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SHADOWS OF FUZZY SETS ON BANACH SPACES

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Abstract. In this paper, we define the shadow of fuzzy sets in a uniformly convex Banach space and prove some theorems for two convex fuzzy sets. Then we shall observe that these results improve the result of Takahashi and Takahashi [3]. Moreover, we prove a theorem concerning the relationship between the shadows defined here and the fuzzy sets defined by Amemiya and Takahashi [1].

1. Introduction

In [5], Zadeh defined the shadow of fuzzy sets in the finite dimensional real Euclidean space and proved a theorem for two convex fuzzy sets to equal each other; see also [6]. It was later shown that this result admitted a counterexample, however, Takahashi and Takahashi [3] proved the following revised version of Zadeh's result:

Let A and B be two convex fuzzy sets in a real Hilbert space X. If both A and B are closed and $S_H[A] = S_H[B]$ for every closed hyperplane H in X, then $A = B$.

Recently, Amemiya and Takahashi [1] studied the shadows of fuzzy sets in a normed linear space. They defined a fuzzy set without using the metric projection, which was in essence a generalization of the shadows of fuzzy sets, and extended the result of Takahashi and Takahashi [3] to that in a normed linear space (see Section 3).

However, their fuzzy set did not give the definition of the shadow of fuzzy sets in a normed linear space.

In this paper, we first define the shadow of fuzzy sets in a uniformly convex Banach space and prove some theorems for two convex fuzzy sets by applying the result of Amemiya and Takahashi [1]. Then, we shall observe that these theorems improve the result of Takahashi and Takahashi [3]. Moreover, we prove a theorem concerning the relationship between the shadows defined here and the fuzzy sets defined by Amemiya and Takahashi.

2. Preliminaries

Throughout this paper, all linear spaces are real and $\mathbb{R}$ denotes the set of real numbers. Also, if X is a normed linear space, then X^* denotes its dual and, for any $f \in X^*$ and any $x \in X$, (x, f) denotes the value of f at x.

Let X be a nonempty set. A fuzzy set in X is a function of X into $[0, 1]$. Let A be a fuzzy set in X. Then, the complement A' of A is a fuzzy set in X, which is defined by the formula

$$A'(x) = 1 - A(x)$$

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for every $x \in X$. Let A and B be two fuzzy sets in X. We write $A \subset B$ if $A(x) \leq B(x)$ for every $x \in X$ and $A = B$ if $A \subset B$ and $B \subset A$. Let $r \in [0,1]$. Then, the r-cut A_r of a fuzzy set A in X is a subset of X defined by

$$A_r = \{x \in X : A(x) \geq r\}.$$

Let X be a topological space and let A be a fuzzy set A in X. Then, A is said to be closed if for each $r \in (0,1]$, A_r is a closed subset of X. This implies that the function $A : X \to [0,1]$ is upper semicontinuous. Let X be a linear space and let A be a fuzzy set A in X. Then, A is said to be convex if for each $r \in (0,1]$, A_r is a convex subset of X. This implies that the function $A : X \to [0,1]$ is quasi-concave.

Let X be a normed linear space and let S_X (respectively B_X) denote the subset $\{x \in X : \|x\| = 1\}$ (respectively $\{x \in X : \|x\| \leq 1\}$) of X. Then, X is said to be strictly convex if for each $x, y \in S_X$,

$$x \neq y \text{ implies } \frac{\|x+y\|}{2} < 1.$$

X is said to be uniformly convex if for each $\varepsilon \in (0,2]$, there exists $\delta > 0$ such that for each $x, y \in S_X$,

$$\|x-y\| \geq \varepsilon \text{ implies } \frac{\|x+y\|}{2} \leq 1 - \delta.$$

Moreover, the norm of X is said to be Gâteaux differentiable (and X is called smooth) if

$$\lim_{t \to \infty} \frac{\|x+ty\| - \|x\|}{t}$$

exists for each $x, y \in S_X$.

We note that, if X is uniformly convex, then X is reflexive and strictly convex and that, if X^* is strictly convex, then X is smooth; see, for instance, [4]. Therefore, if X is uniformly convex, then X^* is smooth.

