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EXISTENCE OF EQUILIBRIA IN N-PERSON GAMES VIA CONNECTEDNESS

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ABSTRACT. The purpose of this paper is to prove a generalization on the preservation of connectedness for single-valued mappings to multi-valued mappings, and next prove some maximal element existence theorem and fixed point theorem for connected settings. As applications, we shall prove three equilibrium existence theorems for connected n-person games without assuming compact convex conditions.

1. Introduction

As is well-known, connectedness can be preserved by continuous mappings, and in the case of $\mathbb{R}$, we can obtain the intermediate value theorem as a consequence. Since multi-valued mappings are very natural extensions of single-valued mappings and useful tools in many real applications, their properties have been extensively studied in many literature, and most results in single-valued mappings have been generalized in multi-valued mappings.

The purpose of this paper is two-fold. First, we provide a proof of a generalization on the preservation of connectedness for single-valued mappings to multi-valued mappings, and next obtain some maximal element existence theorem and fixed point theorem for connected sets. Using those results, we prove three equilibrium existence theorems for connected n-person games without assuming compact convex conditions. We also give some examples that the previous results due to Borglin-Keiding [3], Kim [8], Shafer-Sonnenschein [12], Tian [14], Yannelis-Prabhakar [15] do not work whereas our results do.

2. Preliminaries

We first recall the following notations and definitions. Let A be a non-empty set. We shall denote by 2^A the family of all subsets of A. Let X, Y be non-empty topological spaces and $T : X \to 2^Y$ be a multimap. Then T is said to be *open* or have *open graph* (respectively, *closed* or *closed graph*) if the graph of T ($\operatorname{Gr} T = \{(x, T(x)) \in X \times Y \mid x \in X\}$) is open (respectively, closed) in $X \times Y$. We may call $T(x)$ the *upper section* of T, and $T^{-1}(y)$ $(= \{x \in X \mid y \in T(x)\})$ the *lower section* of T. It is easy to check that if T has open graph, then the upper and lower sections of T are open ; however the converse is not true in general. A multimap $T : X \to 2^Y$ is said to be *closed at* x if for each net $(x_\alpha) \to x$, $y_\alpha \in T(x_\alpha)$ and

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$(y_\alpha) \to y$, then $y \in T(x)$. A multimap T is said to be *closed on* X if it is closed at every point of X. Note that if T is single-valued, then the closedness is equivalent to continuity as a function.

A multimap $T : X \to 2^Y$ is said to be *upper semicontinuous* if for each $x \in X$ and each open set V in Y with $T(x) \subset V$, then there exists an open neighborhood U of x in X such that $T(y) \subset V$ for each $y \in U$. Also $T : X \to 2^Y$ is said to be *lower semicontinuous* if for each $x \in X$ and each open set V in Y with $T(x) \cap V \neq \emptyset$, then there exists an open neighborhood U of x in X such that $T(y) \cap V \neq \emptyset$ for each $y \in U$. A multimap T is said to be *continuous* if T is both upper semicontinuous and lower semicontinuous.

Let $T : X \to 2^Y$ be a multimap; then $\hat{x} \in X$ is called a *maximal element* for T if $T(\hat{x}) = \emptyset$. Indeed, in real applications, the maximal element may be interpreted as the set of those objects in X that are the "best" or "largest" choices.

Let I be a (possibly uncountable) set of agents. For each $i \in I$, let X_i be a non-empty set of actions. A *generalized game* $\Gamma = (X_i, A_i, P_i)_{i\in I}$ is defined as a family of ordered triples (X_i, A_i, P_i) where X_i is a non-empty topological space (a choice set), $A_i : \Pi_{j\in I} X_j \to 2^{X_i}$ is a constraint multimap and $P_i : \Pi_{j\in I} X_j \to 2^{X_i}$ is a preference multimap. An *equilibrium* for Γ is a point $\hat{x} \in X = \Pi_{i\in I} X_i$ such that for each $i \in I$, $\hat{x}_i \in A_i(\hat{x})$ and $P_i(\hat{x}) \cap A_i(\hat{x}) = \emptyset$. In particular, when $I = \{1, \cdots, n\}$, we may call Γ an *n-person game*.

For each $i \in I$ and a given multimap $A_i : X = \Pi_{j\in I} X_j \to 2^{X_i}$, we simply denote a multimap $A'_i : X \to 2^X$, without any confusion of notation, by

$$A'_i(x) := X_1 \times \cdots X_{i-1} \times A_i(x) \times X_{i+1} \times \cdots \times X_n, \quad \text{for each} \quad x \in X.$$

3. Connectedness and maximal elements

The following is a well-known result on the preservation of connectedness via multimaps, e.g., see [4, 9].

