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BEST APPROXIMATION TO COMMON FIXED POINTS OF A SEMIGROUP OF NONEXPANSIVE OPERATORS

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Abstract. We study a sequential algorithm for finding the projection of a given point onto the common fixed points set of a semigroup of nonexpansive operators in Hilbert space. The convergence of such an algorithm was previously established only for finitely many nonexpansive operators. Algorithms of this kind have been applied to the best approximation and convex feasibility problems in various fields of applications.

1. Introduction

Numerous problems in various branches of mathematical and physical sciences can be reduced to finding a common fixed point of a given family of operators. Such problems are usually called *common fixed point problems*. For example, for projection operators onto given closed convex sets C_i $(i \in I)$ in a real Hilbert space, the common fixed points problem becomes the well-known *convex feasibility problem* of finding a point in the intersection $\cap_{i\in I} C_i$, see, e.g., Bauschke and Borwein [3]. Due to the practical and theoretical importance of these problems, algorithms for finding fixed points of operators continue to be a flourishing area of research in fixed point theory.

In this paper we study the convergence properties of infinite families of nonexpansive operators in Hilbert space. Let us denote by $\Re_+$ the set of nonnegative real numbers and by G a given unbounded subset of $\Re_+$. The set of all fixed points of an operator $T : H \to H$ is denoted by $\mathrm{Fix}(T)$. Let $\Gamma = \{T_t \mid t \in G\}$ be a family of nonexpansive operators on a nonempty closed convex subset C of a Hilbert space H. Denote by F the set of common fixed points of Γ, i.e., $F = \cap_{t\in G} \mathrm{Fix}(T_t) = \{x \in C \mid T_t x = x, \text{ for all } t \in G\}$ and assume that F is nonempty. Our purpose in this paper is to find the projection of a given point $u \in C$ onto F, denoted by $P_F u$. In contrast with the common fixed point problem, this problem is the *best approximation problem with respect to* F. To solve this problem, we investigate the following general iterative algorithmic scheme.

Algorithm 1.

Initialization: $x_0 \in C$ *is arbitrary.*

Iterative step: *Given the current iterate* x_n, *calculate the next iterate* x_{n+1} *by*

$$x_{n+1} = \alpha_n u + (1 - \alpha_n) T_{r_n} x_n, \tag{1}$$

for all $n \geq 0$, *where* $0 \leq \alpha_n \leq 1$ *and* $\{r_n\}_{n\geq 0} \subseteq G$ *is some given sequence.*

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In 1967, Halpern [22] first showed that, for $u = 0$, under certain restrictions on the scalars $\{\alpha_n\}_{n\geq 0}$, any sequence generated by (1) converges strongly to $P_F 0$ for a single nonexpansive operator T. Ten years later, Lions [24] investigated a more general case and showed that for any $u \in C$, any sequence generated by (1) converges strongly to $P_F u$, both for a single nonexpansive operator T and for any finite family of firmly nonexpansive operators. However, since the restrictions imposed by both Halpern and Lions on the sequence $\{\alpha_n\}_{n\geq 0}$ excluded the obvious candidate $\alpha_n = 1/n$, Wittmann [34] and, later on, Bauschke [2] studied this algorithm for other sets of conditions on the sequence $\{\alpha_n\}_{n\geq 0}$. For any countable family of firmly nonexpansive operators $\{T_i\}_{i\in I}$, with $F = \cap_{i\in I} \operatorname{Fix}(T_i)$ nonempty, Combettes [11] studied a parallel companion of (1). This parallel algorithm has the following form.

Algorithm 2.

Initialization: *$x_0 \in C$ is arbitrary.*

Iterative step: *Given the current iterate x_n, calculate the next iterate x_{n+1} by*

$$x_{n+1} = \alpha_n u + (1 - \alpha_n) \sum_{i\in I} \omega_i T_i x_n, \tag{2}$$

for all $n \geq 0$, where $0 \leq \alpha_n \leq 1$, $\omega_i \in (0, 1]$, for all $i \in I$ and $\sum_{i\in I} \omega_i = 1$.

Combettes showed that any sequence $\{x_n\}_{n\geq 0}$, generated by this algorithm, converges strongly to $P_F u$ under certain assumptions on the sequence $\{\alpha_n\}_{n\geq 0}$. Motivated by the above results, we show here, that if Γ is a semigroup of nonexpansive operators in a real Hilbert space and satisfies a certain regularity condition, then the projection from u onto F can be constructed iteratively by using (1). Infinite pools of operators are more difficult to handle than finite or even countably infinite ones. Elsner, Koltracht and Neumann [17] have done some work on this for a specially-constructed infinite pool of paracontracting operators and for the common fixed point problem but not for the best approximation problem, as we do here.