The duality map J of X is a set-valued map from X into the class of nonempty subsets of X^* defined by

$$J(x) = \{f \in X^* : (x,f) = \|x\|^2 = \|f\|^2 \}$$

for every $x \in X$. Then, for any $x, y \in X$ and any $f \in J(x)$, it is easy to see that $\|x\|^2 - \|y\|^2 \geq 2(x-y, f)$. J is single-valued if and only if X is smooth. J is continuous from X to X^* supplied with the weak* topology if X is smooth. J is surjective (in the sense that $\bigcup_{x \in X} J(x) = X^*$) if and only if X is reflexive. Moreover, for any $f \in X^*$ and any $x \in J^*(f)$, we have $J(x) = f$ if X is reflexive and smooth, where J^* is the duality map of X^*; see, for instance, [4].

Let $f : X \to (-\infty, \infty\ ]$ be a function. Then, $\mathrm{dom} f$ denotes the subset $\{x \in X : f(x) < \infty\}$ of X. f is said to be proper if $\mathrm{dom} f \neq \emptyset$. Let C be a convex subset of X. Then, f is said to be convex on C if $f\big(\lambda x + (1-\lambda)y\big) \leq \lambda f(x) + (1-\lambda) f(y)$ for any $x, y \in \mathrm{dom} f \cap C$ and any $\lambda \in (0,1)$.

We know the following theorem; see, for instance, [4]:

Let C be a nonempty closed convex subset of a reflexive Banach space. Let $f : C \to (-\infty, \infty]$ be a proper, convex and lower semicontinuous function such that for any sequence $\{x_n\}$ of elements of C, $f(x_n) \to \infty$ whenever $\|x_n\| \to \infty$. Then there exists $x_0 \in C$ such that

$$f(x_0) = \inf\{f(x) : x \in C\}.$$

Using this theorem, we immediately obtain the following; see [4] for instance:

Let C be a nonempty closed convex subset of a uniformly convex Banach space X. Then, for each $x \in X$, there exists a unique element $P_C(x)$ of C such that

$$\|x - P_C(x)\| = \inf\{\|x - y\| : y \in C\}.$$

The mapping P_C is said to be the metric projection from X onto C.

3. Shadows

First of all, we present some definitions and results concerning shadows of fuzzy sets. Let A be a fuzzy set in a Hilbert space X and let H be a closed hyperplane in X. The shadow $S_H[A]$ of A on H is a fuzzy set in H defined by

$$S_H[A](x^*) = \sup\{A(x) : P_H(x) = x^*\}$$

for every $x^* \in H$, where P_H is the metric projection from X onto H; see [3, 5]. In [3], Takahashi and Takahashi proved the following revised version of Zadeh's result [5]:

Let A and B be two convex fuzzy sets in a real Hilbert space X. If both A and B (or A' and B') are closed and $S_H[A] = S_H[B]$ for every closed hyperplane H in X, then $A = B$.

Let X be a normed linear space, let H_0^* be a closed hyperplane in X^* containing $0 \in X^*$ and let A be a fuzzy set in X. In [1], Amemiya and Takahashi defined a fuzzy set $\hat{S}_{H_0^*}[A]$ in X by

$$\hat{S}_{H_0^*}(x) = \sup\{A(z) : z \in \hat{M}_{H_0^*}(x)\}$$

for every $x \in X$, where $\hat{M}_{H_0^*}(x)$ is a subset of X defined by

$$\hat{M}_{H_0^*}(x) = \bigcap_{g \in H_0^*} \{z \in X : (z, g) = (x, g)\}$$

for every $x \in X$.

Then, they proved the following theorem extending the result of Takahashi and Takahashi to that in a normed linear space:

Let A and B be two convex fuzzy sets in a normed linear space X. If both A and B (or A' and B') are closed and $\hat{S}_{H_0^*}[A] = \hat{S}_{H_0^*}[B]$ for every closed hyperplane H_0^* in X^* containing $0 \in X^*$, then $A = B$.

We should mention that the proof of this result implies that, if there exists $x_0 \in X$ such that $A(x_0) < B(x_0)$, there is $H_0^* \in \mathcal{H}_0^*$ satisfying $\hat{S}_{H_0^*}[A](x_0) < \hat{S}_{H_0^*}[B](x_0)$.

We begin with proving the following theorem by using the result of Amemiya and Takahashi described above.