Lemma 1. *Let X, Y be non-empty topological spaces, X a connected set, and $T : X \to 2^Y$ be a multimap such that each $T(x)$ is non-empty connected. If T is either lower semicontinuous or upper semicontinuous, then the image $T(X)$ is a connected set.*

If we assume stronger continuity on T in Lemma 1, then we can prove the following by relaxing the connectedness assumption on $T(x)$ for each $x \in X$:

Proposition 1. *Let X, Y be non-empty topological spaces and $T : X \to 2^Y$ be a continuous multimap such that each $T(x)$ is non-empty. If X is connected and $T(x_0)$ is connected for some $x_0 \in X$, then the image $T(X)$ is a connected set.*

Proof. Suppose the contrary, i.e., $T(X)$ is disconnected. Then there exists a separation for $T(X)$, i.e., there exist two non-empty open sets U, V in $T(X)$ such that $U \cup V = T(X)$ and $U \cap V = \emptyset$. Since $T(x_0)$ is connected, either $T(x_0) \subseteq U$ or $T(x_0) \subseteq V$; hence we may assume $T(x_0) \subseteq U$ without loss of generality.

We now set

$$X_V := \{x \in X \mid T(x) \cap V \neq \emptyset\}$$
$$X_U := \{x \in X \mid T(x) \cap V = \emptyset\};$$

then $X = X_U \cup X_V$. Since $U \cup V = T(X)$ and $U \cap V = \emptyset$, we have that

$$X_U = \{x \in X \mid T(x) \cap V = \emptyset\} = \{x \in X \mid T(x) \subseteq U\}.$$

Then, by the lower semicontinuity of T and the assumption, it is easy to see that X_V is a non-empty open subset of X; and by the upper semicontinuity of T and the assumption that $T(x_0) \subseteq U$, X_U is a non-empty open subset of X.

In the case that $X_U \cap X_V = \emptyset$, then $\{X_U, X_V\}$ is a separation of the connected set X; which is a contradiction.

Next, we assume that there exists a point $\bar{x} \in X_U \cap X_V$. Then we obtain that

$$\bar{x} \in X_U \Rightarrow T(\bar{x}) \subseteq U \quad \text{and} \quad \bar{x} \in X_V \Rightarrow T(\bar{x}) \cap V \neq \emptyset;$$

which contradicts the fact that $U \cap V = \emptyset$. This completes the proof. □

In Theorem 1, the continuity of T and the connectedness of $T(x_0)$ are essential as seen in the following example :

Example 1. Let $X = Y = \mathbb{R}$ be the connected set and the multimap $S, T : X \to 2^X$ be defined as follows :

$$S(x) := \begin{cases} (\frac{1}{x}, 0), & \text{for each } x \in (-\infty, 0), \\ \mathbb{R} \setminus \{0\}, & \text{when } x = 0, \\ (0, \frac{1}{x}), & \text{for each } x \in (0, \infty); \end{cases}$$

$$T(x) := \begin{cases} \{-1, 1\}, & \text{for each } x \in (-\infty, 0), \\ \{1\}, & \text{for each } x \in [0, \infty). \end{cases}$$

Then all hypotheses of Theorem 1 are satisfied except the lower semicontinuity of S at 0 and the upper semicontinuity of T at 0, respectively ; but the both image sets $S(X) = \mathbb{R} \setminus \{0\}$ and $T(X) = \{-1, 1\}$ are disconnected. Therefore, the continuity of T is essential in Theorem 1. Also, the connectedness of $T(x_0)$ are essential in Theorem 1 by using a constant multimap $T : \mathbb{R} \to 2^{\mathbb{R}}$ defined by $T(x) := \{-1, 1\}$ for each $x \in \mathbb{R}$, and the connectedness of the domain X is very essential in Theorem 1.

The following simple example shows that the converse of Theorem 1 need not be true :

Example 2. Let $X = \mathbb{R}$ be the connected set and the multimap $T : X \to 2^X$ be defined as follows :

$$T(x) := [0, \infty) \cup \{-e^{-x}\}, \qquad \text{for each } x \in X.$$

Then we know that T is continuous on X and the image set $T(X) = X$ is connected. However, each $T(x)$ is disconnected. Therefore, the converse of Theorem 1 does not hold.

We shall need the following result in [1] :

Lemma 2. *Let X and Y be two topological spaces, Y be a regular space, and $T : X \to 2^Y$ be an upper semicontinuous multimap such that for each $x \in X$, $T(x)$ is non-empty closed. Then T is a closed multimap.*

For the equilibrium existence theorem, we shall prove the following existence theorem of maximal element :

Lemma 3. *Let X be a non-empty connected subset of a regular topological space E, and $T : X \to 2^E$ be upper semicontinuous at every x where $T(x) \neq \emptyset$ such that*

(1) *$T(x)$ is (possibly empty) closed for each $x \in X$;*

(2) *$T^{-1}(y_0)$ is non-empty open and $y_0 \notin T(y_0)$ for some $y_0 \in X$.*

Then T has a maximal element $\hat{x} \in X$, i.e., $T(\hat{x}) = \emptyset$.

