor, Kirchhoff, monotone, convex, surjective.

We mention the work of Anderson et al. [1], who studied the interconnection of two nonlinear N-ports, and assumed the current voltage relation to be given by the subdifferential of a convex function.

We gain motivation to work on monotone N-ports from the paper [11] by Katzenelson and Unikovski, which shows that the charge on a MOS transistor is a monotone function of the voltage, and thus the transistor is a 4-terminal monotone capacitor. Resistive network theory relates to capacitive networks, and so one is motivated to see how these ideas can apply there.

2. Preliminaries

Our purpose is to study resistive electrical networks with elements which are not necessarily two terminal (one port).

The basic notions and results used in this work can be found in works on circuits and on monotone operators and convexity. For general background on circuits, including N-ports and other mutiterminal resistive devices, see for example [6] and [7]. We suggest [16], [4], [2] and [3] for convexity and related concepts. One must look at the paper [9] by Desoer and Wu, some of whose results we extend. We now define a resistive network, and give some background description of N-ports.

Definition 1. Following Ohtsuki et al. [13], [14], a resistive network is a finite directed graph, whose branches are partitioned into independent sources and resistive N-ports, where a resistive N-port is regarded as a set of N branches in a network, labelled 1 to N, together with a set of allowable current-voltage pairs (i, v), where $v \in \mathbb{R}^N$ and $i \in \mathbb{R}^N$, called the $i - v$ characteristic. (N is not the same for all equivalence classes, so multi-port might be a better word.) An independent source is a current source or a voltage source. For a current source, the current is fixed and the voltage is arbitrary. For a voltage source, the current is arbitrary and the voltage is fixed.

It is often assumed, e.g. in [13], [14] that the allowable current - voltage pairs are given by a "hybrid representation" in which the currents in branches 1 to S and the voltages in branches $S + 1$ to N are a function of the voltages in branches 1 to S and the currents in branches $S + 1$ to N, for some S. We see from Theorem 2 of [14] that one may pass from one hybrid description to another. We will use $S = N$ or zero, to give current a multivalued function of voltage, or the other way round, because of the following result.

Proposition 1. *Suppose B is a maximal monotone operator in $\mathbb{R}^N$, the direct sum $\mathbb{R}^S + \mathbb{R}^{N-S}$, and we consider a hybrid representation; $(i_a, i_b) \in \mathbb{R}^S + \mathbb{R}^{N-S}$ and $(v_a, v_b) \in \mathbb{R}^S + \mathbb{R}^{N-S}$ form an allowable pair iff $(i_a, v_b) \in B(v_a, i_b)$. Define A from $\mathbb{R}^N$ to subsets of $\mathbb{R}^N$ by $(v_a, v_b) \in A(i_a, i_b)$ iff $(i_a, v_b) \in B(v_a, i_b)$. Then A is maximal monotone.*

Proof. Directly from the definition. □

Instead of considering an N-port as some branches of a graph, together with the $i - v$ characteristic, we may alternatively consider a resistive N-port as a device with N pairs of terminals, such that the current into one terminal of any pair must equal the current out of the other, together with a set of allowable current-voltage pairs (i, v), where $v \in \mathbb{R}^N$ and $i \in \mathbb{R}^N$, called the $i - v$ characteristic. The hybrid representation describes the case when we apply voltage sources across the terminals pairs 1 to S and apply current sources between the terminal pairs $S + 1$ to N, and these determine the currents through the voltage sources and the voltages across the current sources. A resistive circuit is obtained by joining together various sets of terminals of N-ports, including current and voltage sources.

The resistive network of Definition 1 is mathematically the same as a network of two terminal resistors, or 1-ports, except that the branch equations do not merely give, for each branch b, a relation between the current i_b and voltage v_b only. Instead, the branches are coupled; for branches $b_1, \dots b_N$, giving an N-port, the set of allowable currents i_{b_1} to i_{b_N} depends on the voltages $v_{b_1} \dots v_{b_N}$.

The following point is important, not for the results of this paper but to reconcile it with other writings. Let D be a resistive device in which there are $N+1$ terminals, with a relationship between the N-vector of voltages between node $N + 1$ and each other node and the N-vector of currents into each node except $N + 1$ (the current out of node $N+1$ being their sum). Then D can be modelled by an N-port together with a short between N port terminals, one from each pair.