Theorem 3.1. *Let A and B be two convex fuzzy sets in a normed linear space X. If both A and B (or A' and B') are closed and if there exists $\varepsilon \in \mathbb{R}$ such that for every closed hyperplane H_0^* in X^* containing $0 \in X^*$ and every $x \in X$, $\hat{S}_{H_0^*}[A](x) - \hat{S}_{H_0^*}[B](x) = \varepsilon$, then $A(x) - B(x) = \varepsilon$ for all $x \in X$.*

Proof. Without loss of generality, we may suppose $\varepsilon > 0$. Let $\mathcal{H}_0^*$ be the class of all closed hyperplanes in X^* containing $0 \in X^*$. Then, by the fact mentioned above, we infer that $A(x) \geq B(x)$ for all $x \in X$. We set $M_A = \sup_{x \in X} A(x)$ and then, define a convex fuzzy set C and D in X by

$$C(x) = \frac{1}{M_A + \varepsilon} A(x)$$

and by

$$D(x) = \frac{1}{M_A + \varepsilon} \big(B(x) + \varepsilon\big)$$

for every $x \in X$, respectively. Then we deduce from assumption that for every $H_0^* \in \mathcal{H}_0^*$ and every $x \in X$,

$$\begin{aligned} \hat{S}_{H_0^*}[C](x) &= \frac{1}{M_A + \varepsilon} \sup\big\{A(z) : z \in \hat{M}_{H_0^*}(x)\big\} \\ &= \frac{1}{M_A + \varepsilon} \hat{S}_{H_0^*}[A](x) = \frac{1}{M_A + \varepsilon} \big(\hat{S}_{H_0^*}[B](x) + \varepsilon\big) = \hat{S}_{H_0^*}[D](x). \end{aligned}$$

Therefore, noting that both C and D (or C' and D') are closed, we have $C = D$, that is, $A(x) = B(x) + \varepsilon$ for every $x \in X$. □

Next, we provide the following result; see, for instance, [4].

Proposition 3.1. *Let M be a linear subspace in a normed linear space X with a Gâteaux differentiable norm, let J be the duality map of X and let $N = M + p$ for any $p \in X$, where $M + p$ denotes the subset $\{m + p \in X : m \in M\}$ of X. Let us fix $x \in X$ and $x_0 \in N$ arbitrarily. Then the following two conditions are equivalent;*

(i) $\big(J(x - x_0), m\big) = 0$ *for all* $m \in M$;

(ii) $\|x - x_0\| = \inf\big\{\|x - u\| : u \in N\big\}$.

Proof. For the sake of completeness, we give a proof.

(i) ⇒ (ii): Since $\|x\|^2 - \|y\|^2 \geq 2\big(x - y, J(y)\big)$ for all $x, y \in X$, it follows from the assumption that for each $u \in N$,

$$\|x - u\|^2 - \|x - x_0\|^2 \geq 2\big(x_0 - u, J(x - x_0)\big) = 0$$

since $x_0 - u \in M$. Therefore, the claim ensues.

(ii) ⇒ (i): Putting $y = x - x_0$, we infer from the assumption that for each $m \in M$ and each $\lambda \in \mathbb{R}$,

$$\|y + \lambda m\| = \|x - (x_0 - \lambda m)\| \geq \|x - x_0\| = \|y\|$$

since $x_0 - \lambda m \in N$. Therefore, we deduce that for any $m \in M$ and any $\lambda > 0$,

$$2\big(\lambda m, J(y + \lambda m)\big) \geq \|y + \lambda m\|^2 - \|y\|^2 \geq 0$$

and thus, that $\big(m, J(y + \lambda m)\big) \geq 0$. Since the duality map J of X is continuous from X to X^* supplied with the weak* topology, the inequality implies, by letting $\lambda \to +0$, that $\big(m, J(y)\big) \geq 0$ for all $m \in M$. Moreover, since M is a linear subspace in X, we have $\big(-m, J(y)\big) \geq 0$, that is, $\big(m, J(y)\big) \leq 0$ for all $m \in M$. This completes the proof. □

Now, we define the shadow of fuzzy sets in a uniformly convex Banach space.