Proof. Suppose the assertion were false. Then $T(x)$ is non-empty for each $x \in X$. By the assumption, since T is upper semicontinuous at x and $T(x)$ is closed, T is closed at x; hence the lower section $T^{-1}(y_o)$ is non-empty closed. In fact, for every net $(x_\alpha)_{\alpha\in\Gamma} \subset T^{-1}(y_o)$ with $(x_\alpha) \to x$, we have $y_o \in T(x_\alpha)$ for each $\alpha \in \Gamma$ and $(x_\alpha) \to x$, so by the closedness of T at x, we have $y_o \in T(x)$. Therefore, $x \in T^{-1}(y_o)$, and hence $T^{-1}(y_o)$ is closed. By the assumption (2), $T^{-1}(y_o)$ is also non-empty open. Therefore, by the connectedness of X, $T^{-1}(y_o) = X$. Hence we have $y_o \in T(x)$ for each $x \in X$ which contradicts the assumption (2). Therefore, T has a maximal element $\hat{x} \in X$, i.e., $T(\hat{x}) = \emptyset$. This completes the proof. □

Remarks. (i) It should be noted that in Lemma 3, we do not need any compact convexity of X nor the convex assumption on $T(x)$ in contrast to the previous many existence theorems for maximal elements (e.g., see [3, 8, 12-15]); however, we shall need the non-empty open lower section at some special point and the connectedness of the domain X.

(ii) In Lemma 3, if we assume that T is closed at every x where $T(x) \neq \emptyset$, by replacing the condition of upper semicontinuity with closed values, then we can obtain the same conclusion without assuming the regularity of E.

If we assume that each $T(x)$ is non-empty in Lemma 3, then we can obtain the following fixed point theorem :

Corollary 1. *Let X be a non-empty connected subset of a Hausdorff topological space E and $T : X \to 2^E$ be a closed multimap such that each $T(x)$ is non-empty. If $T^{-1}(y_o)$ is non-empty open for some $y_o \in E$, then $y_o \in T(x)$ for every $x \in X$. In particular, if $y_o \in X$, then y_o is a fixed point for T.*

Next we shall prove the following :

Proposition 2. *Let X be a non-empty connected subset of a regular topological space E and $T : X \to 2^E$ be upper semicontinuous such that*

(1) *$T(x)$ is non-empty closed for each $x \in X$;*

(2) *$T^{-1}(y)$ is (possibly empty) open in X for each $y \in E$.*

Then there exists a non-empty (closed) subset $K \subseteq E$ such that $T(x) = K$ for each $x \in X$, i.e., T is a constant multimap.

Furthermore, if $T(\bar{x}) \cap X \neq \emptyset$ for some $\bar{x} \in X$, then T has a fixed point $\hat{x} \in X$.

Proof. Suppose the assertion were false. Then we can find distinct points $x_0, x_1 \in X$ and $y_0 \in E$ such that $y_0 \in T(x_0) \setminus T(x_1)$. Since T is upper semicontinuous at x_0 and each $T(x)$ is closed, by repeating the same argument in the proof of Lemma 3, we can show that $T^{-1}(y_o)$ is non-empty closed. By the assumption, $T^{-1}(y_o)$ is open.

Therefore, by the connectedness of X, $T^{-1}(y_o) = X$. Hence we have $y_o \in T(x)$ for each $x \in X$ which contradicts the condition that $y_0 \notin T(x_1)$. Therefore T must be constant, i.e., there exists a non-empty subset $K \subseteq E$ such that $T(x) = K$ for each $x \in X$. Furthermore, if $\hat{x} \in T(\bar{x}) \cap X = K \cap X$, then $\hat{x} \in X$ so that $\hat{x} \in T(\hat{x})$. This completes the proof. □

Remarks. (i) In Proposition 2, the point $\bar{x} \in X$ need not be a fixed point for T but every point in $T(\bar{x}) \cap X$ must be a fixed point for T. And, if we assume that $T : X \to 2^X$ has no fixed point and T is upper semicontinuous with closed values, then we conclude that T must have a maximal element, i.e., there exists an $\hat{x} \in X$ such that $T(\hat{x}) = \emptyset$.

(ii) In Proposition 2, if we assume that T is a closed multimap in place of the upper semicontinuity having closed values, then we can obtain the same conclusion without assuming the regularity of E as in [7].