In some texts we see a putative 2-port made from a three terminal transistor, by shorting one terminal, say the base, to two port terminals, and using the other two transistor terminals as the other port terminals. However, the current into one terminal of a port need not then be the current out of the other. Nevertheless, this transistor is perfectly well modelled by a 2-port with one terminal from each port joined to give the base terminal, as in [14] Figure 2(b) for example.

Suppose B denotes the branch set of a resistive network. We say that $x \in \mathbb{R}^B$ satisfies Kirchhoff's current law, written KCL, to mean the sum of the x_b into any node is zero. We say that $x \in \mathbb{R}^B$ satisfies Kirchhoff's voltage law, written KVL, to mean the sum of the x_b around any loop is zero.

3. Networks of 1-port resistors

We are first going to consider a network of 1-port resistors.

Definition 2. We define a one-port (two terminal) monotone resistor b to have $i - v$ characteristic a maximal monotone set, M_b. Equivalently there is a maximal monotone operator $\hat{v_b}$ from $\mathbb{R}$ to subsets of $\mathbb{R}$ such that $v \in \hat{v_b}(i)$ iff (i, v) is an allowable current-voltage pair. We write $\hat{i_b}$ for the inverse of $\hat{v_b}$, i.e. $i \in \hat{i_b}(v) \Leftrightarrow v \in \hat{v_b}(i)$. Write $D(\hat{v_b})$ for $\{i \in \mathbb{R} : \hat{v_b}(i) \neq \emptyset\}$, and $D(\hat{i_b})$ for $\{v \in \mathbb{R} : \hat{i_b}(v) \neq \emptyset\}$. We say, following [9], that b is of type U (for unbounded) if i and v are unbounded above and below on M_b, i.e. $D(\hat{v_b}) = \mathbb{R}$ and $D(\hat{i_b}) = \mathbb{R}$. We say, it is of type H (for half) if one variable is unbounded above and below on M_b, and the other is bounded below but not above, after reorienting the branch if necessary. We say it is of type B (for bounded) if one variable is unbounded above and below on M, and the other is bounded. We say it is of type Q (for quarter) if i is bounded below on M_b, and v is bounded above on M_b, after reorienting the branch if necessary. We say, following [9] as much as possible, but with multivalued functions making the terminology suspect, that b is current controlled (c.c.) to mean i is not bounded above or below on M_b, i.e. $D(\hat{v_b}) = \mathbb{R}$. We say that b is voltage controlled (v.c.) to mean v is not bounded above or below on M_b, i.e. $D(\hat{i_b}) = \mathbb{R}$.

Let η be a resistive network of monotone 1-ports, with branch set B. A solution consists of $i \in \mathbb{R}^B$ satisfying KCL, and $v \in \mathbb{R}^B$ satisfying KVL, such that $(i_b, v_b) \in M_b$ for all $b \in B$.

Theorem 1. *Let η be a resistive network of monotone 1-ports. For all independent voltage sources with pliers entries, (i.e. in series with branches) and all independent current sources with soldering iron entries (i.e.between nodes in the same component) the network formed from η and the sources has a solution iff:*

(1) *every loop of branches of c.c. resistors and type Q resistors contains a type U branch, or two type H branches oppositely directed with respect to the loop, or two type Q branches oppositely directed, or a Q and an H similarly directed, and*
(2) *every cutset of branches of v.c. resistors and type Q resistors contains a type U branch, or two type H branches oppositely directed with respect to the cutset, or two type Q branches oppositely directed, or a Q and an H oppositely directed.*

Proof. $\Leftarrow$ For each loop L the sum, $\sum_{b \in L} D(\hat{v}_b) = \mathbb{R}$. For each cutset C the sum, $\sum_{b \in C} D(\hat{i}_b) = \mathbb{R}$. By [12], Theorem 4.1, there exist $i^1 : B \to \mathbb{R}$ satisfying KCL, and $v^1 : B \to \mathbb{R}$ satisfying KVL, such that for all $b \in B$, $i^1_b \in D(\hat{i}_b)$, and $v^1_b \in D(\hat{v}_b)$. By [12], Theorem 8.1, the network has a solution.

