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ON THE BEST CONSTANT OF HYERS-ULAM STABILITY

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Dedicated to Professor Ryuichi Ito for his 60th birthday (kanreki)

Abstract. We give a necessary and sufficient condition that the best constant of Hyers-Ulam stability exists. Using the condition, we show that the best constants exist for the weighted composition operators and the first order linear differential operators.

1. Introduction

In 1941, D. H. Hyers solved the stability problem on an approximately additive mapping between two Banach spaces, which had been posed by S. M. Ulam (cf. [4], [13], [14]). From then, various stability problems have studied by many mathematicians (see the references in [5], [6], [7]). In [6], the common property of those problems is formulated as follows:

Definition. Let A and B be linear spaces with gauges ρ_A and ρ_B, respectively. Here a gauge on a linear space L means a function $\rho : L \to [0, \infty]$ satisfying $\rho(\lambda f) = |\lambda|\rho(f)$ for every $f \in L$ and scalar λ. Let T be a mapping from A into B. We say that T has the *Hyers-Ulam stability* if the following condition (1) holds for some constant K:

(1) For any $g_0 \in T(A)$, $\varepsilon > 0$ and $f \in A$ with $\rho_B(Tf - g_0) \leq \varepsilon$, there exists an $f_0 \in A$ such that $Tf_0 = g_0$ and $\rho_A(f - f_0) \leq K\varepsilon$.

We call K *a HUS constant* for T, and denote the infimum of all HUS constants for T by K_T.

It is natural to ask the question: *Is K_T a HUS constant for T?* In this paper, we give an answer to this question when T is linear. If the answer is "Yes", then K_T becomes the best constant of Hyers-Ulam stability, say *the HUS constant* for T.

2. Result

Let A and B be linear spaces with gauges ρ_A and ρ_B, respectively. Let T be a linear operator from A into B. By $\mathcal{N}(T)$, we denote the kernel of T, namely, $\mathcal{N}(T) = \{ h \in A : Th = 0 \}$. For $f \in A$ and $M \subset A$, we put $\mathrm{dist}_A(f, M) = \inf\{ \rho_A(f - h) : h \in M \}$. Our main result is the following:

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Theorem. *Let A, B be linear spaces with gauges ρ_A, ρ_B, respectively, and T be a linear operator from A into B. Suppose that T has the Hyers-Ulam stability. Then K_T is the HUS constant if and only if*

(2) *for any $f \in A$ with $\operatorname{dist}_A(f, \mathcal{N}(T)) = K_T$ and $\rho_B(Tf) \le 1$, there exists an $h \in \mathcal{N}(T)$ such that $\rho_A(f - h) = \operatorname{dist}_A(f, \mathcal{N}(T))$.*

Proof. We first observe that K is a HUS constant for T if and only if

(3) for any $f \in A$ with $\rho_B(Tf) \le 1$, there exists an $f_0 \in \mathcal{N}(T)$ such that $\rho_A(f - f_0) \le K$.

In fact, the linearity of T shows that (3) is equivalent to (1). Hence, if K is a HUS constant for T, then $\operatorname{dist}_A(f, \mathcal{N}(T)) \le K$ for all $f \in A$ with $\rho_B(Tf) \le 1$. If K is taken over all HUS constants for T, then we obtain

$$\operatorname{dist}_A(f, \mathcal{N}(T)) \le K_T \quad \text{for all } f \in A \text{ with } \rho_B(Tf) \le 1. \tag{4}$$

Suppose that K_T is the HUS constant for T. By the above observation,

(5) for any $f \in A$ with $\rho_B(Tf) \le 1$, there exists an $f_0 \in \mathcal{N}(T)$ such that $\rho_A(f - f_0) \le K_T$.

Of course, for any $f \in A$ with $\operatorname{dist}_A(f, \mathcal{N}(T)) = K_T$ and $\rho_B(Tf) \le 1$, we find an $h \in \mathcal{N}(T)$ such that $\rho_A(f - h) \le K_T$. Then we have $\operatorname{dist}_A(f, \mathcal{N}(T)) \le \rho_A(f - h) \le K_T = \operatorname{dist}_A(f, \mathcal{N}(T))$, and so $\rho_A(f - h) = \operatorname{dist}_A(f, \mathcal{N}(T))$. Thus we obtain (2).

