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A NOTE ON THE VON NEUMANN ALTERNATING PROJECTIONS ALGORITHM

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Abstract. We present an elementary geometric proof of von Neumann's classical theorem on alternating orthogonal projections as well as several results regarding this algorithm and its additive counterpart. The nonlinear case is also discussed.

1. Introduction

Let S_1 and S_2 be two closed subspaces of a real Hilbert space $(H, \langle\cdot,\cdot\rangle)$, and let $P_1 : H \to S_1$ and $P_2 : H \to S_2$ be the corresponding orthogonal projections of H onto S_1 and S_2, respectively. Denote by $\mathbb{N} = \{0, 1, 2, \dots\}$ the set of nonnegative integers. Let x_0 be an arbitrary point in H, and define the sequence of alternating projections by

$$x_{2n+1} = P_1 x_{2n} \quad \text{and} \quad x_{2n+2} = P_2 x_{2n+1}, \tag{1}$$

where $n \in \mathbb{N}$.

Theorem 1.1. *The sequence $\{x_n : n = 0, 1, 2, \dots\}$ defined by (1) converges in norm to Px_0, where $P : H \to S$ is the orthogonal projection of H onto the intersection $S = S_1 \cap S_2$.*

This classical result is due to von Neumann; see [Ne], p. 475. It was rediscovered by several other authors; see, for example, [Ar], [Na] and [Wi]. More information concerning this theorem and its many applications can be found in [De] and the references mentioned there. Our main aim in this note is to present an elementary proof of von Neumann's theorem. Our proof is geometric in nature and has some points in common with the proofs presented in [DR], [Sa], [De] and [BMR]. In addition, we present several results concerning von Neumann's algorithm (Section 2) and its additive counterpart (Section 3). The nonlinear case is discussed in Section 4.

2. Products

We begin with a simple lemma on (random) infinite products of projections which may be of independent interest.

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Let $\{S_j : j = 1, 2, \dots, m\}$ be m closed subspaces of H, and let $P_j : H \to S_j$ be the corresponding orthogonal projections of H onto S_j, $j = 1, 2, \dots, m$. Let $r : \{1, 2, \dots\} \to \{1, 2, \dots, m\}$ be a surjective mapping which assumes each one of its values infinitely often. Let x_0 be an arbitrary point in H, and define the sequence $\{x_n : n \in \mathbb{N}\}$ by

$$x_{n+1} = P_{r(n+1)} x_n, \quad n \in \mathbb{N}. \tag{2}$$

Furthermore, let S denote the intersection $\bigcap\{S_j : j = 1, 2, \dots, m\}$ and let $P : H \to S$ be the orthogonal projection of H onto S.

Lemma 2.1. *Let the sequence $\{x_n : n = 0, 1, 2, \dots\}$ be defined by* (2). *Then*

(a) $\langle x_n, s\rangle = \langle x_0, s\rangle$ *for all* $s \in S$ *and* $n \in \mathbb{N}$.
(b) $Px_n = Px_0$ *for all* $n \in \mathbb{N}$.
(c) *If* $\{x_n : n = 0, 1, 2, \dots\}$ *converges weakly to* x_∞, *then* $x_\infty = Px_0$.

Proof. (a) If $\langle x_n, s\rangle = \langle x_0, s\rangle$ for some $n \in \mathbb{N}$ and $s \in S$, then $\langle x_{n+1}, s\rangle = \langle P_{r(n+1)} x_n, s\rangle = \langle x_n, P_{r(n+1)} s\rangle = \langle x_n, s\rangle = \langle x_0, s\rangle$. Therefore this part follows by induction.
(b) Using part (a), we have, for any $s \in S$ and $n \in \mathbb{N}$, $\langle x_n - Px_0, s\rangle = \langle x_n, s\rangle - \langle Px_0, s\rangle = \langle x_0, s\rangle - \langle x_0, Ps\rangle = \langle x_0, s\rangle - \langle x_0, s\rangle = 0$.
(c) Part (b) shows that $Px_\infty = Px_0$. Since $x_\infty \in S$, $Px_\infty = x_\infty = Px_0$, as claimed. Alternatively, we get from part (a) that $\langle x_0 - x_\infty, s\rangle = 0$ for all $s \in S$. Since $x_\infty \in S$, it must coincide with Px_0. □

Remark 2.2. As a matter of fact, it is known [AA] that the sequence $\{x_n : n \in \mathbb{N}\}$ defined by (2) does converge weakly.