4. Existence of equilibria in N-person games

As an application of Corollary 1, we provide a new equilibrium existence theorem for a connected n-person game :

Theorem 2. *Let $\Gamma = (X_i, A_i, P_i)_{i \in I}$ be a generalized game where $I = \{1, \cdots, n\}$ is a finite set of agents such that for each $i \in I$,*

1. *X_i is a non-empty connected subset of a regular topological space E_i, and denote $X = \Pi_{i \in I} X_i$ and $E = \Pi_{i \in I} E_i$;*
2. *the multimap $A_i : X \to 2^{E_i}$ is upper semicontinuous such that $A_i(x)$ is a non-empty closed subset of E_i for each $x \in X$;*
3. *the multimap $A_i \cap P_i$ is upper semicontinuous at every x where $(A_i \cap P_i)(x) \neq \emptyset$, and $(A_i \cap P_i)(x)$ is (possibly empty) closed for each $x \in X$;*
4. *the set $W_i := \{x \in X : (A_i \cap P_i)(x) \neq \emptyset\}$ is (possibly empty) open in X, and for each $x \in W_i$, $x_i \notin P_i(x)$;*
5. *$A_i^{-1}(\bar{y}_i)$ and $(X \setminus W_i) \cup P_i^{-1}(\bar{y}_i)$ are open in X for some $\bar{y}_i \in X_i$.*

If the set $\bigcap_{i \in I} (A_i \cap P_i)^{-1}(\bar{y}_i)$ is non-empty, then $\bar{y} = (\bar{y}_1, \cdots, \bar{y}_n) \in X$ is actually an equilibrium for Γ, i.e., for each $i \in I$,

$$\bar{y}_i \in A_i(\bar{y}) \quad \text{and} \quad A_i(\bar{y}) \cap P_i(\bar{y}) = \emptyset.$$

Proof. Since each X_i is connected, $X = \Pi_{i \in I} X_i$ is also connected. Let $I_o = \{i \in I : W_i \neq \emptyset\}$. For each $i \in I_o$, we now define a multimap $\phi_i : X \to 2^{E_i}$ by

$$\phi_i(x) = \begin{cases} A_i(x), & \text{if } \ x \notin W_i, \\ (A_i \cap P_i)(x), & \text{if } \ x \in W_i. \end{cases}$$

Then for each $x \in X$, $\phi_i(x)$ is a non-empty closed subset of E_i. To show ϕ_i is upper semicontinuous, we must show that the set $U := \{x \in X : \phi_i(x) \subset V\}$ is open in X for every open subset V of E_i. Now we have

$$\begin{aligned} U &= \{x \in W_i : \phi_i(x) \subset V\} \cup \{x \in X \setminus W_i : \phi_i(x) \subset V\} \\ &= \{x \in W_i : (A_i \cap P_i)(x) \subset V\} \cup \{x \in X \setminus W_i : A_i(x) \subset V\} \\ &= \{x \in W_i : (A_i \cap P_i)(x) \subset V\} \cup \{x \in X : A_i(x) \subset V\}. \end{aligned}$$

Since W_i is open and $A_i \cap P_i$ is upper semicontinuous on W_i, U is open ; and hence ϕ_i is upper semicontinuous such that each $\phi_i(x)$ is non-empty closed. Therefore, by Lemma 3, ϕ_i is a closed multimap on X.

Next we shall show that $\phi_i^{-1}(\bar{y}_i)$ is a non-empty open subset of X. In fact, since $(X \setminus W_i) \cup P_i^{-1}(\bar{y}_i)$ is open, we have

$$\begin{aligned}
\phi_i^{-1}(\bar{y}_i) &= \{x \in X : \bar{y}_i \in \phi_i(x)\} \\
&= \{x \in W_i : \bar{y}_i \in \phi_i(x)\} \cup \{x \in X \setminus W_i : \bar{y}_i \in \phi_i(x)\} \\
&= [W_i \cap (A_i \cap P_i)^{-1}(\bar{y}_i)] \cup [(X \setminus W_i) \cap A_i^{-1}(\bar{y}_i)] \\
&= A_i^{-1}(\bar{y}_i) \cap [P_i^{-1}(\bar{y}_i) \cup ((X \setminus W_i) \cap A_i^{-1}(\bar{y}_i))] \\
&= A_i^{-1}(\bar{y}_i) \cap [P_i^{-1}(\bar{y}_i) \cup (X \setminus W_i)]
\end{aligned}$$

is non-empty open in X.

Finally, we define $\Psi : X \to 2^E$, by

$$\Psi(x) := \Pi_{i \in I}\ \psi_i(x) \quad \text{for each } x \in X,$$

where

$$\psi_i(x) = \begin{cases} \phi_i(x), & \text{if} \quad i \in I_o, \\ A_i(x), & \text{if} \quad i \notin I_o. \end{cases}$$

Since $E = \Pi_{i \in I} E_i$ is a regular space, Ψ is a closed multimap on X such that each $\Psi(x)$ is a non-empty closed subset of E. For $\bar{y} = (\bar{y}_1, \cdots, \bar{y}_n) \in X$, we now obtain

$$\begin{aligned}
x \in \Psi^{-1}(\bar{y}) &\Leftrightarrow \bar{y} \in \Psi(x) = \Pi_{i \in I} \psi_i(x) \\
&\Leftrightarrow \bar{y}_i \in \psi_i(x), \quad \text{for each } \ i \in I \\
&\Leftrightarrow x \in \psi_i^{-1}(\bar{y}_i), \quad \text{for each } \ i \in I \\
&\Leftrightarrow x \in \cap_{i \in I} \psi_i^{-1}(\bar{y}_i) \\
&\Leftrightarrow x \in \cap_{i \notin I_0} A_i^{-1}(\bar{y}_i) \bigcap \cap_{i \in I_0} \phi_i^{-1}(\bar{y}_i);
\end{aligned}$$