Conversely, suppose that (2) holds. To verify that K_T is the HUS constant, it suffices to show (5), by the first observation. Choose $f \in A$ so that $\rho_B(Tf) \le 1$. By (4), $\operatorname{dist}_A(f, \mathcal{N}(T)) \le K_T$. If $\operatorname{dist}_A(f, \mathcal{N}(T)) < K_T$, then the definition of dist_A shows that there exists an $h \in \mathcal{N}(T)$ such that $\rho_A(f - h) < K_T$. On the other hand, if $\operatorname{dist}_A(f, \mathcal{N}(T)) = K_T$, then (2) says that there exists an $h \in \mathcal{N}(T)$ such that $\rho_A(f - h) = \operatorname{dist}_A(f, \mathcal{N}(T)) = K_T$. Thus we obtain (5), which was to be proved. □

Let us consider the case that A is a normed space, where the gauge ρ_A is a norm of A. A subspace M of A is said to be *proximinal*, if for any $f \in A$, there exists an $h \in M$ such that $\|f - h\| = \operatorname{dist}_A(f, M)$ $(= \inf\{\|f - g\| : g \in M\})$. If A is reflexive and M is closed, or if M is finite-dimensional, then M is proximinal (see [2, §V.4]). The next corollary is an immediate consequence of Theorem.

Corollary. *Let T be a linear operator from a normed space into a linear space with gauge. Suppose that T has the Hyers-Ulam stability. If $\mathcal{N}(T)$ is proximinal, then K_T is the HUS constant for T.*

We remark that K_T is not necessarily a HUS constant for T. Here is such an example.

Example. It is known that some Banach space A admits a closed subspace M which is not proximinal. Then there exists an $f_0 \in A$ such that no $h \in M$ satisfies $\|f_0 - h\| = \operatorname{dist}_A(f_0, M)$. Without loss of generality, we may assume $\operatorname{dist}_A(f_0, M) = 1$. For example, take $A = C[0, 1]$, the Banach space of all continuous functions on $[0, 1]$ with the supremum norm, $f_0(x) = 4x$ and $M = \{\, h \in C[0, 1] : \int_0^{1/2} h(x)\, dx = \int_{1/2}^1 h(x)\, dx \,\}$ ([2, Example V.4.8]).

Let B be the quotient space A/M with the usual norm; $\|f+M\| = \inf\{\, \|f+h\| : h \in M\,\} = \mathrm{dist}_A(f, M)$. Define an operator T from A into B by $Tf = f + M$ for all $f \in A$. Then $\mathcal{N}(T) = M$. If $\tilde{T}$ is the one-to-one operator from $A/\mathcal{N}(T)$ into B induced by T, then

$$\tilde{T}(f + \mathcal{N}(T)) = Tf = f + M = f + \mathcal{N}(T)$$

for all $f \in A$. In other words, $\tilde{T}$ is the identity operator of A/M. Hence $\tilde{T}$ is invertible and $\|\tilde{T}^{-1}\| = 1$. By [10, Theorem 2], T has the Hyers-Ulam stability and $K_T = \|\tilde{T}^{-1}\| = 1$. Thus we have

$$\mathrm{dist}_A(f_0, \mathcal{N}(T)) = \mathrm{dist}_A(f_0, M) = 1 = K_T,$$
$$\|Tf_0\| = \|f_0 + M\| = \mathrm{dist}_A(f_0, M) = 1.$$

But there is no $h \in \mathcal{N}(T) = M$ such that $\|f_0 - h\| = \mathrm{dist}_A(f_0, \mathcal{N}(T))$. Hence Theorem shows that K_T is not a HUS constant for T.

3. Application to weighted composition operators

In this section, we apply Corollary to weighted composition operators. The studies about weighted composition operators may be found in the book [9].

