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ITERATIVE SCHEME FOR FINDING A COMMON POINT OF INFINITELY MANY CONVEX SETS IN A BANACH SPACE

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Abstract. In this paper, we introduce an iterative scheme for finding a common point of a countable infinite family of closed convex subsets of a uniformly convex Banach space by using the hybrid method in mathematical programming. Then, we prove that the sequence converges strongly to an element of the intersection set.

1. Introduction

Let $\{C_i\}_{i=1}^{\infty}$ be a countable infinite family of closed convex subsets of a Banach space E such that $\bigcap_{i=1}^{\infty} C_i \neq \emptyset$. Then we study the problem of finding an element of $\bigcap_{i=1}^{\infty} C_i$ by an iterative scheme. Such a problem is connected with the *convex feasibility problem.* In fact, if $f_1, f_2, \cdots : E \to \mathbb{R}$ are continuous and convex functions, then the convex feasibility problem is to find an element of the following set:

$$\bigcap_{i=1}^{\infty} \{x \in E : f_i(x) \leq 0\}.$$

There are some weak convergence theorems for a finite family of closed convex subsets of a Banach space which were proved in Alber [1] and Reich [10], assuming that the duality mapping is weakly sequentially continuous. On the other hand, using the notion of W-mapping, Kimura and Takahashi [7] and Shimoji and Takahashi [11] proved weak and strong convergence theorems for a countable infinite family of nonexpansive mappings in a Banach space, respectively. Then they applied their results to the problem of finding a common point of a countable infinite family of nonexpansive retracts; see also Atsushiba and Takahashi [2], Takahashi [13] and Takahashi and Shimoji [16].

Recently, using the hybrid method in mathematical programming, Kamimura and Takahashi [6] obtained a strong convergence theorem for maximal monotone operators in a Banach space, which is a generalization of Solodov and Svaiter [12]; see also Ohsawa and Takahashi [9] for another generalization of the result of Solodov and Svaiter [12]. More recently, using the notion of Bregman distance, Bauschke and Combettes [3] also proved a strong convergence theorem extending the result of Solodov and Svaiter [12].

In this paper, motivated by Kamimura and Takahashi [6] and Nakajo and Takahashi [8], we study the convergence of the following iterative sequence $\{x_n\}$ for finding an element of $\bigcap_{i=1}^{\infty} C_i$ in a smooth and uniformly convex Banach space:

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$x_1 = x \in E$ and

$$\begin{cases} y_n = P_1 P_2 \cdots P_n(x_n); \\ X_n = \{z \in E : \phi(z, y_n) \le \phi(z, x_n)\}; \\ Y_n = \{z \in E : \langle z - x_n, Jx - Jx_n\rangle \le 0\}; \\ x_{n+1} = P_{X_n \cap Y_n}(x) \qquad\qquad (n = 1, 2, \dots), \end{cases}$$

where $P_i : E \to C_i$ is the generalized projection [1, 6] onto C_i for each $i \in \mathbb{N}$,

$$\phi(u, v) = \|u\|^2 - 2\langle u, Jv\rangle + \|v\|^2$$

for all $u, v \in E$ and J is the duality mapping from E into E^*. Then we prove that the sequence $\{x_n\}$ converges strongly to an element of $\bigcap_{i=1}^{\infty} C_i$. We finally study a convex minimization problem in a Banach space.

2. Preliminaries

Let E be a (real) Banach space with norm $\|\cdot\|$ and let E^* denote the Banach space of all continuous linear functionals on E. We denote the strong convergence and the weak convergence of vectors in E by $\to$ and $\rightharpoonup$, respectively. For all $x \in E$ and $x^* \in E^*$, we denote $x^*(x)$ by $\langle x, x^*\rangle$. We denote by $\mathbb{R}$ and $\mathbb{N}$ the set of all real numbers and the set of all positive integers, respectively. The *duality mapping* J from E into 2^{E^*} is defined by