The origin of our work lies in a recent publication by Dominguez Benavides, Lopez Acedo and Xu [15] who attempted to construct sunny nonexpansive retractions in Banach spaces in an iterative manner. While reading their work, we discovered that, regrettably, their Lemma 2.1, which is a central tool in their work, is erroneous. Subsequently, we started this study which does not replace their work because it is formulated in a Hilbert space rather then in a Banach space, as Dominguez Benavides, Lopez Acedo and Xu attempted to do. Nonetheless, their approach and techniques motivated and inspired us in the present paper which is organized as follows. Section 2 contains some known preliminary results in fixed point theory, in Section 3 we state and prove our main theorem and in Section 4 we give counterexamples which show that Lemma 2.1 in [15] is not true. The main result of our present work is the development of Algorithm 1 and its convergence proof. This is a sequential algorithm for solving the best approximation problem with respect to common fixed points set of infinitely countable or non-countable families of nonexpansive operators in a real Hilbert space. As mentioned above, the validity of this algorithm was previously established only for families of finitely many nonexpansive operators. An interesting direction for future research is to combine the approach presented here with the work of Deutsch and Yamada [13] who study a generalized version of the best approximation problem where the objective function belongs to a certain family of convex functions. A final comment about the novelty of our results. The main applicational value is for projection algorithms, where the result is new even if stated in the Euclidean space. This is so because infinite pools of projection operators play a role if one wishes to include relaxation parameters which may change as the projection iterations proceed.

2. Preliminaries

In this section we present definitions and some tools that will be used later on in the proof of our main theorem. Throughout our work, we denote by $\Re_+$ the set of nonnegative real numbers, by N the set of nonnegative integers and by H a real Hilbert space with inner product $\langle \cdot\ ,\cdot\rangle$ and induced norm $\|\cdot\|$. The projection of a point $x \in H$ onto a subset C, is a point in C nearest to x. Generally one cannot assure the existence or uniqueness of such a point, but if C is a nonempty closed convex set then the projection of x onto C, denoted by $P_C x$, exists and is unique.

2.1. Projection operators and nonexpansive operators.

Definition 3. Given a nonempty closed convex subset C of a Hilbert space H, the operator that sends every point $x \in H$ to its nearest point in C (in the norm induced by the inner product of H) is called the `projection onto` C and denoted by P_C, that is, $y = P_C x$ if and only if

$$y \in C \text{ and } \|x - y\| = \inf\{\|x - c\| \mid c \in C\}. \tag{3}$$

Projection operators are characterized by the following result, see, e.g., Goebel and Kirk [18, Lemma 12.1].

Proposition 4. *Suppose that C is a nonempty closed convex subset of a Hilbert space H and $x \in H$. Then $y = P_C x$, if and only if $y \in C$ and, for all $c \in C$,*

$$\langle c - y, x - y\rangle \leq 0. \tag{4}$$

Seven different characterizations of (nearest point) projections in Hilbert space (including the one given above) are assembled and proved in Goebel and Reich [20, Proposition 3.5]. Projection operators belong to the broader class of nonexpansive operators.

Definition 5. Let C be some given nonempty subset of a Hilbert space H. An operator $T : C \to H$ is called `Lipschitzian` if there exists a constant $k \geq 0$ such that, for all $x, y \in C$,

$$\|Tx - Ty\| \leq k\|x - y\|. \tag{5}$$

The smallest k for which (5) holds is said to be the `Lipschitz constant` for T and is denoted by $k(T)$.

Definition 6. Let C be some given nonempty subset of a Hilbert space H. The Lipschitzian operator $T : C \to H$ is (i) `nonexpansive` if its Lipschitz constant $k(T)$ does not exceed 1, that is $\|Tx - Ty\| \leq \|x - y\|$, for all $x, y \in C$, (ii) `strictly contractive` if its Lipschitz constant $k(T) < 1$, that is, there exists

$0 \leq k < 1$ such that, $\|Tx - Ty\| \leq k\|x - y\|$, for all $x, y \in C$, (iii) `contractive` if $\|Tx - Ty\| < \|x - y\|$, for all $x, y \in C$, such that $x \neq y$, (iv) `firmly nonexpansive` if, $\|Tx - Ty\|^2 \leq \langle x - y, Tx - Ty \rangle$, for all $x, y \in C$.

Obviously, the class of nonexpansive operators includes all contractions and strictly contractive operators. Moreover, it contains all isometries (including the identity operator, denoted by I) and the projection operators (projection operators are firmly nonexpansive operators which in turn are nonexpansive, see, for example, [20, Theorem 3.6]). Observe that if T is nonexpansive then all its iterates T^n (that is, T composed with itself n times, $n \in N$ and $T^0 = I$) are nonexpansive as well. Now we turn to the existence and structure of the fixed points set of nonexpansive operators.