By $C(X)$, we denote the Banach space of all continuous functions on a compact Hausdorff space X with the supremum norm. We deal with two spaces $C(X)$ and $C(Y)$, where X and Y are compact Hausdorff spaces. For any $u \in C(Y)$, we put $S(u) = \{\, y \in Y : u(y) \neq 0\,\}$. Take a function $u \in C(Y)$ and a mapping φ form Y into X which is continuous on $S(u)$. Then u and φ induce a bounded linear operator uC_φ from $C(X)$ into $C(Y)$ defined by

$$(uC_\varphi f)(y) = u(y)\, f(\varphi(y)) \qquad (y \in Y) \tag{6}$$

for all $f \in C(X)$. We call uC_φ a *weighted composition operator* from $C(X)$ into $C(Y)$. As in [10, Theorem 3], we characterize the Hyers-Ulam stability of uC_φ, as follows:

Theorem A ([10, Theorem 3])**.** *Let uC_φ be a weighted composition operator from $C(X)$ into $C(Y)$. Then uC_φ has the Hyers-Ulam stability if and only if there exists a positive constant r such that*

$$\varphi(\{y \in Y : |u(y)| \geq r\}) = \varphi(S(u)). \tag{7}$$

If uC_φ has the Hyers-Ulam stability, then K_{uC_φ} is the reciprocal of the supremum of all r satisfying (7)*.*

Theorem A does not mention whether K_{uC_φ} is the HUS constant of uC_φ. The next proposition answers this question.

Proposition 1. *Let uC_φ be as in Theorem A. If uC_φ has the Hyers-Ulam stability, then K_{uC_φ} is the HUS constant for uC_φ.*

Proof. Pick $f \in C(X)$. As in [10, Lemma], we can show that

$$\|f + \mathcal{N}(uC_\varphi)\| = \sup\{\, |f(x)| : x \in \varphi(S(u))\,\},$$

where the left side is the norm of $f + \mathcal{N}(uC_\varphi)$ in the quotient space $C(X)/\mathcal{N}(uC_\varphi)$. Let F be the closure of $\varphi(S(u))$. We use Tietze's extension theorem ([8, Theorem 20.4]) for the restriction of f to F, and we find a $g \in C(X)$ such that

$$g(x) = f(x) \quad (x \in F) \quad \text{and} \quad \|g\| = \sup\{\,|f(x)| : x \in F\,\}.$$

Hence we have

$$\begin{aligned}\operatorname{dist}_{C(X)}(f, \mathcal{N}(uC_\varphi)) = \|f + \mathcal{N}(uC_\varphi)\| &= \sup\{\,|f(x)| : x \in \varphi(S(u))\,\} \\ &= \sup\{\,|f(x)| : x \in F\,\} = \|g\| = \|f - (f - g)\|.\end{aligned}$$

Moreover, $(f - g)(x) = 0$ for all $x \in \varphi(S(u))$, which implies $f - g \in \mathcal{N}(uC_\varphi)$. Thus we establish the existence of a function $h \in \mathcal{N}(uC_\varphi)$ such that $\|f - h\| = \operatorname{dist}_{C(X)}(f, \mathcal{N}(uC_\varphi))$. Hence $\mathcal{N}(uC_\varphi)$ is proximinal. By Corollary, K_{uC_φ} is the HUS constant for uC_φ. □

Next, we generalize Proposition 1 by considering uniform algebras instead of $C(X)$ and $C(Y)$. A *uniform algebra* (or a *function algebra*) on X means a uniformly closed subalgebra of $C(X)$ which contains the constants and separates the points of X. Let $\mathcal{A}$ and $\mathcal{B}$ be uniform algebras on X and Y, respectively. For any subset E of X, we put $\ker E = \{\, f \in \mathcal{A} : f(x) = 0 \text{ for all } x \in E\,\}$ and $\overline{E}^{\mathcal{A}} = \{\, x \in X : f(x) = 0 \text{ for all } f \in \ker E\,\}$. The set $\overline{E}^{\mathcal{A}}$ is nothing but the closure of E with respect to the hull-kernel topology on X. A closed subset F of X is called a *peak set* for $\mathcal{A}$, if there exists an $f \in \mathcal{A}$ such that $f(x) = 1$ for $x \in F$ and $|f(x)| < 1$ for $x \in X \setminus F$. The intersection of some collection of peak sets for $\mathcal{A}$ is called a *generalized peak set* (or a *peak set in the weak sense*) for $\mathcal{A}$. For the details on uniform algebras, see the books [1], [3].