$\Rightarrow$ As in [9], we suppose the second condition fails, so there is a cutset C such that the sum of the currents across C is bounded above, after reorienting C if required. Then apply a current source between nodes in the two components given by C so that the current flow across C cannot be zero, contradicting the existence of a solution. If the first condition fails we obtain a contradiction too. □

Note that our two terminal monotone resistors are partitioned into the types U, H, B and Q, but in [9] all resistors were assumed to be U, H or B, and M_b was assumed to be the graph of a function from $\mathbb{R}$ to $\mathbb{R}$. We leave untreated the question of uniqueness and continuity of solutions, which was answered in [9] for their network.

4. N-port resistors: Preliminaries

Now we extend the idea of types U and H to N-port resistors. Definition 3 is consistent with Definition 2. Given $\psi : \mathbb{R}^N \to \mathbb{R} \cup \{\infty\}$, we say ψ is proper if there exists $x \in \mathbb{R}^N$ such that $\psi(x) < \infty$, we say ψ is lower semicontinuous to mean that for all $k \in \mathbb{R}$, $\{x \in \mathbb{R}^N : \psi(x) \leq k\}$ is closed, and we say ψ is convex to mean that for all $x \in \mathbb{R}^N$, all $y \in \mathbb{R}^N$, and all $\lambda \in [0, 1]$, $\psi(\lambda x + (1-\lambda)y) \leq \lambda\psi(x) + (1-\lambda)\psi(y)$. Suppose $\psi : \mathbb{R}^N \to \mathbb{R} \cup \{\infty\}$ is a lower semicontinuous proper convex function. For $x \in \mathbb{R}^N$, we let $\partial\psi(x) = \{w \in \mathbb{R}^N : \text{ for all } y \in \mathbb{R}^N, \psi(y) \geq \psi(x) + \langle w, y - x\rangle\}$. This gives $\partial\psi$ as a multivalued operator from $\mathbb{R}^N$ to $\mathbb{R}^N$, called the subdifferential of ψ.

The book [2] by Bazaraa et al. gives a background on convex analysis and includes, in Chapter 1.2, an application to electrical networks. Also [4], Section 2.1 gives an exercise on resistive networks.

Definition 3. We define a monotone resistive N-port, or resistor, where we have labelled the branches $1 \dots N$, to have as $i - v$ characteristic the graph of $\hat{v}$, the subdifferential of a lower semicontinuous proper convex function $\psi : \mathbb{R}^N \to \mathbb{R} \cup \{\infty\}$.

Thus $\hat{v}$ is a multivalued maximal monotone operator from $\mathbb{R}^N$ to $\mathbb{R}^N$, and the current i and voltage v satisfy $v \in \hat{v}(i)$. The inverse of $\hat{v}$ is written $\hat{i}$, $i \in \hat{i}(v)$. We write $D(\hat{v})$ for $\{i \in \mathbb{R}^N : \hat{v}(i) \neq \emptyset\}$, the "domain" of $\hat{v}$, and $R(\hat{v})$ for $\{v \in \mathbb{R}^N : \text{there exists } i \in \mathbb{R}^N, v \in \hat{v}(i)\}$, the "range" of $\hat{v}$. This usage of "domain" and "range" extends to all multivalued operators, including $\hat{i}$. Thus $R(\hat{v}) = D(\hat{i})$. We say that an N-port is c.c. to mean $D(\hat{v}) = \mathbb{R}^N$, and v.c. to mean $D(\hat{i}) = \mathbb{R}^N$. In the next section we will consider the set D of resistors in a network, and for all $d \in D$, we have $\hat{v}_d$, $\hat{i}_d$, and ψ_d.

For $x \in \mathbb{R}^N$ and $y \in \mathbb{R}^N$ we write $\langle x, y\rangle$ for the dot product $\Sigma_i x_i y_i$. We use the Euclidean norm. In the next definition and in the sequel we write $\langle \hat{i}(v), v\rangle/\|v\| \to \infty$ to mean $\langle i, v\rangle/\|v\| \to \infty$ for all $i \in \hat{i}(v)$.

Definition 4. We say that a branch $b \in \{1 \dots N\}$, in a v.c. monotone resistive N-port is of type U to mean $\langle \hat{i}(v), v\rangle/\|v\| \to \infty$ as $v_b \to \pm\infty$, v_b being the voltage for branch b. We say that a branch b in a v.c. resistor is of type H to mean $\langle \hat{i}(v), v\rangle/\|v\| \to \infty$ as $v_b \to \infty$, after choosing an orientation for b, but it is not true that $\langle \hat{i}(v), v\rangle/\|v\| \to \infty$ as $v_b \to -\infty$. Analogously we define a branch b in a c.c. resistor d to be of type U or type H.