and hence, by the assumption, $\Psi^{-1}(\bar{y})$ is a non-empty open subset of X. Therefore, by applying Corollary 1 to Ψ, we know that $\bar{y} \in \Psi(\bar{y})$, i.e., for each $i \in I$, $\bar{y}_i \in \psi_i(\bar{y})$. For each $i \in I_o$, $\bar{y}_i \in \psi_i(\bar{y}) = \phi_i(\bar{y})$. If $\bar{y} \in W_i$, then

$$\bar{y}_i \in \phi_i(\bar{y}) = (A_i \cap P_i)(\bar{y}) \subseteq P_i(\bar{y}),$$

which is a contradiction. Therefore for each $i \in I_o$, $\bar{y} \notin W_i$, i.e., $\bar{y}_i \in \psi_i(\bar{y}) = A_i(\bar{y})$ and $(A_i \cap P_i)(\bar{y}) = \emptyset$. Next, in case $i \notin I_o$, then $W_i = \emptyset$ and $\psi_i = A_i$. Therefore $\bar{y}_i \in \psi_i(\bar{y}) = A_i(\bar{y})$ and $P_i(\bar{y}) \cap A_i(\bar{y}) = \emptyset$. This completes the proof. □

Remarks. (i) As we can see in the conclusion, W_i must be a proper subset of X, and the set $\bigcap_{i \in I}(A_i \cap P_i)^{-1}(\bar{y}_i)$ is not closed. In fact, if it is closed, then by the connectedness of X, $\bigcap_{i \in I}(A_i \cap P_i)^{-1}(\bar{y}_i) \equiv X$, and hence this contradicts the conclusion. Also, note that $\bar{y} \notin \bigcap_{i \in I}(A_i \cap P_i)^{-1}(\bar{y}_i)$.

(ii) Theorem 2 is quite different from the previous many equilibrium existence theorems (e.g. [3, 5, 8, 11-15]). In fact, in Theorem 2, we do not require compact convex assumption on the choice set X_i, but we only use the connectedness of X_i. Also we do not need the convexity of the values $A_i(x)$ and $P_i(x)$, and strong open lower section assumptions; just the weaker open lower section and intersection property at some special point are necessary.

(iii) In the assumption (5) of Theorem 4 in [7], W_i should be open as in the assumption (4) of Theorem 2, and also the assumption (5) and the non-emptiness of $\bigcap_{i\in I}(A_i \cap P_i)^{-1}(\bar{y}_i)$ of Theorem 2 should be needed to obtain the conclusion in [7].

Next, we give a simple example of a connected (non-compact non-convex) 1-person game where Theorem 2 is applicable but the previous known results are not available :

Example 3. Let $\Gamma = (X, A, P)$ be an 1-person game where $X = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x,\ 0 \leq y \leq \frac{1}{x}\}$ be a connected set and the multimaps $A, P : X \to 2^X$ be defined as follows:

$$A(x,y) := \begin{cases} \{(s,t) \in X \mid s = y, 0 \leq t \leq \dfrac{1}{y} \text{ or } 0 \leq s \leq y, t = 0\}, \\ \qquad\qquad\qquad\qquad \text{if } (x,y) \in X \text{ with } y \geq \dfrac{1}{2}, \\ \{(s,t) \in X \mid s = y, 0 \leq t \leq 2 \text{ or } 0 \leq s \leq y, t = 0\}, \\ \qquad\qquad\qquad\qquad \text{if } (x,y) \in X \text{ with } y < \dfrac{1}{2}; \end{cases}$$

$$P(x,y) := \begin{cases} \emptyset, \quad \text{if } (x,y) \in C := \{(x,y) \in X \mid 0 \leq x \leq 1,\ x \leq y\}, \\ \{(s,t) \mid s = y, 0 \leq t \leq \dfrac{1}{y} \quad \text{or} \quad 0 \leq s \leq y, t = 0\}, \\ \qquad\qquad\qquad\qquad \text{if } (x,y) \in X \setminus C \text{ with } y \geq \dfrac{1}{2}, \\ \{(s,t) \mid s = y, 0 \leq t \leq 2 \quad \text{or} \quad 0 \leq s \leq y, t = 0\}, \\ \qquad\qquad\qquad\qquad \text{if } (x,y) \in X \setminus C \text{ with } y < \dfrac{1}{2}. \end{cases}$$