Returning to the sequence $\{x_n : n = 0, 1, 2, \dots\}$ defined by (1), we first note that

$$|x_n|^2 = |x_n - x_{n+1}|^2 + |x_{n+1}|^2, \quad n \in \mathbb{N}, \tag{3}$$

by the Pythagorean Theorem. Here $|x| = \sqrt{\langle x, x\rangle}$ denotes the norm of $x \in H$ induced by the inner product.

Now we formulate and prove our key lemma. Given two integers $l \geq k \geq 1$, we denote by $m = m(k, l) = [(k + l + 1)/2]$ the integer part of $(k + l + 1)/2$; that is, $m = (k + l)/2$ if $k + l$ is even and $m = (k + l + 1)/2$ if $k + l$ is odd.

Lemma 2.3. *Let the sequence $\{x_n : n \in \mathbb{N}\}$ be defined by* (1), *let $l \geq k \geq 1$ be integers, and let $m = m(k, l) = [(k + l + 1)/2]$. Then*

$$|x_k - x_l|^2 = |x_k|^2 - 2|x_m|^2 + |x_l|^2. \tag{4}$$

Proof. We use induction on $n = l - k$. It is clear that (4) is true for $n = 0$ (that is, $k = l$) and $n = 1$ (by (3)). Now suppose (4) holds for some $n \in \mathbb{N}$ and assume that $l - k = n + 1$.

If n is even, then x_{k+1} and x_l belong to the same subspace and therefore the inner product $\langle x_k - x_{k+1}, x_{k+1} - x_l\rangle = 0$. Also, $m(k+1, l) = m(k, l) = (k + l + 1)/2$. Hence

$$\begin{aligned} |x_k - x_l|^2 &= |x_k - x_{k+1} + x_{k+1} - x_l|^2 \\ &= |x_k - x_{k+1}|^2 + |x_{k+1} - x_l|^2 \end{aligned}$$

$$= |x_k|^2 - |x_{k+1}|^2 + |x_{k+1}|^2 - 2|x_{m(k+1,l)}|^2 + |x_l|^2$$
$$= |x_k|^2 - 2|x_{m(k,l)}|^2 + |x_l|^2,$$

where we have used (3) and the induction hypothesis.

If n is odd, then the points x_k and x_l belong to the same subspace, the inner product $\langle x_{l-1} - x_l, x_k - x_l \rangle = 0$, and $m(k, l-1) = m(k, l) = (k+l)/2$. Hence, again by the induction hypothesis and (3),

$$|x_k - x_{l-1}|^2 = |x_k - x_l + x_l - x_{l-1}|^2$$
$$= |x_k - x_l|^2 + |x_l - x_{l-1}|^2,$$

and so

$$|x_k - x_l|^2 = |x_k - x_{l-1}|^2 - |x_l - x_{l-1}|^2$$
$$= |x_k|^2 - 2|x_{m(k,l-1)}|^2 + |x_{l-1}|^2 - (|x_{l-1}|^2 - |x_l|^2)$$
$$= |x_k|^2 - 2|x_{m(k,l)}|^2 + |x_l|^2,$$

as required. □

Proof of Theorem 1.1. Since the numerical sequence $\{|x_n| : n \in \mathbb{N}\}$ is decreasing and $k \leq m(k,l) \leq l$, equality (4) immediately implies the inequality

$$(5) \qquad |x_k - x_l|^2 \leq |x_k|^2 - |x_l|^2, \quad l \geq k \geq 1.$$

Thus the sequence $\{x_n : n \in \mathbb{N}\}$ is Cauchy and converges strongly to Px_0 by part (c) of Lemma 2.1. □

The following rather curious proposition is also a consequence of Lemma 2.3.