Fix a function $u \in \mathcal{B}$ and a mapping φ from Y into X which is continuous on the set $S(u)$. Then the equation (6) with $f \in \mathcal{A}$ defines a bounded linear operator uC_φ from $\mathcal{A}$ into $C(Y)$. If uC_φ maps $\mathcal{A}$ into $\mathcal{B}$, then we call uC_φ a *weighted composition operator* from $\mathcal{A}$ into $\mathcal{B}$. The Hyers-Ulam stability of this type of operator is investigated in [11]. Here we show the following fact:

Proposition 2. *Let uC_φ be a weighted composition operator from $\mathcal{A}$ into $\mathcal{B}$. Suppose that there is a generalized peak set F for $\mathcal{A}$ such that $\varphi(S(u)) \subset F \subset \overline{\varphi(S(u))}^{\mathcal{A}}$. If uC_φ has the Hyers-Ulam stability, then K_{uC_φ} is the HUS constant for uC_φ.*

Proof. Pick $f \in \mathcal{A}$. As in [11, Lemma 2], we can show that

$$\|f + \mathcal{N}(uC_\varphi)\| = \sup\{\,|f(x)| : x \in F\,\}.$$

Put $\alpha = \sup\{\,|f(x)| : x \in F\,\}$ and $G = \{\, x \in X : |f(x)| \le \alpha\,\}$. Then G is a G_δ-set containing F. Since F is a generalized peak set, there is a peak set F' such that $F \subset F' \subset G$. We use [1, Theorem 2.4.1] to find a $g \in \mathcal{A}$ such that

$$g(x) = f(x) \quad (x \in F') \quad \text{and} \quad |g(x)| < \|g\| \quad (x \in X \setminus F').$$

Then we have

$$\begin{aligned}\|g\| &= \sup\{\,|g(x)| : x \in F'\,\} = \sup\{\,|f(x)| : x \in F'\,\} \\ &= \sup\{\,|f(x)| : x \in F\,\},\end{aligned}$$

where the last equality follows from $\sup\{\,|f(x)| : x \in F\,\} = \alpha = \sup\{\,|f(x)| : x \in G\,\}$ and $F \subset F' \subset G$. By repeating the argument in the proof of Proposition 1, we see that there exists an $h \in \mathcal{N}(uC_\varphi)$ such that $\|f - h\| = \operatorname{dist}_{\mathcal{A}}(f, \mathcal{N}(uC_\varphi))$. Hence $\mathcal{N}(uC_\varphi)$ is proximinal. By Corollary, K_{uC_φ} is the HUS constant for uC_φ. □

Let us consider the case that $\mathcal{A} = \mathcal{B} = A(\mathbb{D})$: the disc algebra. Let $\mathbb{D}$ be the open unit disc in the complex plane and let $\overline{\mathbb{D}}$ and $\mathbb{T}$ be its closure and boundary, respectively. The disc algebra $A(\mathbb{D})$ is the uniform algebra of all continuous complex functions on $\overline{\mathbb{D}}$ which are analytic on $\mathbb{D}$. Let $u, \varphi \in A(\mathbb{D})$ and suppose $\varphi(\overline{\mathbb{D}}) \subset \overline{\mathbb{D}}$. Then a *weighted composition operator* uC_φ on $A(\mathbb{D})$ is defined by

$$(uC_\varphi f)(z) = u(z)\, f(\varphi(z)) \qquad (z \in \overline{\mathbb{D}})$$

for all $f \in A(\mathbb{D})$. In order to exclude the trivial case, we assume that u is nonzero and that φ is nonconstant on $\overline{\mathbb{D}}$. Under this assumption, uC_φ is said to be *nontrivial*.

Theorem B ([11, Corollary 2]). *Let uC_φ be a nontrivial weighted composition operator on $A(\mathbb{D})$. Then uC_φ has the Hyers-Ulam stability if and only if there exists a positive constant r such that*

$$\varphi(\{z \in \mathbb{T} : |u(x)| \geq r\}) \supset \mathbb{T}. \tag{8}$$

If uC_φ has the Hyers-Ulam stability, then K_{uC_φ} is the reciprocal of the supremum of all r satisfying (8)*.*

We here show the following:

Proposition 3. *Let uC_φ be as in Theorem B. If uC_φ has the Hyers-Ulam stability, then K_{uC_φ} is the HUS constant for uC_φ.*