Let C be a nonempty subset of a Hilbert space H and let $T : C \to H$ be an operator. A point $x \in C$ is called a *fixed point* of T if $Tx = x$. The set of all fixed points of T is denoted by $\mathrm{Fix}(T)$, namely, $\mathrm{Fix}(T) = \{x \in C \mid Tx = x\}$. If T does not have a fixed point, then it is called a *fixed point free* operator. A nonexpansive operator may be fixed point free (e.g., the operator $Tx = x + y$ $(y \neq 0)$ is nonexpansive and does not have any fixed point) and, obviously, when such an operator has a fixed point it need not be unique (e.g., the identity operator). Moreover, in some spaces the fixed points sets of nonexpansive operators may not be convex and may even be disconnected (e.g., [18, Example 3.6 and Example 3.7]). The quest for geometrical conditions on C which will guarantee the existence of at least one fixed point for each nonexpansive self-operator has led to an extensive fixed point theory for nonexpansive operators. It turns out that in a Hilbert space there is a useful property that the nonexpansive operators have, as the following proposition shows.

Proposition 7. [18, Lemma 3.4]. *Suppose that T is a nonexpansive self-operator of a nonempty closed convex subset of a Hilbert space H. Then* $\mathrm{Fix}(T)$ *is closed and convex.*

Another useful tool that we use in our work is the following Opial's demiclosedness principle. We denote strong or weak convergence by "$\to$" or "$\rightharpoonup$", respectively.

Proposition 8. [26, Lemma 2]. *Let C be some given nonempty closed convex subset of a Hilbert space H and let $T : C \to H$ be a nonexpansive operator. If $\{x_n\}_{n \geq 0}$ is a sequence in C converging weakly to $\widetilde{x}$ and if $x_n - Tx_n \to 0$, then $\widetilde{x}$ is a fixed point of T.*

We conclude this subsection with two useful tools. One relates the divergence of an infinite product to the divergence of an infinite series and the other is the well-known property which characterizes every bounded sequence in a Hilbert space.

Proposition 9. [14, Lemma 7.24]. *Suppose that $\{\alpha_n\}_{n \geq 0}$ is a sequence in the interval $[0, 1)$, converging to 0. Then*

$$\sum_{n=0}^{\infty} \alpha_n = \infty \quad \textit{(i.e., } \lim_{k \to \infty} \sum_{n=0}^{k} \alpha_n = \infty\textit{)} \tag{6}$$

if and only if

$$\prod_{n=0}^{\infty}(1-\alpha_n)=0 \qquad (i.e., \ \lim_{k\to\infty}\prod_{n=0}^{k}(1-\alpha_n)=0. \tag{7}$$

Proposition 10. [14, Theorem 9.12]. *Every bounded sequence in a Hilbert space H possesses a weakly convergent subsequence.*

2.2. **Semigroups of nonexpansive operators.** We consider special families of nonexpansive operators which are called semigroups of nonexpansive operators. Such families appear, for example, in the problem of stability of the fixed point property for nonexpansive operators (see, e.g., Kuczumov [23], Gornicki [21] and Reich [32]).

Definition 11. Let G be an unbounded subset of $\Re_+$ such that

$$\begin{aligned} &t+s\in G, \quad \text{for all } t,s\in G,\\ &t-s\in G, \quad \text{for all } t,s\in G \text{ with } t\geq s, \end{aligned} \tag{8}$$

and let $\Gamma=\{T_t \mid t\in G\}$ be a family of self-operators on a nonempty closed convex subset C of a Hilbert space H. The family Γ is called a `semigroup of nonexpansive operators on` C if the following conditions hold:
(*i*) T_t is a nonexpansive self-operator on C, for all $t\in G$,
(*ii*) $T_{t+s}x=T_tT_sx$, for all $t,s\in G$ and all $x\in C$.

Example 12. In the particular case $G=N$, the family of nonexpansive operators on a nonempty closed convex subset C of a Hilbert space H is the semigroup of iterates

$$\Gamma=\{T_t \mid t\in G\}=\{T_1^n \mid n\in N\}. \tag{9}$$

Semigroups of nonexpansive operators appeared early on in Brezis' book [5]. They were studied (in a more general form as uniformly Lipschitzian semigroups) by Goebel, Kirk and Thele [19]. The basic result of [19] asserts that there exists a $k_0>1$ such that, whenever C is a nonempty bounded closed convex subset of a Hilbert space H and Γ is a uniformly k-Lipschitzian semigroup of self-operators on C with $k<k_0$, then Γ has a common fixed point in C. Another fixed point result [21, Corollary 1] asserts that if Γ is a semigroup of nonexpansive operators on a nonempty closed convex subset C of a Hilbert space, which satisfies certain regularity condition and there exists an $\widetilde{x}$ in C such that the orbit $\{T_t\widetilde{x} \mid t\in G\}$ is bounded, then Γ has a common fixed point in C. It is worthwhile to note that similar conditions ensure existence of a fixed point of a single nonexpansive operator, see, for example, [20, Proposition 1.4 and Theorem 3.2]. In this connection, see also Reich [28]. In [27], Pazy studied the asymptotic behavior of a semigroup $\Gamma=\{T_t \mid t\in\Re_+\}$ of nonexpansive operators on a nonempty closed convex subset C of a real Hilbert space. He proved that, under certain conditions, the trajectory T_tx tends to a limit, when t tends to infinity, for each $x\in C$. Reich [32, Corollary 3] pointed out that if a semigroup Γ on the Hilbert ball is generated by some nonexpansive operator T then Γ has a common fixed point if and only if T has a fixed point. Since the asymptotic behavior of nonexpansive operators yields information on the