Proposition 2.4. *Let the sequence* $\{x_n : n \in \mathbb{N}\}$ *be defined by* (1). *Then*

$$\sum_{n=1}^{\infty} |x_n - x_{n+2}|^2 = |x_1|^2 - |x_2|^2.$$

Proof. Using induction, we obtain from Lemma 2.3

$$\sum_{n=1}^{N} |x_n - x_{n+2}|^2 = |x_1|^2 - |x_2|^2 - |x_{N+1}|^2 + |x_{N+2}|^2,$$

for any $N \geq 2$, whence the result follows when $N \to \infty$. □

In a similar vein we note another such result.

Proposition 2.5. *Let the sequence* $\{x_n : n \in \mathbb{N}\}$ *be defined by* (1). *Then*

$$\sum_{n=1}^{\infty} |x_n - x_{n+4}|^2 = |x_1|^2 + |x_2|^2 - |x_3|^2 - |x_4|^2.$$

Proof. Again using induction and Lemma 2.3, we get this time

$$\sum_{n=1}^{N} |x_n - x_{n+4}|^2 = |x_1|^2 + |x_2|^2 - |x_3|^2 - |x_4|^2$$
$$- |x_{N+1}|^2 - |x_{N+2}|^2 + |x_{N+3}|^2 + |x_{N+4}|^2$$

for any $N \geq 4$. □

More generally, the following identity can also be established by induction.

Theorem 2.6. *Let the sequence* $\{x_n : n \in \mathbb{N}\}$ *be defined by* (1). *Then for each* $k \geq 1$,

$$\sum_{n=1}^{\infty} |x_n - x_{n+2k}|^2 = \sum_{i=1}^{k} |x_i|^2 - \sum_{i=1}^{k} |x_{k+i}|^2.$$

3. Sums

Instead of composing P_1 and P_2 we can also form their (midpoint) average and, starting from an arbitrary point $y_0 \in H$, consider the sequence $\{y_n : n \in \mathbb{N}\}$ defined by

$$y_{n+1} = (P_1 y_n + P_2 y_n)/2. \tag{6}$$

More generally, consider again $\{S_j : j = 1, 2, \ldots, m\}$, m closed subspaces of H, with corresponding orthogonal projections $P_j : H \to S_j$, $j = 1, 2, \ldots, m$, and let $\{a_j : j = 1, 2, \ldots, m\}$ be m positive numbers such that $\sum_{j=1}^{m} a_j = 1$. Given a point $y_0 \in H$, define the sequence $\{y_n : n = 0, 1, 2, \ldots\}$ by

$$y_{n+1} = \sum_{j=1}^{m} a_j P_j y_n, \quad n \in \mathbb{N}. \tag{7}$$

As we shall see shortly, the following theorem (cf. [La] and [Re2]) can be deduced from Theorem 1.1.

Theorem 3.1. *The sequence* $\{y_n : n = 0, 1, 2, \ldots\}$ *defined by* (7) *converges in norm to* Py_0, *where* $P : H \to S$ *is the orthogonal projection of* H *onto the intersection* $S = \bigcap\{S_j : j = 1, 2, \ldots, m\}$.

Proof. We equip the m-th power H^m of H (that is, $H^m = H \times H \times \cdots \times H$, m times) with the inner product

$$\langle (u_1, u_2, \ldots, u_m), (v_1, v_2, \ldots, v_m) \rangle = \sum_{j=1}^{m} a_j \langle u_j, v_j \rangle,$$

and consider the two closed subspaces $S_1 \times S_2 \times \cdots \times S_m$ and $D = \{(u, u, \ldots, u) \in H^m : u \in H\}$ of H^m with the corresponding orthogonal projections (cf. [Pi])

$$Q_1 : H^m \to S_1 \times S_2 \times \cdots \times S_m$$

and

$$Q_2 : H^m \to D.$$

Since

$$Q_1(u_1, u_2, \ldots, u_m) = (P_1 u_1, P_2 u_2, \ldots, P_m u_m)$$

and

$$Q_2(u_1, u_2, \ldots, u_m) = (v, v, \ldots, v),$$

where $v = \sum_{j=1}^{m} a_j u_j$, we have

$$z_n := (y_n, y_n, \ldots, y_n) = (Q_2Q_1)^n (y_0, y_0, \ldots, y_0) = (Q_2Q_1)^n z_0$$

for all $n \in \mathbb{N}$.
Therefore Theorem 1.1 implies that the sequence $\{z_n : n \in \mathbb{N}\}$ converges in norm to Qz_0, where Q is the orthogonal projection of H^m onto the intersection

$$(S_1 \times S_2 \times \cdots \times S_m) \cap D = S \times S \times \cdots \times S.$$