Let X be a uniformly convex Banach space and let A be a fuzzy set in X. Let H^* be a closed hyperplane in X^* and let H_0^* be a closed hyperplane in X^* containing $0 \in X^*$ and being parallel to H^* (in the sense that there exists $f_0 \in X^*$ such that $H^* = H_0^* + f_0$). Then, the shadow $S_{H^*}[A]$ of A on H^* is a fuzzy set in H^* defined by

$$S_{H^*}[A](f) = \sup\big\{A(z) : z \in M_{H^*}(f)\big\}$$

for every $f \in H^*$, where $M_{H^*}(f)$ is a subset of X defined by

$$M_{H^*}(f) = \bigcap_{g \in H_0^*} \Big\{ z \in X : \big(z - J^*(f), g\big) = 0 \Big\}$$

and J^* is the duality map of X^*.

Before stating a theorem concerning the shadows of fuzzy sets in a uniformly convex Banach space, we provide the following result; see, for instance, [2].

Proposition 3.2. *Let X be a normed linear space, let J be the duality map of X and let $x_0 \in X$ and $f \in X^*$ be fixed arbitrarily. Then the following two conditions are equivalent;*

(i) *There exists $f_0 \in J(x_0)$ such that $(u, f_0 - f) \geq 0$ for all $u \in X$;*
(ii) *for any $u \in X$ and any $\lambda \geq 0$, $\|x_0 + \lambda u\|^2 - \|x_0\|^2 \geq 2\lambda(u, f)$.*

Proof. For the sake of completeness, we give a proof.

(i) $\Rightarrow$ (ii): Let $u \in X$. Then, since $f_0 \in J(x_0)$ and $(u, f_0 - f) \geq 0$, we deduce that for any $\lambda \geq 0$, $\|x_0 + \lambda u\|^2 - \|x_0\|^2 \geq 2(\lambda u, f_0) \geq 2(\lambda u, f)$. Therefore, the claim ensues.

(ii) $\Rightarrow$ (i): Let $u \in X$. If there exists $\lambda_0 > 0$ such that $\|x_0 + \lambda_0 u\| = 0$, that is, $x_0 = -\lambda_0 u$, we have $(-\lambda_0 u, f_0) = (x_0, f_0) = \|x_0\|^2 = \lambda_0^2 \|u\|^2$ for every $f_0 \in J(x_0)$. Therefore, it follows from the assumption that for any $\lambda > 0$ and any $f_0 \in J(x_0)$,

$$\begin{aligned} 2\lambda(u, f) &\leq \|x_0 + \lambda u\|^2 - \|x_0\|^2 = |\lambda - \lambda_0|^2 \|u\|^2 - |\lambda_0|^2 \|u\|^2 \\ &= \lambda^2 \|u\|^2 - 2\lambda\lambda_0 \|u\|^2 = \lambda^2 \|u\|^2 + 2\lambda(u, f_0) \end{aligned}$$

and hence, that $\lambda\|u\|^2 + 2(u, f_0 - f) \geq 0$. Letting $\lambda \to +0$, we have $(u, f_0 - f) \geq 0$. Otherwise, putting $g_\lambda = \dfrac{f_\lambda}{\|f_\lambda\|}$ for any $\lambda > 0$ and any $f_\lambda \in J(x_0 + \lambda u)$, we see at once that the elements g_λ belong to the subset B_{X^*} of X^*. Since B_{X^*} is a compact subset of X^* supplied with the weak* topology, there exists a subnet $\{g_{\lambda_\alpha}\}$ of $\{g_\lambda\}$ such that g_{λ_α} converges to some $g \in B_{X^*}$ in the weak* topology. Therefore, we deduce that

$$\begin{aligned} \|x_0\| - \lambda_\alpha \|u\| \leq \|x_0 + \lambda_\alpha u\| &= (x_0 + \lambda_\alpha u, g_{\lambda_\alpha}) \\ &= (x_0, g_{\lambda_\alpha}) + \lambda_\alpha (u, g_{\lambda_\alpha}) \\ &\leq \|x_0\| + \lambda_\alpha (u, g_{\lambda_\alpha}) \leq \|x_0\| + \lambda_\alpha \|u\| \end{aligned}$$

and thus, that $\|x_0\| = (x_0, g)$ and $\|g\| = 1$. Further, we infer by assumption that

$$\begin{aligned} \big(\|x_0 + \lambda_\alpha u\| + \|x_0\|\big) \cdot \lambda_\alpha (u, g_{\lambda_\alpha}) &\geq \big(\|x_0 + \lambda_\alpha u\| + \|x_0\|\big) \cdot \big(\|x_0 + \lambda_\alpha u\| - \|x_0\|\big) \\ &= \|x_0 + \lambda_\alpha u\|^2 - \|x_0\|^2 \geq 2\lambda_\alpha (u, f) \end{aligned}$$

and consequently, that

$$\big(\|x_0 + \lambda_\alpha u\| + \|x_0\|\big) \cdot (u, g_{\lambda_\alpha}) \geq 2(u, f).$$

This implies that $\big(u, \|x_0\|g\big) \geq (u, f)$. Hence, putting $f_0 = \|x_0\|g$, we have $(u, f_0 - f) \geq 0$ and $f_0 \in J(x_0)$. This completes the proof. □

Using Proposition 3.2, we obtain the following lemma.