$$J(x) = \{x^* \in E^* : \langle x, x^*\rangle = \|x\|^2 = \|x^*\|^2\}$$

for all $x \in E$. A function $f : E \to (-\infty, \infty]$ is said to be *proper* if the set $\{x \in E : f(x) \in \mathbb{R}\}$ is nonempty. A proper function $f : E \to (-\infty, \infty]$ is said to be *convex* if

$$f(\alpha x + (1-\alpha)y) \le \alpha f(x) + (1-\alpha) f(y)$$

for all $x, y \in E$ and $\alpha \in (0, 1)$. Also f is said to be *lower semicontinuous* if the set $\{x \in E : f(x) \le r\}$ is closed in E for all $r \in \mathbb{R}$.

A Banach space E is said to be *strictly convex* if $\|x\| = \|y\| = 1$ and $x \ne y$ imply $\|(x+y)/2\| < 1$. Also, E is said to be *uniformly convex* if for each $\varepsilon \in (0, 2]$, there exists $\delta > 0$ such that

$$\|x\| = \|y\| = 1,\ \|x - y\| \ge \varepsilon$$

imply $\|(x+y)/2\| \le 1 - \delta$. It is also said to be *smooth* if the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{1}$$

exists for all $x, y \in S(E) = \{z \in E : \|z\| = 1\}$. Further, a Banach space E is said to have a *uniformly Gâteaux differentiable norm* if for each $y \in S(E)$, the limit (1) exists uniformly in $x \in S(E)$. We know that if E is smooth, strictly convex and reflexive, then the duality mapping J is single-valued, one to one and onto. We also know the following; see Takahashi [14, 15] for details:

(1) If E is uniformly convex, then it is reflexive;
(2) if E is uniformly convex, then $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$ imply $x_n \to x$;
(3) if E has a uniformly Gâteaux differentiable norm, then J is uniformly norm to weak* continuous on each bounded subset of E.

Let E be a smooth, strictly convex and reflexive Banach space and let $\phi : E \times E \to [0,\infty)$ be the function defined as follows:

$$\phi(x,y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2$$

for all $x, y \in E$. It is easy to see that $(\|x\| - \|y\|)^2 \le \phi(x,y)$ for all $x, y \in E$. If C is a nonempty closed convex subset of E, then for each $x \in E$, there exists a unique $x_0 \in C$ such that $\phi(x_0, x) = \min_{y \in C} \phi(y,x)$ and we denote the point x_0 by $P_C(x)$. The mapping P_C is called the *generalized projection* from E onto C; see Alber [1] or Kamimura and Takahashi [6]. It is known that for each $x \in E$, $x_0 = P_C(x)$ is equivalent to the following:

$$\langle y - x_0, Jx - Jx_0\rangle \le 0$$

for all $y \in C$; see [1, 6] for details. We also know the following:

Lemma 2.1 ([1]; see also [6])**.** *Let C be a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space E. Then*

$$\phi(u, P_C x) + \phi(P_C x, x) \le \phi(u, x)$$

for all $u \in C$ and $x \in E$.

Lemma 2.2 (Kamimura-Takahashi [6])**.** *Let E be a smooth and uniformly convex Banach space and let $\{x_n\}$ and $\{y_n\}$ be sequences in E such that either $\{x_n\}$ or $\{y_n\}$ is bounded. If* $\lim_{n\to\infty} \phi(x_n, y_n) = 0$*, then* $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

3. Strong Convergence Theorem

Now, we prove a strong convergence theorem for finding a common point of closed convex subsets in a Banach space.