In the next lemma, we consider an arbitrary branch in an N-port to be the first one, so x_1 refers to the first component of $x \in \mathbb{R}^N$. The result is like part of Proposition 2.14 of [3]. We use r.i.(K) to denote the relative interior of a convex set K.

Lemma 1. *Suppose $\psi : \mathbb{R}^N \to \mathbb{R} \cup \{\infty\}$ is a lower semicontinuous proper convex function. The following are equivalent:*

(i) *For all $u \in D(\psi)$, $\langle y, x - u\rangle/\|x\| \to \infty$ as $x_1 \to \infty$, $y \in \partial\psi(x)$.*

(ii) *$\psi(x)/\|x\| \to \infty$ as $x_1 \to \infty$.*

(iii) *There exists $u \in r.i.(D(\psi))$, $\langle y, x - u\rangle/\|x\| \to \infty$ as $x_1 \to \infty$, $y \in \partial\psi(x)$.*

Proof. (i) $\Rightarrow$ (iii) Nothing to prove, since the relative interior is nonempty in finite dimensions. (iii) $\Rightarrow$ (ii). Let $u \in$ r.i.$(D(\psi))$ and let $w \in \partial\psi(u)$. Replace $\psi(x)$ by $\phi(x) = \psi(x + u) - \langle w, x\rangle - \psi(u)$, giving $\phi(0) = 0$, and $\partial\phi(0) \ni 0$, so ϕ is minimised at 0. Then for $y \in \partial\psi(x)$, there is $z \in \partial\psi(x + u)$ such that $\langle y, x\rangle/\|x\| = \langle z - w, (x + u) - u\rangle/\|x\| \geq \langle z, (x + u) - u\rangle/\|x\| - \|w\|$. By (iii), $\langle y, x\rangle/\|x\| \to \infty$ for $x_1 \to \infty$, and $y \in \partial\psi(x)$.

Now there is an increasing function $c(t) \geq 0$ defined for all $t \geq 0$ such that $c(t) \to \infty$ as $t \to \infty$, and for all $x \in D(\partial\phi)$ with $x_1 \geq 0$, and all $y \in \partial\phi(x)$, $\langle y, x\rangle \geq c(x_1)\|x\|$. Take $x \in D(\phi)$ with $x_1 > 0$ and parametrise the line from 0 to x using the first variable. Let $\xi = x_1^{-1}x$ and then $r(t) = t\xi$ for $t \in [0, x_1]$. Now $\phi(r(t))$ is absolutely continuous on $[0, x_1]$, giving $\phi(r(x_1)) = \phi(r(0)) + \int_0^{x_1} \frac{d}{dt}\phi(r(t))dt$ giving $\phi(x) = \phi(0) + \int_0^{x_1}\langle y_t, \xi\rangle dt$, where $y_t \in \partial\phi(t\xi)$ for all t, by Exercise 3.1.29 of [4].

Hence,

$$\frac{\phi(x)}{\|x\|} \geq \frac{1}{\|x\|}\int_0^{x_1} c(t)\|\xi\|dt$$

$$= \frac{1}{x_1}\int_0^{x_1} c(t)dt$$
$$\to \infty \text{ as } x_1 \to \infty.$$

Now for $y \in \mathbb{R}^N$, $\psi(y) = \phi(y-u) + \langle w, y-u\rangle + \psi(u)$, so $\psi(y)/\|y\| \to \infty$ as $y_1 \to \infty$.

(ii) $\Rightarrow$ (i) . Given $x \in D(\partial(\psi))$, and $y \in \partial\psi(x)$, $\psi(u) + \langle y, x-u\rangle \geq \psi(x)$. Hence (ii) $\Rightarrow$ (i) □

In the next lemma, e_1 refers to $(1, 0, \ldots, 0)$, the first element of the usual basis of $\mathbb{R}^N$.

Lemma 2. *Suppose $\psi : \mathbb{R}^N \to \mathbb{R}$ is a lower semicontinuous convex function, i.e. $D(\psi) = \mathbb{R}^N$. Suppose $\psi(x)/\|x\| \to \infty$ as $x_1 \to \infty$. Then for all $a \in \mathbb{R}^N$ there is $t_a > 0$ such that for all $t > t_a$, $a + te_1 \in \operatorname{int}(R(\partial\psi))$, the interior of the range of $\partial\psi$.*

Proof. Let $b \in \mathbb{R}^N$ be given.