Proof. For a nontrivial weighted composition operator uC_φ on $A(\mathbb{D})$, we have $\mathcal{N}(uC_\varphi) = \{0\}$, and so $\mathcal{N}(uC_\varphi)$ is proximinal. Hence the result follows from Corollary. □

4. Application to differential operator

Let $C(\mathbb{R})$ be the linear space of all continuous functions on the real line $\mathbb{R}$, and $C^1(\mathbb{R})$ the subspace of $C(\mathbb{R})$ consisting of differential functions on $\mathbb{R}$ whose derivatives are continuous on $\mathbb{R}$. For any $f \in C(\mathbb{R})$, we put $\|f\|_\infty = \sup\{\,|f(x)| : x \in \mathbb{R}\,\}$, admitting the value ∞. Clearly, $\|\ \|_\infty$ is a gauge on $C(\mathbb{R})$ and $C^1(\mathbb{R})$.

For any $u \in C(\mathbb{R})$, we define a linear differential operator T_u from $C^1(\mathbb{R})$ into $C(\mathbb{R})$ by

$$(T_u f)(t) = f'(t) + u(t)f(t) \qquad (t \in \mathbb{R}) \tag{9}$$

for all $f \in C^1(\mathbb{R})$. In [7] and [12], we completely describe the Hyers-Ulam stability of T_u. The next fact is one of the results.

Proposition 4 ([7], [12]). *Let T_u be the linear differential operator defined by* (9)*. If T_u has the Hyers-Ulam stability, then K_{T_u} is the HUS constant for T_u.*

We here deduce Proposition 4 from Theorem. To do this, we recall the following fact (cf. [7], [12]): If we set $\tilde{u}(t) = 1\big/\exp\int_0^t u(s)\,ds$ for $t \in \mathbb{R}$, then

$$\mathcal{N}(T_u) = \{\, c\,\tilde{u} : c \text{ is any scalar}\,\}. \tag{10}$$

Proof. By Theorem, it is enough to show that if $f \in C^1(\mathbb{R})$ and $\mathrm{dist}_{C^1(\mathbb{R})}(f, \mathcal{N}(T_u)) = K_{T_u}$, then there exists an $h \in \mathcal{N}(T_u)$ such that $\|f - h\|_\infty = \mathrm{dist}_{C^1(\mathbb{R})}(f, \mathcal{N}(T_u))$ (we need not assume that $\|T_u f\|_\infty \le 1$). Let f be as above. Since K_{T_u} is finite by hypothesis, the definition of $\mathrm{dist}_{C^1(\mathbb{R})}$ gives a sequence $\{h_n\}$ in $\mathcal{N}(T_u)$ such that $K_{T_u} \le \|f - h_n\|_\infty < K_{T_u} + 1/n$. By (10), each h_n is written as $h_n = c_n \tilde{u}$ for some scalar c_n. Noting $\tilde{u}(0) = 1$, we have

$$|c_n| = |h_n(0)| \le |f(0)| + |f(0) - h_n(0)| \\ \le |f(0)| + \|f - h_n\|_\infty < |f(0)| + K_{T_u} + 1,$$

for $n = 1, 2, \ldots$. Hence $\{c_n\}$ is a bounded sequence of scalars, and so it has a subsequence $\{c_{n'}\}$ converging to some scalar c. Put $h = c\,\tilde{u}$. By (10), h is in $\mathcal{N}(T_u)$. Let m be an arbitrary positive integer. For each $t \in \mathbb{R}$ and $n' \ge m$,

$$|f(t) - c_{n'}\tilde{u}(t)| \le \|f - h_{n'}\|_\infty < K_{T_u} + \frac{1}{n'} \le K_{T_u} + \frac{1}{m}.$$

Letting $n' \to \infty$, we obtain $|f(t) - c\tilde{u}(t)| \le K_{T_u} + 1/m$ for all $t \in \mathbb{R}$. Hence $\|f - h\|_\infty \le K_{T_u} + 1/m$. Since m was arbitrary, we get $\|f - h\|_\infty \le K_{T_u}$, and so $\|f - h\|_\infty = \mathrm{dist}_{C^1(\mathbb{R})}(f, \mathcal{N}(T_u))$. Thus h is the desired function, and the proposition was proved. □

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