asymptotic behavior of nonexpansive semigroups and vice versa, some known fixed point results for a single nonexpansive operator can be reformulated in terms of semigroups. For a general overview of theorems concerning existence of a common fixed point of uniformly Lipschitzian semigroups and their connections to the fixed point results established for a single nonexpansive operator see also Budzyńska, Kuczumov and Reich [9, 10], Downing and Ray [16]. For results concerning the asymptotic behavior of semigroups see Bruck [7], Pazy [27], Brezis [4], Reich [32], Crandall and Pazy [12], Reich [29], Nevanlinna and Reich [25] and Reich [30, 31].

2.3. **Uniformly asymptotically regular semigroups of nonexpansive operators.** We consider asymptotically regular operators and provide a generalization of this concept for a semigroup of nonexpansive operators. The concepts of asymptotic regularity and uniform asymptotic regularity appear in Browder and Petryshyn [6] and Schu [33, Definition 1.2], respectively.

Definition 13. Let C be a nonempty closed convex subset of a Hilbert space H. An operator $T : C \to C$ is said to be `asymptotically regular` if, for all $x \in C$,

$$\lim_{n\to\infty} \left\|T^{n+1}x - T^n x\right\| = 0. \tag{10}$$

Definition 14. Let C be a nonempty closed convex subset of a Hilbert space H. An operator $T : C \to C$ is said to be `uniformly asymptotically regular` if

$$\lim_{n\to\infty} \sup_{x\in C} \left\|T^{n+1}x - T^n x\right\| = 0. \tag{11}$$

Browder and Petryshyn [6] proved that whenever C is a nonempty bounded closed convex subset of a Hilbert space, T is a nonexpansive operator on C with a nonempty set of fixed points and I is the identity operator, then the operator $T_t = tI + (1-t)T$ is nonexpansive and asymptotically regular for each $0 < t < 1$ and has the same set of fixed points as T. Thus, the problem of the fixed point existence for a nonexpansive operator is equivalent to the same problem for a nonexpansive and asymptotically regular operator. Using this idea, we can see that fixed points theorems for nonexpansive and asymptotically regular operators provide fixed point results for nonexpansive operators (see, e.g., [6, 9, 10, 21, 23]). The concept of asymptotic regularity was further generalized for semigroups of nonexpansive operators (see, e.g., [9, 10, 21]).

Definition 15. Let $\Gamma = \{T_t \mid t \in G\}$ be a semigroup of nonexpansive operators on a nonempty closed convex subset C of a Hilbert space H. The family Γ is called an `asymptotically regular semigroup of nonexpansive operators` on C if, for all $x \in C$ and all $s \in G$,

$$\lim_{r\to\infty} \|T_s T_r x - T_r x\| = 0. \tag{12}$$

Definition 16. Let $\Gamma = \{T_t \mid t \in G\}$ be a semigroup of nonexpansive operators on a nonempty closed convex subset C of a Hilbert space H. The family Γ is called a `uniformly asymptotically regular semigroup of nonexpansive operators` on C if

$$\lim_{r\to\infty} \left(\sup_{x\in C} \|T_s T_r x - T_r x\|\right) = 0, \tag{13}$$

uniformly for all $s \in G$.

The following examples show how these notions can be used in our context.

Example 17. (See [20, Theorem 15.2]). Let T be a linear firmly nonexpansive self-operator on a nonempty convex compact subset C of a Hilbert space H, let $G = N$, and $\Gamma = \{T^n \mid n \in G\}$ be a semigroup of iterates of T. It is known (Baillon [1]) that if $C = -C$ and T is odd, then $\{T^n x\}_{n \geq 0}$ converges strongly for all $x \in C$. Fix $\varepsilon > 0$. Then there exist $x_1, x_2, \ldots, x_k \in C$ such that $C \subseteq \bigcup_{i=1}^{k} B(x_i, \varepsilon)$, where

$$B(x_i, \varepsilon) = \{x \in H \mid \|x - x_i\| < \varepsilon\}, \tag{14}$$

for all $i = 1, 2, \ldots, k$, and n_0 such that

$$\|T^n x_i - T^m x_i\| < \varepsilon, \tag{15}$$

for all $n, m > n_0$, $i = 1, 2, \ldots, k$. Take $x \in C$ and x_i such that $\|x - x_i\| \leq \varepsilon$. Then

$$\|T^n x - T^m x\| \leq \|T^n (x - x_i)\| + \|(T^n - T^m)(x_i)\| + \|T^m (x_i - x)\| \leq 3\varepsilon, \tag{16}$$

for all $n, m > n_0$. That is, Γ is a uniformly asymptotically regular semigroup of iterates of T.