We claim there is a sequence x_n with $\partial\psi(x_n) = b + t_n(b)e_1$ with $t_n \to \infty$. Consider the function $\psi_b(x) = \psi(x) - \langle b, x\rangle$. Take a sequence $k_n \nearrow \infty$ with $\psi_b(x) \geq \|x\|$ if $x_1 \geq k_1$. For each n, take $a^n \in \mathbb{R}^N$ with first component $a^n_1 > k_n$. Define $K_n = \{x \in \mathbb{R}^N : x_1 \geq k_n, \psi_b(x) \leq \psi_b(a^n)\}$. For $x \in K_n$, $\|x\| \leq \psi_b(x) \leq \psi_b(a^n)$. Hence K_n is nonempty, closed and bounded. Let the function $x \mapsto x_1$ be maximised on K_n at z^n. Let $t_n(b)$ be the Lagrange multiplier, defined by $\partial\psi_b(z^n) \ni t_n(b)e_1$, (Prop 2.1.1 of [4]). Note $z^n_1 > k_n$, so $t_n(b) = \|\partial\psi_b(z_n)\| \to \infty$ as $n \to \infty$, proving the claim.

Let $a \in \mathbb{R}^N$ be given. For $i = 2 \ldots n$, let $b_i = a - e_i$ and $B_i = a + e_i$ and write B for the the set of all the b_i and B_i. For each $b \in B$, there is a sequence x_n with $\partial\psi_b(x_n) = t_n(b)e_1$ with $t_n \to \infty$. Let $t_a = \max\{t_1(b) : b \in B\}$. Then the convex hull of the range of $\partial\psi$ contains $b + [t_a, \infty)e_1$ for all $b \in B$. Therefore $\operatorname{int}(R(\partial\psi))$ is nonempty, and by Proposition 2.9 of [3], the interior of the convex closure of the range of $\partial\psi$ is the interior of the range of $\partial\psi$. Hence $a + (t_a, \infty)e_1$ is contained in the interior of the range of $\partial\psi$. □

The next example shows that Lemma 2 need not hold when the domain of ψ is a general convex set.

Example 1. Suppose $\psi(x_1, x_2) = \sec(x_1) + x_2$ for $x_1 \in (-\pi/2, \pi/2)$. Then $b + te_1 = \nabla\psi(z)$ only if $b_2 = 1$. Here $\nabla\psi(z)$ denotes the gradient of ψ at z.

Lemma 3. *Suppose $\psi : \mathbb{R}^N \to \mathbb{R}$ is a lower semicontinuous convex function. Suppose $\psi(x)/\|x\| \to \infty$ as $x_1 \to \pm\infty$. Then $\partial\psi$ is coercive, (there exists x_0 such that $\langle \partial\psi^\circ(x), x - x_0\rangle/\|x\| \to \infty$ as $\|x\| \to \infty$* [3] *Cor 2.4). Here $\partial\psi^\circ(x)$ denotes the element of least norm in $\partial\psi(x)$.*

Proof. Add to the previous proof, to see that for $a \in \mathbb{R}^N$, $b - te_1 \in R(\partial\psi)$ for arbitrarily large t and all $b \in B$. Hence, $\operatorname{cl}(R(\partial\psi))$ contains $B + \mathbb{R}e_1$. Since $\operatorname{int}(R(\partial\psi)) = \operatorname{int}(\operatorname{cl}(R(\partial\psi)))$, $a + \mathbb{R}e_1 \subset R(\partial\psi)$. Thus $\partial\psi$ is onto $\mathbb{R}^N$, and by [3], Proposition 2.14, $\partial\psi$ is coercive. □

Corollary 1. *A v.c. monotone resistive N-port containing a type U branch is c.c., and vice versa.*

5. Existence of Solutions

The main result, Theorem 2, extends the sufficient conditions of Theorem 1 of Desoer and Wu [8], by allowing N-ports.

Note that in a type Q 1-port resistor, there are intervals $[a, \infty) \subset D(\hat{v}) \subset [A, \infty)$, and we now use this to give a definition of weak type Q.