Theorem 3.1. *Let E be a uniformly convex Banach space with a uniformly Gâteaux differentiable norm and let $\{C_i\}_{i=1}^{\infty}$ be a countable infinite family of closed convex subsets of E such that $\bigcap_{i=1}^{\infty} C_i \ne \emptyset$. Let P_i be the generalized projection from E onto C_i for each $i \in \mathbb{N}$ and let $\{x_n\}$ be a sequence defined as follows: $x_1 = x \in E$ and*

$$\begin{cases} y_n = P_1 P_2 \cdots P_n(x_n); \\ X_n = \{z \in E : \phi(z, y_n) \le \phi(z, x_n)\}; \\ Y_n = \{z \in E : \langle z - x_n, Jx - Jx_n\rangle \le 0\}; \\ x_{n+1} = P_{X_n \cap Y_n}(x) \qquad\qquad (n = 1, 2, \dots). \end{cases}$$

Then the sequence $\{x_n\}$ converges strongly to $P_{\bigcap_{i=1}^{\infty} C_i}(x)$.

Proof. Put $C_0 = \bigcap_{i=1}^{\infty} C_i$. We denote the mapping P_{C_0} by P_0. Note that for each $n \in \mathbb{N}$,

$$X_n = \{z \in E : 2\langle z, Jx_n - Jy_n\rangle \le \|x_n\|^2 - \|y_n\|^2\}$$

is closed and convex. We first prove that $\{x_n\}$ is well-defined. Let $u \in C_0$ be given. Then by Lemma 2.1 we have

$$\phi(u, y_1) = \phi(u, P_1 x_1) \le \phi(u, x_1).$$

Hence $C_0 \subset X_1$. Since $Y_1 = E$, we have $C_0 \subset X_1 \cap Y_1$ and hence $x_2 = P_{X_1 \cap Y_1}(x)$ and $y_2 = P_1 P_2 x_2$ are defined. If $C_0 \subset X_{n-1} \cap Y_{n-1}$ for some $n \geq 2$, then $x_n = P_{X_{n-1} \cap Y_{n-1}}(x)$ and $y_n = P_1 \cdots P_n x_n$ are defined. For each $u \in C_0$, we have

$$\begin{aligned} \phi(u, y_n) &= \phi(u, P_1 \cdots P_n x_n) \\ &\leq \phi(u, P_2 \cdots P_n x_n) \\ &\leq \cdots \\ &\leq \phi(u, x_n). \end{aligned}$$

Hence $u \in X_n$. And it holds from $x_n = P_{X_{n-1} \cap Y_{n-1}}(x)$ and $u \in X_{n-1} \cap Y_{n-1}$ that

$$\langle u - x_n, Jx - Jx_n \rangle \leq 0.$$

Thus $u \in Y_n$ and hence we have $C_0 \subset X_n \cap Y_n$. Thus $x_{n+1} = P_{X_n \cap Y_n}(x)$ and $y_{n+1} = P_1 \cdots P_{n+1} x_{n+1}$ are defined. Therefore the sequence $\{x_n\}$ is well-defined and $C_0 \subset X_n \cap Y_n$ for all $n \in \mathbb{N}$.

Let us prove that $\lim_{n \to \infty} \|x_n - x_{n+1}\| = 0$. For any $n \in \mathbb{N}$, we have from $x_{n+1} \in Y_n$ and Lemma 2.1 that

$$\begin{aligned} \phi(x_{n+1}, x_n) + \phi(x_n, x) &= \phi(x_{n+1}, P_{Y_n} x) + \phi(P_{Y_n} x, x) \\ &\leq \phi(x_{n+1}, x) \\ &= \phi(P_{X_n \cap Y_n}(x), x) \\ &\leq \phi(P_0 x, x). \end{aligned} \tag{2}$$

Hence the sequence $\{\phi(x_n, x)\}$ is nondecreasing and bounded from above. Therefore the limit of $\phi(x_n, x)$ exists. Since $(\|x_n\| - \|u\|)^2 \leq \phi(x_n, x)$, $\{x_n\}$ is bounded. It also holds from $\phi(u, y_n) \leq \phi(u, x_n)$ that $\{y_n\}$ is bounded. By the existence of $\lim_{n \to \infty} \phi(x_n, x)$ and (2), we have