Observe that condition (13) implies that there exists a monotone sequence $\{r_n\}_{n \geq 0} \subseteq G$ such that

$$0 \leq r_0 \leq r_1 \leq \cdots \leq r_n \leq \cdots \quad , \text{ and } \lim_{n \to \infty} r_n = \infty, \tag{17}$$

and

$$\sum_{n=0}^{\infty} \sup_{x \in C} \|T_s T_{r_n} x - T_{r_n} x\| < \infty, \tag{18}$$

uniformly for all $s \in G$. For results on the convergence of the iterates of linear firmly nonexpansive mappings see Bruck and Reich [8, Corollary 2.1, p. 465] and the discussion following it.

Example 18. Let the following assumptions hold. C is a nonempty bounded closed convex subset of a Hilbert space H, $T : C \to C$ is a contraction operator with Lipschitz constant $k < 1$, $G = N$, and $\Gamma = \{T^n \mid n \in G\}$ is a semigroup of iterates of T. Then for all $n, m \in G$ we have

$$\begin{aligned} \left\|T^{m+n} x - T^n x\right\| &\leq \sum_{i=0}^{m-1} \left\|T^{n+i+1} x - T^{n+i} x\right\| \\ &\leq \sum_{i=0}^{m-1} k^{n+i} \|Tx - x\| \\ &\leq \frac{k^n}{1-k} \|Tx - x\|, \end{aligned} \tag{19}$$

therefore,

$$\lim_{n \to \infty} \left(\sup_{x \in C} \left\|T^{m+n} x - T^n x\right\| \right) = 0, \tag{20}$$

uniformly for all $m \in G$. That is, Γ is a uniformly asymptotically regular semigroup of iterates of T. Additionally, we observe, that

$$\sum_{n=0}^{\infty} \sup_{x \in C} \left\| T^{m+n}x - T^{n}x \right\| < \infty, \tag{21}$$

uniformly for all $m \in G$, that is, (18) holds for every strictly increasing sequence $\{r_n\}_{n \geq 0} \subseteq G$.

3. Main Result

The main result of our work is the next convergence theorem for Algorithm 1. We need the following definition.

Definition 19. If a real sequence $\{\alpha_n\}_{n \geq 0}$ has the following three properties

$$\alpha_n \in [0,1)\ , \text{ for all } n \geq 0, \text{ and } \lim_{n \to \infty} \alpha_n = 0, \tag{22}$$

$$\sum_{n=0}^{\infty} \alpha_n = +\infty \text{ (or, equivalently, } \prod_{n=0}^{\infty} (1-\alpha_n) = 0), \tag{23}$$

$$\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \tag{24}$$

then it is called a `steering sequence`.

Theorem 20. *Let C be a nonempty closed convex subset of a real Hilbert space H, let $u \in C$ be a given point, let $\Gamma = \{T_t \mid t \in G\}$ be a uniformly asymptotically regular semigroup of nonexpansive operators on C such that $F = \cap_{t \in G} \operatorname{Fix}(T_t) \neq \emptyset$, let $\{\alpha_n\}_{n \geq 0}$ be a steering sequence, and let $\{r_n\}_{n \geq 0} \subseteq G$ be a sequence such that (17)–(18) hold. Then any sequence $\{x_n\}_{n \geq 0}$, generated by Algorithm 1, converges strongly to $P_F u$.*

Proof. Proposition 4 and Proposition 7 ensure that the point $P_F u$ is well-defined. We prove the result first for the special case $x_0 = u$ and extend it later to the general case. The proof is divided into a sequence of separate claims.

Claim 1. For all $n \geq 0$ and every $f \in F$

$$\|x_n - f\| \leq \|u - f\|. \tag{25}$$

We proceed by induction on n. Fix $f \in F$. Clearly, (25) holds for $n = 0$. If $\|x_n - f\| \leq \|u - f\|$ then

$$\begin{aligned} \|x_{n+1} - f\| &\leq \alpha_n \|u - f\| + (1-\alpha_n) \|T_{r_n} x_n - f\| \\ &\leq \alpha_n \|u - f\| + (1-\alpha_n) \|x_n - f\| \\ &\leq \|u - f\|, \end{aligned} \tag{26}$$

as requested.

Claim 2. The following strong convergence holds

$$x_{n+1} - T_{r_n} x_n \to 0. \tag{27}$$

This is true because (25) guarantees that $\{x_n\}_{n\geq 0}$ is bounded, which, in turn, implies that $\{T_{r_n}x_n\}_{n\geq 0}$ is bounded, because $\|T_{r_n}x_n - f\| \leq \|x_n - f\|$ due to non-expansivity of T_{r_n}. The boundedness of $\{T_{r_n}x_n\}_{n\geq 0}$ together with (22) imply, in view of (1), the requested result.