Then it is easy to show that the multimap A is upper semicontinuous on X such that each $A(x, y)$ is non-empty closed, and the fixed point set $\mathcal{F}$ of A is exactly the diagonal of X, i.e., $\mathcal{F} = \{(x, x) \in \mathbb{R}^2 \mid 0 \leq x \leq 1\}$. Also we have that P is upper semicontinuous on $X \setminus C$ and $P(x, y)$ is non-empty closed at every point in $X \setminus C$. It is clear that $W = X \setminus C$ is open in X and $(x, y) \notin P(x, y)$ for each $(x, y) \in W$. Note that $A^{-1}(0, 0) = X$ and $(A \cap P)^{-1}(0, 0) = X \setminus C$ are non-empty open in X, and $(X \setminus W) \cup P^{-1}(0, 0) \equiv X$ is open in X. Therefore, all assumptions of Theorem 2 are satisfied, so that we can obtain an equilibrium point $(0, 0) \in X$ such that $(0, 0) \in A(0, 0)$ and $A(0, 0) \cap P(0, 0) = \emptyset$. Note that since X is neither compact nor convex, previous many equilibrium existence theorems (e.g., [3, 5, 8, 11-15]) are not available in this game.

In the assumption (5) of Theorem 2, if we replace the condition "$X \setminus W_i \cup P_i^{-1}(\bar{y}_i)$ is open" by "$W_i \subseteq P_i^{-1}(\bar{y}_i)$" (or "$A_i^{-1}(\bar{y}_i) \subseteq W_i$"), then we can obtain the same conclusion under weaker assumptions as follows :

Theorem 3. *Let* $\Gamma = (X_i, A_i, P_i)_{i\in I}$ *be a generalized game where* $I = \{1, \cdots, n\}$ *is a finite set of agents such that for each* $i \in I$,

(1) *X_i is a non-empty connected subset of a regular topological space E_i, and denote $X = \Pi_{i\in I} X_i$ and $E = \Pi_{i\in I} E_i$;*
(2) *the multimap $A_i : X \to 2^{E_i}$ is upper semicontinuous such that $A_i(x)$ is non-empty closed subset of X_i for each $x \in X$;*
(3) *for each $x \in W_i := \{x \in X : (A_i \cap P_i)(x) \neq \emptyset\}$, $x_i \notin P_i(x)$;*
(4) *there exists some $\bar{y}_i \in X_i$ such that $A_i^{-1}(\bar{y}_i)$ is open and $W_i \subseteq P_i^{-1}(\bar{y}_i)$.*

If $\bigcap_{i\in I} A_i^{-1}(\bar{y}_i)$ is non-empty, then Γ has an equilibrium choice $\bar{y} = (\bar{y}_1, \cdots, \bar{y}_n) \in X$, i.e., for each $i \in I$,

$$\bar{y}_i \in A_i(\bar{y}) \quad \text{and} \quad A_i(\bar{y}) \cap P_i(\bar{y}) = \emptyset.$$

Proof. Since each X_i is connected, $X = \Pi_{i\in I} X_i$ is also connected. Since each A_i is upper semicontinuous with closed values, A_i is a closed multimap on X. And note that the multimap $A := \Pi_{i\in I} A_i : X \to 2^E$, defined by $A(x) = \Pi_{i\in I} A_i(x)$ for each $x \in X$, is also a closed multimap on X. If we let $\bar{y} = (y_1, \cdots, y_n) \in X$, then by the assumption, $A^{-1}(\bar{y}) = \cap_{i\in I} A_i^{-1}(\bar{y}_i)$ is non-empty open. By applying Corollary 1 to the multimap A, we have a fixed point $\bar{y} = (y_1, \cdots, y_n) \in X$ such that $\bar{y} \in A(\bar{y})$, i.e., for each $i \in I$, $\bar{y}_i \in A_i(\bar{y})$. First consider the case that $W_i = \emptyset$ for all $i \in I$. Then $A_i(\bar{y}) \cap P_i(\bar{y}) = \emptyset$ for each $i \in I$, and hence we have done.

Suppose that I_o be the maximal non-empty subset of I such that $W_i \neq \emptyset$ for all $i \in I_o$. In case that $\bar{y} \notin W_i$ for each $i \in I_0$, then we have done. Suppose that $\bar{y} \in W_i$ for some $i \in I_0$, then by the assumption (3), $\bar{y}_i \notin P_i(\bar{y})$; however, by the assumption (4), we have $\bar{y} \in W_i \subseteq P_i^{-1}(\bar{y}_i)$, which is a contradiction. Therefore, $\bar{y} \notin W_i$ for every $i \in I_0$, which implies that $A_i(\bar{y}) \cap P_i(\bar{y}) = \emptyset$ for each $i \in I$. This completes the proof. □ □

Remark. The assumption (4) clearly implies the assumption (5) in Theorem 2. In fact, the set $(X \setminus W_i) \cup P_i^{-1}(\bar{y}_i)$ is equal to the whole set X, and hence it is open. Therefore, we can also obtain the equilibrium in Example 1 by applying Theorem 3.