Lemma 3.1. *Let X be a reflexive Banach space with a Gâteaux differentiable norm and let H_0 be a closed hyperplane in X containing $0 \in X$. Then, for each $x \in X$, there exists $x_0 \in H_0$ such that*

$$\big(u, J(x) - J(x_0)\big) = 0$$

for all $u \in H_0$, where J is the duality map of X.

Proof. Let $x \in X$ be fixed arbitrarily. Then, we define a convex, continuous and real-valued function φ on H_0 by

$$\varphi(z) = \frac{1}{2}\|z\|^2 - \big(z, J(x)\big)$$

for every $z \in H_0$. Since

$$\varphi(z) \geq \frac{1}{2}\|z\|^2 - \|z\|\|x\| = \|z\|\left(\frac{1}{2}\|z\| - \|x\|\right),$$

we infer that, for any sequence $\{x_n\}$ of elements of X, $\varphi(x_n) \to \infty$ whenever $\|x_n\| \to \infty$. Therefore, we have $x_0 \in H_0$ such that

$$\varphi(z) \geq \varphi(x_0)$$

for all $z \in H_0$. Replacing z by $x_0 + \lambda u$ in the above inequality, we deduce that for any $u \in H_0$ and any $\lambda \geq 0$,

$$\|x_0 + \lambda u\|^2 - \|x_0\|^2 \geq 2\lambda\big(u, J(x)\big).$$

Hence, by Proposition 3.2, we have

$$\big(u, J(x_0) - J(x)\big) \geq 0$$

for all $u \in H_0$. Moreover, since $-u \in H_0$, we also have $\big(u, J(x_0) - J(x)\big) \leq 0$ for all $u \in H_0$. This completes the proof. □

Applying Theorem 3.1 and Lemma 3.1, we prove the following theorem.

Theorem 3.2. *Let A and B be two convex fuzzy sets in a uniformly convex Banach space X. If both A and B (or A' and B') are closed and if there exists $\varepsilon \in \mathbb{R}$ such that for every closed hyperplane H_0^* in X^* containing $0 \in X^*$ and every $x^* \in H_0^*$, $S_{H_0^*}[A](x^*) - S_{H_0^*}[B](x^*) = \varepsilon$, then $A(x) - B(x) = \varepsilon$ for all $x \in X$.*

Proof. Let J (respectively J^*) be the duality map of X (respectively X^*) and let $\mathcal{H}_0^*$ be the class of all closed hyperplanes in X^* containing $0 \in X^*$. By Theorem 3.1, it is sufficient to show that for each $H_0^* \in \mathcal{H}_0^*$ and each $x \in X$,

$$\hat{S}_{H_0^*}[A](x) = \hat{S}_{H_0^*}[B](x) + \varepsilon.$$

Let $x \in X$ and let $f \in J(x)$. Then, by Lemma 3.1, there exists $f_0 \in H_0^*$ such that $\big(J^*(f) - J^*(f_0), g\big) = 0$, that is, $\big(x - J^*(f_0), g\big) = 0$ for all $g \in H_0^*$. Therefore, we infer that

$$\begin{aligned} z \in \hat{M}_{H_0^*}(x) &\Leftrightarrow (z - x, g) = 0 \text{ for all } g \in H_0^* \\ &\Leftrightarrow \big(z - J^*(f_0) + J^*(f_0) - x, g\big) = 0 \text{ for all } g \in H_0^* \\ &\Leftrightarrow \big(z - J^*(f_0), g\big) = 0 \text{ for all } g \in H_0^* \\ &\Leftrightarrow z \in M_{H_0^*}(f_0). \end{aligned}$$

and consequently, that $\hat{M}_{H_0^*}(x) = M_{H_0^*}(f_0)$. Hence, it follows that

$$\begin{aligned} \hat{S}_{H_0^*}[A](x) &= \sup\big\{A(z) : z \in \hat{M}_{H_0^*}(x)\big\} \\ &= \sup\big\{A(z) : z \in M_{H_0^*}(f_0)\big\} \\ &= S_{H_0^*}[A](f_0) = S_{H_0^*}[B](f_0) + \varepsilon = \hat{S}_{H_0^*}[B](x) + \varepsilon. \end{aligned}$$

This completes the proof. □

As a direct consequence of Theorem 3.2, we have the following theorem which improves the result of Takahashi and Takahashi [3].