Claim 3. The differences of consecutive iterates converges strongly to zero, namely,

$$x_{n+1} - x_n \to 0. \tag{28}$$

Let $\widetilde{C}$ be any bounded subset of C which contains the sequence $\{x_n\}_{n\geq 0}$. By the boundedness of $\{x_n\}_{n\geq 0}$ and $\{T_{r_n}x_n\}_{n\geq 0}$ there exists some constant $L \geq 0$ such that $\|x_{n+1} - x_n\| \leq L$ and $\|u - T_{r_n}x_n\| \leq L$, for all $n \geq 0$. Therefore, for all $n \geq 1$, we get

$$\begin{aligned} &\|x_{n+1} - x_n\| \\ &= \left\|(\alpha_n - \alpha_{n-1})(u - T_{r_{n-1}}x_{n-1}) + (1-\alpha_n)(T_{r_n}x_n - T_{r_{n-1}}x_{n-1})\right\| \\ &\leq \left\|(\alpha_n - \alpha_{n-1})(u - T_{r_{n-1}}x_{n-1})\right\| + \|(1-\alpha_n)(T_{r_n}x_n - T_{r_n}x_{n-1})\| \\ &+ \left\|(1-\alpha_n)(T_{r_n}x_{n-1} - T_{r_{n-1}}x_{n-1})\right\| \\ &\leq |\alpha_n - \alpha_{n-1}| \left\|u - T_{r_{n-1}}x_{n-1}\right\| + (1-\alpha_n)\|x_n - x_{n-1}\| \\ &+ (1-\alpha_n)\left\|T_{r_n}x_{n-1} - T_{r_{n-1}}x_{n-1}\right\| \\ &\leq |\alpha_n - \alpha_{n-1}| L + (1-\alpha_n)\|x_n - x_{n-1}\| + \left\|T_{r_n}x_{n-1} - T_{r_{n-1}}x_{n-1}\right\|. \end{aligned} \tag{29}$$

Since Γ is a semigroup, and by using (17), we are able to rewrite the last term as follows

$$\left\|T_{r_n}x_{n-1} - T_{r_{n-1}}x_{n-1}\right\| = \left\|T_{r_n - r_{n-1}}T_{r_{n-1}}x_{n-1} - T_{r_{n-1}}x_{n-1}\right\|, \tag{30}$$

for all $n \geq 1$, that is,

$$\begin{aligned} \|x_{n+1} - x_n\| \leq |\alpha_n - \alpha_{n-1}| L + (1-\alpha_n)\|x_n - x_{n-1}\| \\ + \left\|T_{r_n - r_{n-1}}T_{r_{n-1}}x_{n-1} - T_{r_{n-1}}x_{n-1}\right\|, \end{aligned} \tag{31}$$

for all $n \geq 1$. Thus, inductively,

$$\begin{aligned} \|x_{n+1} - x_n\| \leq L\sum_{k=m}^{n} |\alpha_k - \alpha_{k-1}| + \|x_m - x_{m-1}\| \prod_{k=m}^{n}(1-\alpha_k) \\ + \sum_{k=m}^{n}\left\|T_{r_k - r_{k-1}}T_{r_{k-1}}x_{k-1} - T_{r_{k-1}}x_{k-1}\right\|, \end{aligned} \tag{32}$$

for all $n \geq m \geq 1$. Hence, by taking the limit as n tends to $+\infty$, we have

$$\begin{aligned} \limsup_{n\to\infty}\|x_{n+1} - x_n\| \leq L\sum_{k=m}^{\infty} |\alpha_k - \alpha_{k-1}| + L\prod_{k=m}^{\infty}(1-\alpha_k) \\ + \sum_{k=m}^{\infty}\left\|T_{r_k - r_{k-1}}T_{r_{k-1}}x_{k-1} - T_{r_{k-1}}x_{k-1}\right\| \end{aligned} \tag{33}$$

$$\leq L\sum_{k=m}^{\infty}|\alpha_k-\alpha_{k-1}|+L\prod_{k=m}^{\infty}(1-\alpha_k)$$
$$+\sum_{k=m}^{\infty}\sup_{x\in\widetilde{C}}\left\|T_{r_k-r_{k-1}}T_{r_{k-1}}x-T_{r_{k-1}}x\right\|.$$

On the other hand, conditions (22), (23), (24) and (18) imply

$$\lim_{m\to\infty}\sum_{k=m}^{\infty}|\alpha_k-\alpha_{k-1}|=0, \tag{34}$$
$$\lim_{m\to\infty}\prod_{k=m}^{\infty}(1-\alpha_k)=0$$

and

$$\lim_{m\to\infty}\sum_{k=m}^{\infty}\sup_{x\in\widetilde{C}}\left\|T_{r_k-r_{k-1}}T_{r_{k-1}}x-T_{r_{k-1}}x\right\|=0. \tag{35}$$

Altogether, by letting m tend to $+\infty$, we conclude $x_{n+1}-x_n\to 0$, as requested.