$$\lim_{n \to \infty} \phi(x_{n+1}, x_n) = 0. \tag{3}$$

Then Lemma 2.2 ensures that

$$\lim_{n \to \infty} \|x_{n+1} - x_n\| = 0. \tag{4}$$

Let $\{x_{n_i}\}$ be any subsequence of $\{x_n\}$. Since $\{x_n\}$ is bounded, we have a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ such that $x_{n_{i_j}} \rightharpoonup v$ as $j \to \infty$. Then to show that $x_n \to P_0 x$, it is sufficient to show that $x_{n_{i_j}} \to P_0 x$ as $j \to \infty$. Without loss of generality, let us denote $\{x_{n_{i_j}}\}$ by $\{x_{n_j}\}$. We prove that $v \in C_i$ for all $i \in \mathbb{N}$ by induction. We have from the definition of X_n that

$$\phi(x_{n+1}, y_n) \leq \phi(x_{n+1}, x_n).$$

By (3), we get $\lim_{n \to \infty} \phi(x_{n+1}, y_n) = 0$. Then Lemma 2.2 implies that

$$\lim_{n \to \infty} \|x_{n+1} - y_n\| = 0. \tag{5}$$

By (4) and (5), we obtain

$$\lim_{n \to \infty} \|x_n - y_n\| = 0. \tag{6}$$

Hence the subsequence $\{y_{n_j}\}$ converges weakly to v. From $y_n = P_1 \cdots P_n(x_n)$, we have that $y_n \in C_1$ for all $n \in \mathbb{N}$. Since C_1 is weakly closed, we have $v \in C_1$. Set $z_n^{(2)} = P_2 \cdots P_n x_n$ for all $n \geq 2$ and let $u \in C_0$ be given. Then it holds that

$$\begin{aligned}\phi(u, y_n) + \phi(y_n, z_n^{(2)}) &= \phi(u, P_1(z_n^{(2)})) + \phi(P_1(z_n^{(2)}), z_n^{(2)}) \\ &\leq \phi(u, z_n^{(2)}) \\ &\leq \phi(u, x_n)\end{aligned}$$

for all $n \geq 2$. Thus we get

$$\phi(y_n, z_n^{(2)}) \leq \phi(u, x_n) - \phi(u, y_n) \tag{7}$$

for all $n \geq 2$. Since the norm of E is uniformly Gâteaux differentiable, we have from (6) that

$$\begin{aligned}\phi(y_n, z_n^{(2)}) &\leq \phi(u, x_n) - \phi(u, y_n) \\ &= \|x_n\|^2 - \|y_n\|^2 + 2\langle u, Jy_n - Jx_n\rangle \\ &\leq 2M\|x_n - y_n\| + 2\langle u, Jy_n - Jx_n\rangle \\ &\to 0\end{aligned}$$

as $n \to \infty$, where $M > 0$ is a real number satisfying $\|x_n\| \leq M$ and $\|y_n\| \leq M$ for all $n \in \mathbb{N}$. Then by Lemma 2.2, we have

$$\lim_{n\to\infty} \|y_n - z_n^{(2)}\| = 0.$$

Hence we have $z_{n_j}^{(2)} \rightharpoonup v$ as $j \to \infty$. Since $z_n^{(2)} \in C_2$ for all $n \geq 2$, we get $v \in C_2$. Put

$$z_n^{(3)} = P_3 \cdots P_n x_n$$

for all $n \geq 3$. By the above method, we have

$$\lim_{n\to\infty} \|z_n^{(2)} - z_n^{(3)}\| = 0.$$

Further, we have $z_{n_j}^{(3)} \rightharpoonup v$ as $j \to \infty$. Thus we have $v \in C_3$. Similarly, we have $v \in C_n$ for all $n \geq 4$.