As an application of Lemma 3, we shall prove another equilibrium existence theorem for a connected n-person game where the decisive agent exists :

Theorem 4. *Let $\Gamma = (X_i, A_i, P_i)_{i\in I}$ be an n-person game such that for each $i \in I = \{1, \cdots, n\}$,*

(1) *X_i is a non-empty connected subset of a regular topological space E_i, and denote $X = \Pi_{i\in I} X_i$ and $E = \Pi_{i\in I} E_i$;*
(2) *the multimap $A_i : X \to 2^{E_i}$ is upper semicontinuous such that for each $x \in X$, $A_i(x)$ is non-empty closed, and the fixed point set $\mathcal{F}(A)$ of $A = \Pi_{i\in I} A_i$ is connected ;*
(3) *there exists some $\bar{y}_i \in X_i$ such that $A_i^{-1}(\bar{y}_i)$ is non-empty open in X.*

Furthermore, assume that there exists some decisive agent $j \in I$ such that

(4) *the multimap $A_j \cap P_j : X \to 2^{E_j}$ is closed at x where $(A_j \cap P_j)(x) \neq \emptyset$;*
(5) *$P_j^{-1}(\bar{y}_j) \cap \mathcal{F}(A)$ is non-empty open in $\mathcal{F}(A)$, and $\bar{y}_j \notin P_j(\bar{y})$ for $\bar{y} = (\bar{y}_1, \cdots, \bar{y}_n)$;*
(6) *$(A_j \cap P_j)(x) = \emptyset$ implies $(A_i \cap P_i)(x) = \emptyset$ for each $i \in I$.*

Then Γ has an equilibrium choice $\hat{x} \in X$, i.e., for each $i \in I$,

$$\hat{x}_i \in A_i(\hat{x}) \quad \text{and} \quad A_i(\hat{x}) \cap P_i(\hat{x}) = \emptyset.$$

Proof. For each $i \in I$, we define multimap $(A_i \cap P_i)' : X \to 2^E$ by

$$(A_i \cap P_i)'(x) := E_1 \times \cdots \times E_{i-1} \times (A_i \cap P_i)(x) \times E_{i+1} \times \cdots \times E_n, \quad \text{for each} \quad x \in X.$$

Since each X_i is connected, $X = \Pi_{i \in I} X_i$ is also connected. By the assumption (2), using the same argument of the proof in Lemma 3, we can see that each $A_i^{-1}(\bar{y}_i)$ is closed. Hence by the assumption (3), $A_i^{-1}(\bar{y}_i) \equiv X$ since X is connected. Let the multimap $A := \Pi_{i \in I} A_i : X \to 2^E$ be defined by $A(x) := \Pi_{i \in I} A_i(x)$ for each $x \in X$. Then, for $\bar{y} = (\bar{y}_1, \cdots, \bar{y}_n) \in X$, $A^{-1}(\bar{y}) = \cap_{i \in I} A_i^{-1}(\bar{y}_i) = X$ is non-empty open. Hence we have $\bar{y} \in A(\bar{y})$, and so the fixed point set $\mathcal{F}(A) := \{x \in X \mid x \in A(x)\} \equiv \bigcap_{i \in I} \{x \in X \mid x_i \in A_i(x)\}$ is a non-empty subset of X. By the assumption (2), $\mathcal{F}(A)$ is a non-empty connected set and $\bar{y} \in \mathcal{F}(A)$. Denote $\mathcal{F}(A)$ by $\mathcal{F}$ for simplicity.

For $j \in I$, we define a multimap $\phi_j : \mathcal{F} \to 2^E$ by

$$\phi_j(x) := (A_j \cap P_j)'(x), \qquad \text{for each} \quad x \in \mathcal{F}.$$

Then, by the assumption (4), we have that ϕ_j is closed at every x where $\phi_j(x) \neq \emptyset$. In fact, for every net $(x_\alpha)_{\alpha \in \Gamma} \subset \mathcal{F}$ with $(x_\alpha) \to x$, and $y_\alpha \in \phi_j(x_\alpha)$ with $(y_\alpha) \to y$, we have $y_\alpha^j \in (A_j \cap P_j)(x_\alpha)$ with $(y_\alpha^j) \to y_j$. Here, y_α^j denotes the j-th component of the element $y_\alpha \in E$. Since $A_j \cap P_j$ is closed at x, we have $y_j \in (A_j \cap P_j)(x)$. It is clear that $y_i \in E_i$ for each $i \neq j$; and hence $y \in (A_j \cap P_j)'(x) = \phi_j(x)$. Therefore, ϕ_j is closed at every x where $\phi_j(x) \neq \emptyset$.