Claim 4. For each fixed $s\in G$

$$T_sx_n-x_n\to 0. \tag{36}$$

Indeed, let $\widetilde{C}$ be any bounded subset of C which contains the sequence $\{x_n\}_{n\geq 0}$. Then,

$$\begin{aligned}&\|T_sx_n-x_n\|\\&\leq\|T_sx_n-T_sT_{r_n}x_n\|+\|T_sT_{r_n}x_n-T_{r_n}x_n\|+\|T_{r_n}x_n-x_n\|\\&\leq 2\|x_n-T_{r_n}x_n\|+\sup_{x\in\widetilde{C}}\|T_sT_{r_n}x-T_{r_n}x\|\\&\leq 2(\|x_n-x_{n+1}\|+\|x_{n+1}-T_{r_n}x_n\|)+\sup_{x\in\widetilde{C}}\|T_sT_{r_n}x-T_{r_n}x\|.\end{aligned} \tag{37}$$

Combining (28), (27) and (18) yields $T_sx_n-x_n\to 0$.

Claim 5.

$$\limsup_{n\to\infty}\langle x_{n+1}-P_Fu,u-P_Fu\rangle\leq 0. \tag{38}$$

From boundedness of $\{x_n\}_{n\geq 0}$ follows that there exists a subsequence $\{n_k\}_{k\geq 0}$ such that

$$\begin{aligned}&\limsup_{n\to\infty}\langle x_{n+1}-P_Fu,u-P_Fu\rangle\\&=\lim_{n_k\to\infty}\langle x_{n_k+1}-P_Fu,u-P_Fu\rangle.\end{aligned} \tag{39}$$

By using Proposition 10 we obtain (after passing to another subsequence if necessary) that $x_{n_k+1}\rightharpoonup x$. From (36) follows that $T_sx_{n_k+1}-x_{n_k+1}\to 0$, for each $s\in G$, hence Proposition 8 implies $x\in F$. Consequently, by Proposition 4,

$$\begin{aligned}&\limsup_{n\to\infty}\langle x_{n+1}-P_Fu,u-P_Fu\rangle\\&=\lim_{n_k\to\infty}\langle x_{n_k+1}-P_Fu,u-P_Fu\rangle\end{aligned} \tag{40}$$

$$= \langle x - P_F u, u - P_F u \rangle$$
$$\leq 0,$$

as requested.

Now we can conclude the proof for the special case $x_0 = u$. Since

$$(x_{n+1} - P_F u) - \alpha_n (u - P_F u) = (1 - \alpha_n)(T_{r_n} x_n - P_F u), \tag{41}$$

we have

$$\begin{aligned} \|x_{n+1} - P_F u\|^2 \leq{}& (1 - \alpha_n) \|T_{r_n} x_n - P_F u\|^2 \\ &+ 2\alpha_n \langle x_{n+1} - P_F u, u - P_F u \rangle, \end{aligned} \tag{42}$$

which yields

$$\begin{aligned} \|x_{n+1} - P_F u\|^2 \leq{}& (1 - \alpha_n) \|x_n - P_F u\|^2 \\ &+ 2(1 - (1 - \alpha_n)) \langle x_{n+1} - P_F u, u - P_F u \rangle, \end{aligned} \tag{43}$$

for all $n \geq 0$. Let $\varepsilon > 0$. By (38) there exists an integer n_ε such that

$$\langle x_{n+1} - P_F u, u - P_F u \rangle \leq \frac{\varepsilon}{2}, \tag{44}$$

for all $n \geq n_\varepsilon$. Then,

$$\begin{aligned} \|x_{n+n_\varepsilon} - P_F u\|^2 \leq{}& \prod_{k=n_\varepsilon}^{n+n_\varepsilon-1} (1 - \alpha_k) \|x_k - P_F u\|^2 \\ &+ \left(1 - \prod_{k=n_\varepsilon}^{n+n_\varepsilon-1} (1 - \alpha_k)\right) \varepsilon, \end{aligned} \tag{45}$$

for all $n \geq 1$. Hence, from conditions (22) and (23) follows

$$\limsup_{n\to\infty} \|x_n - P_F u\|^2 = \limsup_{n\to\infty} \|x_{n+n_\varepsilon} - P_F u\|^2 \leq \varepsilon. \tag{46}$$

Since ε is an arbitrary positive real number, we conclude $x_n \to P_F u$, that is, the special case is verified.

Now we extend the proof to the general case. Let $\{x_n\}_{n\geq 0}$ be the sequence generated by (1) with a starting point x_0 (possibly different from u) and let $\{y_n\}_{n\geq 0}$ be another sequence generated by (1) with a starting point $y_0 = u$. On the one hand, by the special case,

$$y_n \to P_F u. \tag{47}$$

On the other hand, it is easily checked that

$$\|x_n - y_n\| \leq \|x_0 - y_0\| \prod_{k=0}^{n-1} (1 - \alpha_k), \tag{48}$$

for all $n \geq 1$. Thus, $x_n - y_n \to 0$ and, altogether, $x_n \to P_F u$, completing the proof. □

4. Counter-Examples

In this section we give two counter-examples showing that Lemma 2.1 in the paper by Dominguez Benavides, Lopez Acedo and Xu [15] is not true. Our first counter-example is contradictory to the zero convergence claim of the lemma. The second counter-example is even stronger in that it shows that the conditions presented in the lemma do not necessarily guarantee convergence to any finite limit. We first quote this lemma precisely as formulated in that paper.