Definition 5. We say a monotone resistive N-port is of weak type Q to mean its $i - v$ characteristic is the graph of the operator $\hat{v}$, defined on $D(\hat{v})$, such that after orienting branches, there are $a \in \mathbb{R}^N$ and $A \in R^N$ such that for all $i \in D(\hat{v})$, $i \geq A$, and for all $i \geq a$, $i \in D(\hat{v})$. We refer to its branches as weak type Q branches. Here we used the usual partial ordering on $\mathbb{R}^N$, $i \geq a$ means that for all j, $i_j \geq a_j$.

Example 2. Suppose b is a one-port resistor, v.c. and of type H, then b is weak Q, but not Q since v is not bounded above.

Condition 1. We suppose that the resistive network each monotone resistive N-port d is v.c. or c.c. or is of weak type Q. We suppose also that $\text{int}(D(\hat{v_d}))$ is nonempty.

Definition 6. A solution to a resistive network is a set of branch currents, satisfying Kirchhoff's current law, and branch voltages, satisfying Kirchhoff's voltage law, such that for each resistive N-port, the current-voltage pair is in the $i - v$ characteristic, and also for each source.

Theorem 2. *Let η be a resistive network of monotone resistors, satisfying Condition 1. Suppose:*

(1) *Every loop of branches of c.c. resistors and weak type Q resistors contains a type U branch, or at least two type H branches oppositely directed with respect to the loop, or two weak type Q branches oppositely directed, or a weak Q and a H similarly directed, and*
(2) *Every cutset of branches of v.c. resistors and weak type Q resistors contains a type U branch, or two type H branches oppositely directed with respect to the cutset, or two weak type Q branches oppositely directed, or a weak Q and a H oppositely directed.*

For all independent voltage sources with pliers entries, (i.e. in series with branches) and all independent current sources with soldering iron entries (i.e. between nodes in the same component), the network formed from η and the sources has a solution.

Proof. Note that we may replace each current source by a finite sequence of current sources, each one across a port. Hence we may suppose each independent source is zero, by incorporating it into the characteristic by translation. Let D denote the set of resistors. For $d \in D$, the translates of $\hat{i}_d$ and $\hat{v}_d$ satisfy Condition 1, and they also satisfy the conditions on type U, weak Q, and H elements in loops and cutsets. Let B be the branch set of η, N the node set, and write $B(d)$ for the branch set of a resistor d. Let $K = \{i \in \mathbb{R}^B : i \text{ satisfies KCL }\}$ be the vector space of the branch-currents, $K^\perp = \{v \in \mathbb{R}^B : v \text{ satisfies KVL }\}$, the vector space of the branch-voltages. The plan is to define the function ψ in (1), and then minimize it on K.

For all resistors $d \in D$, $\hat{v}_d$ is the subdifferential of ψ_d, which is proper, lower semicontinuous, and convex. Note we have $B = \cup_{d \in D} B(d)$, so that any vector $x = \{x_b\}_{b \in B} \in \mathbb{R}^B$ gives, and is given by, a collection of vectors $x^d \in \mathbb{R}^{B(d)}$, for all $d \in D$. We define $\psi : \mathbb{R}^B \to (-\infty, \infty]$, by

$$\psi(x) = \sum_{d \in D} \psi_d(x^d). \tag{1}$$

We claim

$$\psi(x)/\|x\| \to \infty \text{ as } \|x\| \to \infty, \quad x \in K. \tag{2}$$

Let $\langle x^n \rangle$ be a sequence in K, and suppose $\|x^n\| \to \infty$ as $n \to \infty$. We claim there is a loop $\mathcal{L}$ and $\delta > 0$ such that

$$x_b^n \geq \delta \|x^n\| \tag{3}$$

for all $b \in \mathcal{L}$, after replacing $\{x^n\}$ by a subsequence, and positive current flows the same way in each branch of $\mathcal{L}$ for each n. Since B is finite, there is a branch b_1 and a subsequence, written $\langle x^n \rangle$ again, such that $|x_{b_1}^n| = \max\{|x_e^n| : e \in B\}$, and the direction of positive current flow is the same for all n. If b_1 is incident to only one node we are done. Otherwise, for each node n_1 that b_1 is incident to, KCL shows that there is a second branch b_2 incident to this node, for which $|x_{b_2}^n| \geq x_{b_1}^n/(\text{degree}(n_1) - 1)$, after replacing $\{x^n\}$ by a subsequence, and the direction of positive current flow is into n_1 on one branch and out of n_1 on the other. Continuing, since there are only a finite number of branches, we obtain a loop $\mathcal{L}$ as claimed.