Next we shall show that $\phi_j^{-1}(\bar{y})$ is non-empty open in $\mathcal{F}$. In fact, since $P_j^{-1}(\bar{y}_j) \cap \mathcal{F}$ is non-empty open in $\mathcal{F}$ and $A_j^{-1}(\bar{y}_j) \equiv X$, we have

$$\begin{aligned} \phi_j^{-1}(\bar{y}) &= \{x \in \mathcal{F} : \bar{y} \in \phi_j(x)\} = \mathcal{F} \cap (A_j \cap P_j)^{-1}(\bar{y}_j) \\ &= \mathcal{F} \cap A_j^{-1}(\bar{y}_j) \cap P_j^{-1}(\bar{y}_j) = \mathcal{F} \cap P_j^{-1}(\bar{y}_j) \end{aligned}$$

is non-empty open in $\mathcal{F}$. By the assumption (5), it is clear that $\bar{y} \notin \phi_j(\bar{y})$.

Therefore, by Lemma 3, there exists a point $\hat{x} \in \mathcal{F}$ such that $\phi_j(\hat{x}) = \emptyset$. Since $\hat{x} \in \mathcal{F}$, we have $\hat{x}_i \in A_i(\hat{x})$ for each $i \in I$. Also, since $\phi_j(\hat{x}) = \emptyset$, $A_j(\hat{x}) \cap P_j(\hat{x}) = \emptyset$. By the assumption (6), we have $A_i(\hat{x}) \cap P_i(\hat{x}) = \emptyset$ for each $i \in I$. Therefore, $\hat{x} \in \mathcal{F}$ is the desired equilibrium for the game Γ. This completes the proof. □

Remarks. (i) Theorem 4 is also different from the previous many equilibrium existence theorems (e.g., [3, 5, 8, 11-15]), and note that $\bar{y} \in X$ in the assumption might be an equilibrium for the game Γ.

(ii) As we remarked, the condition (6) means that the j-th agent is decisive in this game. This is a rather strong assumption, but Theorem 4 is suitable for an incomplete market having monopolistic agent in a real economy. Also, for 1-person game, this assumption is automatically satisfied.

We now give an example of a connected n-person game where Theorem 4 is applicable but the previous known results are not available :

Example 4. Let $\Gamma = (X_i, A_i, P_i)_{i \in I}$ be an n-person game where for each $k \in I = \{1, 2, \cdots, n\}$, let $X_k = [0, \infty)$ be a connected set and the multimaps A_k, P_k :

$\Pi_{i\in I}X_i \to 2^{\mathbb{R}}$ be defined as follows: for each $x \in X = \Pi_{i\in I}X_i$,

$$A_k(x) := \Big[0, \frac{2k+1}{3k}\Big]$$

$$P_k(x) := \begin{cases} \Big\{\dfrac{k+1}{2k}\Big\}, & \text{if } x \in \Pi_{i\in I}\Big[0, \dfrac{2i+1}{3i}\Big), \\ \Big[0, \dfrac{k+1}{2k}\Big], & \text{if } x \in \Pi_{i\in I}(1,2), \\ \{2\}, & \text{if } x = (2,2,\cdots,2), \\ \Big\{\dfrac{2k+1}{3k}\Big\}, & \text{if } x \in \Pi_{i\in I}(2,\infty), \\ \emptyset, & \text{otherwise.} \end{cases}$$

Then it is easy to show that the multimap A_k is upper semicontinuous on X such that each $A_k(x)$ is non-empty closed, and the fixed point set $\mathcal{F}(A)$ of $A = \Pi_{k\in I}A_k$ is equal to $\Pi_{k\in I}[0, \frac{2k+1}{3k}]$. Also it is easy to check that for each $k \in I$, $A_k \cap P_k$ is closed at every point x where $A_k \cap P_k(x) \neq \emptyset$. If we let $\bar{y} := (1, \frac{3}{4}, \frac{4}{6}, \cdots, \frac{n+1}{2n}) \in X$, then $P_1^{-1}(\bar{y}_1) \cap \mathcal{F}(A) \equiv \Pi_{i\in I}[0, \frac{2i+1}{3i})$ is non-empty open in $\mathcal{F}(A)$, and $1 = \bar{y}_1 \notin P_1(\bar{y}) = \emptyset$. Note that each $A_k^{-1}(\bar{y}_k)$ is equal to X, and hence $A_k^{-1}(\bar{y}_k)$ is non-empty open. And we know that the preference multimap P_k is decreasing and the first agent is decisive in this game. In fact, if $(A_1 \cap P_1)(x) = \emptyset$, then $(A_k \cap P_k)(x) = \emptyset$ for each $k \in I$. Hence the assumptions (4)-(6) are satisfied for the first index $1 \in I$. Therefore, all assumptions of Theorem 4 are satisfied, so that we can obtain an equilibrium point $\hat{x} = (1, \frac{5}{6}, \frac{7}{9}, \cdots, \frac{2n+1}{3n}) \in X$ such that $\hat{x}_k \in A_k(\hat{x})$ and $A_k(\hat{x}) \cap P_k(\hat{x}) = \emptyset$ for each $k \in I$. Finally, note that $\bar{y} \in A(\bar{y})$ and $(A_k \cap P_k)(\bar{y}) = \emptyset$ for each $k \in I$; and hence $\bar{y}$ can be another equilibrium for this game.

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