Since the function ϕ is weakly lower semicontinuous in its first variable, we get

$$\begin{aligned}\phi(P_0x, x) &\leq \phi(v, x) \\ &\leq \liminf_{j\to\infty} \phi(x_{n_j}, x) \\ &\leq \limsup_{j\to\infty} \phi(x_{n_j}, x) \\ &\leq \phi(P_0x, x).\end{aligned}$$

Hence we have

$$\lim_{j\to\infty} \phi(x_{n_j}, x) = \phi(v, x) = \phi(P_0x, x)$$

and hence $v = P_0x$. This also implies that $\lim_{j\to\infty} \|x_{n_j}\| = \|P_0x\|$. Then the uniform convexity of E implies

$$\lim_{j\to\infty} x_{n_{i_j}} = \lim_{j\to\infty} x_{n_j} = P_0x.$$

Therefore the sequence $\{x_n\}$ converges strongly to P_0x as $n \to \infty$. This completes the proof. □

As a direct consequence of Theorem 3.1, we obtain the following:

Corollary 3.2. *Let H be a Hilbert space and let $\{C_i\}_{i=0}^{\infty}$ be a countable infinite family of closed convex subsets of H such that $\bigcap_{i=1}^{\infty} C_i \neq \emptyset$. Let P_i be the metric projection from H onto C_i for each $i \in \mathbb{N}$ and let $\{x_n\}$ be a sequence defined as follows: $x_1 = x \in H$ and*

$$\begin{cases} y_n = P_1 P_2 \cdots P_n(x_n); \\ X_n = \{z \in H : \|z - y_n\| \leq \|z - x_n\|\}; \\ Y_n = \{z \in H : \langle z - x_n, x - x_n \rangle \leq 0\}; \\ x_{n+1} = P_{X_n \cap Y_n}(x) \qquad (n = 1, 2, \dots). \end{cases}$$

Then the sequence $\{x_n\}$ converges strongly to $P_{\bigcap_{i=1}^{\infty} C_i}(x)$.

4. Application to a Convex Minimization Problem

We next consider a convex minimization problem. Let E be a Banach space, let $f : E \to (-\infty, \infty]$ be a proper lower semicontinuous convex function and let $f_1, f_2, \cdots : E \to \mathbb{R}$ be a sequence of continuous convex functions. Then we study the problem of finding an element u of

$$C = \bigcap_{i=1}^{\infty} \{x \in E : f_i(x) \leq 0\}$$

such that

$$f(u) = \min_{y \in C} f(y) = \alpha.$$

If the *optimal set* $M = \{x \in E : f(x) = \alpha\}$ is nonempty and the *optimal value* α is known, then we can define a proper lower semicontinuous convex function $f_0 : E \to (-\infty, \infty]$ by $f_0(x) = f(x) - \alpha$ for all $x \in E$. Then we have

$$M = \bigcap_{i=0}^{\infty} \{x \in E : f_i(x) \leq 0\}.$$

Therefore, using Theorem 3.1, we obtain the following strong convergence theorem:

Theorem 4.1. *Let E be a uniformly convex Banach space with a uniformly Gâteaux differentiable norm. Let $f : E \to (-\infty, \infty]$ be a proper lower semicontinuous convex function and let $f_1, f_2, \cdots : E \to \mathbb{R}$ be a sequence of continuous convex functions such that*

$$M = \{x \in C : f(x) = \inf_{y \in C} f(y)(= \alpha)\}$$

is nonempty, where $C = \bigcap_{i=1}^{\infty} \{x \in E : f_i(x) \leq 0\}$. Let $f_0(x) = f(x) - \alpha$ for all $x \in E$ and let P_i be the generalized projection from E onto $\{x \in E : f_i(x) \leq 0\}$ for *each* $i = 0, 1, 2, \ldots$. *Let* $\{x_n\}$ *be a sequence defined as follows:* $x_0 = x \in E$ *and*

$$\begin{cases} y_n = P_0 P_1 \cdots P_n(x_n); \\ X_n = \{z \in E : \phi(z, y_n) \leq \phi(z, x_n)\}; \\ Y_n = \{z \in E : \langle z - x_n, Jx - Jx_n \rangle \leq 0\}; \\ x_{n+1} = P_{X_n \cap Y_n}(x) \qquad\qquad (n = 0, 1, 2, \ldots). \end{cases}$$

Then the sequence $\{x_n\}$ *converges strongly to* $P_M(x)$.

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