Lemma 21. [15, Lemma 2.1]. *If $\{s_n\}_{n\geq 0}$ is a sequence of nonnegative numbers satisfying*

$$s_{n+1} \leq (1-\alpha_n)(s_n+\beta_n)+\alpha_n\gamma_n, \tag{49}$$

for all $n \geq 0$, where $\{\alpha_n\}_{n\geq 0}$, $\{\beta_n\}_{n\geq 0}$, $\{\gamma_n\}_{n\geq 0}$ are sequences of real numbers such that

$$0 \leq \alpha_n \leq 1, \ \lim_{n\to\infty} \alpha_n = 0, \ \sum_{n=0}^{\infty} \alpha_n = \infty, \tag{50}$$

$$\limsup_{n\to\infty} \beta_n \leq 0, \tag{51}$$

$$\limsup_{n\to\infty} \gamma_n \leq 0, \tag{52}$$

then $\lim_{n\to\infty} s_n = 0$.

Example 22. We show that there exists a sequence, defined by (49), which fulfills all conditions of Lemma 21 but does not converge to zero. Take $\alpha_n = \beta_n = \gamma_n = 1/(n+1)$ for all $n \geq 0$, let b_0 be an arbitrary nonnegative real number and define the sequence $\{b_n\}_{n\geq 0}$ by

$$b_{n+1} := (1-\frac{1}{n+1})(b_n+\frac{1}{n+1})+\frac{1}{n+1}\cdot\frac{1}{n+1}, \tag{53}$$

for all $n \geq 0$. It is easy to see that conditions (50)–(52) are satisfied, thus the sequence $\{b_n\}_{n\geq 0}$ is one of the many sequences that can be generated by (49). By induction on n we see that $b_n = 1$ for all $n \geq 1$, thus, $\lim_{n\to\infty} b_n = 1$, in contradiction to the claim of Lemma 21 .

Example 23. This example shows that the sequence defined by (49) and the other conditions in Lemma 21 may be even divergent. Take $\alpha_n = \dfrac{1}{n+1}$, $\beta_n = \dfrac{1}{\sqrt{n+1}}$, $\gamma_n = 0$, for all $n \geq 0$, let b_0 be an arbitrary nonnegative real number and define the sequence $\{b_n\}_{n\geq 0}$ by

$$b_{n+1} := (1-\frac{1}{n+1})(b_n+\frac{1}{\sqrt{n+1}}) \tag{54}$$

for all $n \geq 0$. It is easy to see that conditions (50)–(52) are satisfied, thus, the sequence $\{b_n\}_{n\geq 0}$ is one of the many sequences that can be generated by (49). From (54) we get

$$b_{n+1} = (1-\frac{1}{n+1})b_n+\frac{1}{\sqrt{n+1}}-\frac{1}{(n+1)\sqrt{n+1}} \tag{55}$$

$$= (\frac{n}{n+1})b_n + \frac{1}{\sqrt{n+1}} - \frac{1}{(n+1)\sqrt{n+1}}. \tag{56}$$

Multiplying by $(n+1)$ yields

$$(n+1)b_{n+1} = nb_n + \sqrt{n+1} - \frac{1}{\sqrt{n+1}}. \tag{57}$$

By denoting $z_n = nb_n$ we have

$$z_{n+1} = z_n + \sqrt{n+1} - \frac{1}{\sqrt{n+1}}, \tag{58}$$

for all $n \geq 0$. Thus,

$$\begin{aligned} z_1 - z_0 &= \sqrt{1} - \frac{1}{\sqrt{1}}, \\ z_2 - z_1 &= \sqrt{2} - \frac{1}{\sqrt{2}}. \end{aligned} \tag{59}$$

Similarly,

$$z_n - z_{n-1} = \sqrt{n} - \frac{1}{\sqrt{n}}. \tag{60}$$

Adding these n sequences yields

$$z_n - z_0 = \sum_{k=1}^{n}(\sqrt{k} - \frac{1}{\sqrt{k}}), \tag{61}$$

that is,

$$z_n = \sum_{k=1}^{n}(\sqrt{k} - \frac{1}{\sqrt{k}}) + z_0, \tag{62}$$

or

$$b_n = \frac{1}{n}\sum_{k=1}^{n}(\sqrt{k} - \frac{1}{\sqrt{k}}) + \frac{z_0}{n}. \tag{63}$$

Since $\lim_{k\to\infty}(\sqrt{k} - \frac{1}{\sqrt{k}}) = \infty$, we have $\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n}(\sqrt{k} - \frac{1}{\sqrt{k}}) = \infty$, implying $\lim_{n\to\infty} b_n = \infty$, in contradiction to the claim of Lemma 21.

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