Note that there are $k_1 > 0$ and $k_2 > 0$ such that for all resistors e and all $x^e \in R^{B(e)}$,

$$\psi_e(x^e) \geq -k_1 \|x^e\| - k_2. \tag{4}$$

Suppose there is a branch $b \in \mathcal{L}$ with b in a v.c. resistor d. Since $\hat{v}_d$ is onto, and in a finite dimensional space,

$$\psi_d(x^d)/\|x^d\| \to \infty \tag{5}$$

as $\|x^d\| \to \infty$, by Rem 2.3 of Brezis [3]. By (1),(3) (4) and (5), $\psi(x^n)/\|x^n\| \to \infty$ as $n \to \infty$, and hence (2) holds.

Suppose all branches in $\mathcal{L}$ are c.c. or weak type Q.

Suppose the branch b of $\mathcal{L}$ is of type U, and in the resistor d . Then d is v.c. by Cor 1, giving (2) again.

Suppose there are two type H branches in $\mathcal{L}$, say b_1 and b_2, in resistors d_1 and d_2, oriented opposite ways. One of them say b_1, is oriented the same as the positive current. Then $\psi_{d_1}((x^n)^{d_1})/\|(x^n)^{d_1}\| \to \infty$ as $n \to \infty$. Hence (2) holds again.

Suppose there are two branches in $\mathcal{L}$, say b_1 and b_2, in weak type Q resistors d_1 and d_2, oriented opposite ways. This bounds the currents in b_1 and b_2, contradicting our supposition that the current diverged to infinity.

Suppose there are two branches in $\mathcal{L}$, say b_1 and b_2, of type H and weak type Q respectively, in resistors d_1 and d_2, oriented the same way. Since the current $x_{b_2}^n$ is bounded below, $x_{b_2}^n \to \infty$, and b_2 is oriented as the positive current. Then the type H branch, b_1, is oriented the same as the positive current, and then $\psi_{d_1}((x^n)^{d_1})/\|(x^n)^{d_1}\| \to \infty$ as $n \to \infty$. Hence (2) holds as claimed.

We claim ψ is proper on K, i.e. there is $i \in K$ with $\psi(i) < \infty$; even more, there is $i \in K$ with $i^d \in \operatorname{int}(D(\partial\psi_d))$ for all d. We recall from Theorem 4.1 of Minty [12] the following. Consider a finite directed graph. Let us assign to each branch b an interval I_b. There is an element $i \in K$ such that $i_b \in I_b$ for all b $\in B$ iff for every oriented cutset C, there is $i_b \in I_b$ for each $b \in C$ such that the flow of i across C is zero, i.e. $\sum_{b\in C} \epsilon_b i_b = 0$, where $\epsilon_b = 1$ if the orientation agrees with that of C, and -1 otherwise.

We assign I_b to each b as follows. If $b \in d$ and d is c.c., then set $I_b = \mathbb{R}$. Suppose d is v.c. and not c.c. If no branches $b \in d$ are of type H, then take $a \in \operatorname{int}(D(\hat{v}_d))$ and set $I_b = \{a_b\}$ for each $b \in d$. Supposing some branches $b \in d$ are of type H, then set d_H the set of type H branches in d, and take, by repeated applications of Lemma 2, $a \in \operatorname{int}(D(\hat{v}_d))$ such that $a + \sum_{b\in d_H}[0,\infty)e_b \subset \operatorname{int}(D(\hat{v}_d))$ and set $I_b = \{a_b\}$ for each $b \notin d_H$, otherwise $I_b = a_b + [0,\infty)$. Supposing d is of weak type Q, we take a such that if $i \geq a$, then $i \in \operatorname{int}(D(\hat{v}_d))$, and for each branch b of d, take $I_b = a_b + [0,\infty)$.