Theorem 3.3. *Let A and B be two convex fuzzy sets in a Hilbert space X. If both A and B (or A' and B') are closed and $S_{H_0}[A] = S_{H_0}[B]$ for every closed hyperplane H_0 in X containing $0 \in X$, then $A = B$.*

Using Theorem 3.3, we have the following theorem for two closed convex subsets of a Hilbert space.

Theorem 3.4. *Let A and B be two closed convex subsets of a Hilbert space X. If $A \cap H_0 = B \cap H_0$ for every closed hyperplane H_0 in X containing $0 \in X$, then $A = B$.*

Proof. Let $\mathcal{H}_0$ be the class of all closed hyperplanes in X containing $0 \in X$. We define a closed convex fuzzy set f_A in X by

$$f_A(x) = \mathbf{1}_A(x)$$

and f_B in X by

$$f_B(x) = \mathbf{1}_B(x)$$

for every $x \in X$. By Theorem 3.3, it is sufficient to show that $S_{H_0}[f_A] = S_{H_0}[f_B]$ for every closed hyperplane $H_0 \in \mathcal{H}_0$. In order to do it, we suppose that there exist $H_0 \in \mathcal{H}_0$ and $x \in H_0$ such that $S_{H_0}[f_A](x) \neq S_{H_0}[f_B](x)$. Without loss of generality, we may assume that $S_{H_0}[f_A] = 1$ and $S_{H_0}[f_B] = 0$. Then there exists $x_0 \in X$ with $P_{H_0}(x_0) = x$ such that $x_0 \in A$ and $x_0 \notin B$. If $x_0 = 0$, then $x_0 \in A \cap H_0 = B \cap H_0$. This is a contradiction. So, we suppose that $x_0 \neq 0$. Then we have $p \neq 0$ such that $(x_0, p) = 0$. Putting $H = \big\{x \in X : (x, p) = 0\big\}$, it follows that $H \in \mathcal{H}_0$ and $x_0 \in H$. Therefore we have $x_0 \in A \cap H_0 = B \cap H_0$. This is a contradiction. □

At the end of this paper, we state a theorem dealing with the relationship between the shadows of fuzzy sets defined here and the fuzzy sets due to Amemiya and Takahashi [1] in a uniformly convex Banach space.

Let X be a uniformly convex Banach space, let H_0^* be a closed hyperplane in X^* containing $0 \in X^*$ and let J^* be the duality map of X^*. Then we observe by Lemma 3.1 that for each $x \in X$, there exists $f_0 \in H_0^*$ such that

$$\big(x - J^*(f_0), g\big) = 0 \text{ for all } g \in H_0^*.$$

On the other hand, we see at once that for each $x \in X$, there exists $f_0 \in X^*$ such that $J^*(f_0) = x$, that is,

$$\big(x - J^*(f_0), g\big) = 0 \text{ for all } g \in X^*;$$

(see Section 2.) Moreover, it is obvious from the definitions that for each $f \in H_0^*$, $M_{H_0^*}(f) = \hat{M}_{H_0^*}\big(J^*(f)\big)$. Thus, we have proved the following theorem.

Theorem 3.5. *Let X be a uniformly convex Banach space, let H_0^* be a closed hyperplane in X^* containing $0 \in X^*$ and let A be a fuzzy set in X. Then the following (*i*) and (*ii*) hold:*

(i) *For each $x \in X$, there exists $f_0 \in H_0^*$ such that*

$$\hat{S}_{H_0^*}[A](x) = S_{H_0^*}[A](f_0);$$

(ii) *for each $f \in H_0^*$, there exists $x_0 \in X$ such that*

$$S_{H_0^*}[A](f) = \hat{S}_{H_0^*}[A](x_0).$$

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