Let C be a cutset. Suppose it has a branch b_0 in a current controlled resistor, or a type U branch b_0 in a voltage controlled resistor. Then $I_{b_0} = \mathbb{R}$, and hence after choosing any $i_b \in I_b$ for the other branches $b \in C$, we may choose i_{b_0} so that the flow across C is zero. Supposing C contains branches in v.c. resistors, including two type H branches b_1 and b_2, not similarly oriented with respect to C. After choosing any $i_b \in I_b$ for the other branches $b \in C$, we may choose i_{b_1} and i_{b_2} so that the flow across C is zero. Likewise, if we have in $\mathcal{C}$ two opposite directed weak Q branches, or if there is a weak Q and a voltage controlled H oppositely directed.

Hence there is an element z of the current space K with $z_b \in I_b$ for all branches b. Hence for all $d \in D$, $z^d \in \operatorname{int}(D(\hat{v}_d))$, and so ψ is proper.

Now $\psi : \mathbb{R}^B \to (-\infty,\infty]$ is proper, convex and lower semicontinuous, and $\psi|_K(x) \to \infty$ as $\|x\| \to \infty$. Hence there is $i \in K$ at which $\psi|_K$ is minimized. Write $\phi_d(x) = \psi_d(x^d)$ for $x \in \mathbb{R}^B$ and $d \in D$. Now $\partial\psi = \sum_d \partial\phi_d$ is maximal monotone; for this we could use [3] Proposition 2.17, noting that for d and e distinct resistors, $\phi_d((I + \lambda\partial\phi_e)^{-1}x) \leq \phi_d(x)$ for all $x \in \mathbb{R}^B$ and $\lambda > 0$. And $\partial\psi + \partial I_K$ is maximal monotone by [3] Cor 2.7, since the point z from the previous paragraph is in $\operatorname{int}(D(\psi))$. Hence $\sum_{d\in D} \partial\phi_d(i) + K^\perp \ni 0$, where $K^\perp$ is defined in the first paragraph of this proof. Then take $v \in K^\perp$, such that for all d, $v^d \in \hat{v}_d(i^d)$, so that (i, v) is a solution. □

Remark. There are topics that this paper does not address. Can one give a version of Theorem 2 without assuming $\hat{v}$ is a subdifferential, but merely assuming it to be monotone? Can one obtain solutions for a network constructed of multiterminal devices, as in [5], but without coercivity? Can one find conditions which give uniqueness and continuity of the solution as a funtion of the sources? Can one give necessary conditions for existence of solutions, as is possible for 1-port networks? Further questions arise about the topics addressed in this paper. Can one extend the results of Section 4? Can one give other sufficient conditions for existence of solutions?

The next example shows that our conditions are not necessary.

Example 3. Consider η to be two coupled resistors in parallel, between nodes 1 and 2. Suppose i_1 and i_2 are the currents from node 1 to node 2. Let

$$v_1 = i_1 - i_2 + \tanh(i_1 + i_2),$$

$$v_2 = i_2 - i_1 + \tanh(i_1 + i_2).$$

Then $\hat{v}$ is a strictly monotone continuous gradient map, the gradient of $0.5(i_1 - i_2)^2 + \log(\cosh(i_1 + i_2))$. Given any current source i_s between the nodes, and any voltage sources e_1 and e_2 in series with the resistors, (current i_1 through source e_1), there is a unique solution; choosing one orientation of the sources gives

$$i_1 + i_2 = i_s,$$

$$2(i_1 - i_2) = e_2 - e_1.$$

But we do not have $\langle \hat{v}(i), i \rangle / \|i\| \to \infty$ as $i_1 \to \infty$ or $-\infty$, nor as $i_2 \to \infty$ or $-\infty$. For then the boundedness of tanh would give $(i_1 - i_2)^2 / \|i\| \to \infty$, as i_1 or i_2 diverges to ∞ or $-\infty$, which is false.

The next example shows that we cannot weaken the requirements on $\hat{v}$, even if linear, to being a monotone isomorphism.

Example 4. Consider η to be two coupled resistors in parallel, between nodes 1 and 2. Suppose i_1 and i_2 are the currents from node 1 to node 2. Let

$$v_1 = i_2,$$

$$v_2 = -i_1.$$

The map $\hat{v}$ is a monotone linear isomorphism. Given a voltage source e in series with one of the resistors, there is no solution unless $e = 0$, and in this case there are infinitely many solutions.

Acknowledgement. I thank Professor Jacob Katzenelson, Technion – Israel Institute of Technology, for his considerable contribution to this paper. I thank the referees and editors for their suggestions.

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Manuscript received July 23, 2004
revised December 6, 2004

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