In other words, the sequence $\{y_n : n \in \mathbb{N}\}$ converges in norm to Py_0, as asserted. □

Applying Proposition 2.4 and Proposition 2.5, we now obtain the following result for the sequence $\{y_n : n \in \mathbb{N}\}$ defined by (7).

Proposition 3.2. *Let the sequence $\{y_n : n \in \mathbb{N}\}$ be defined by* (7). *Then*

(a) $\sum_{n=0}^{\infty} \{|y_n - y_{n+1}|^2 + \sum_{j=1}^{m} a_j |P_j y_n - P_j y_{n+1}|^2\} = |y_0|^2 - \sum_{j=1}^{m} a_j |P_j y_0|^2.$

(b) $\sum_{n=0}^{\infty} \{|y_n - y_{n+2}|^2 + \sum_{j=1}^{m} a_j |P_j y_n - P_j y_{n+2}|^2\}$

$$= |y_0|^2 + \sum_{j=1}^{m} a_j |P_j y_0|^2 - |y_1|^2 - \sum_{j=1}^{m} a_j |P_j y_1|^2.$$

Applying Theorem 2.6 with $k = 2q$, $q \geq 1$, we also arrive at the following identity for the iteration (7).

Proposition 3.3. *Let the sequence $\{y_n : n \in \mathbb{N}\}$ be defined by* (7). *Then for each $k = 2q$, $q \geq 1$,*

$$\sum_{n=0}^{\infty} \{|y_n - y_{n+k}|^2 + \sum_{j=1}^{m} a_j |P_j y_n - P_j y_{n+k}|^2\}$$

$$= \sum_{i=0}^{q-1} \{|y_i|^2 + \sum_{j=1}^{m} a_j |P_j y_i|^2 - |y_{q+i}|^2 - \sum_{j=1}^{m} a_j |P_j y_{q+i}|^2\}.$$

4. Nonlinear Projections

Returning to the composition of orthogonal projections, we first recall that a linear contraction $T : H \to H$ is said to satisfy condition (S) if $\{x_n - Tx_n\} \to 0$ whenever the sequence $\{x_n\}$ is bounded and $\{|x_n| - |Tx_n|\} \to 0$. Since each orthogonal projection obviously satisfies condition (S) ($|x - Px|^2 = |x|^2 - |Px|^2$), and since the class of linear operators satisfying condition (S) is closed under composition, it follows that the composition $T = P_m P_{m-1} ... P_1$ of m orthogonal projections also satisfies condition (S) and hence is asymptotically regular: $\lim_{n \to \infty} (T^n x - T^{n+1} x) = 0$ for each $x \in H$.

The Yosida mean ergodic theorem can now be used to show that the sequence defined by (2) converges strongly whenever the mapping $r : \{1, 2, ...\} \to \{1, 2, ..., m\}$ is periodic ([Ha], [AA]). As a matter of fact, this result is susceptible to a nonlinear generalization even in certain Banach spaces; see Theorem 4.5 in [MR] and the references cited there. We mention, in particular, the following nonlinear analog of Theorem 1.1, where this time C_1 and C_2 are two closed convex subsets of the Hilbert space H with corresponding nearest point projections $P_1 : H \to C_1$ and $P_2 : H \to C_2$. We denote by $d(C_1, C_2)$ the distance between C_1 and C_2. In this connection, see also [BB1] and [BB2].

Theorem 4.1. *Let the sequence $\{x_n : n \in \mathbb{N}\}$ be defined by* (1).

(a) *If $d(C_1, C_2)$ is attained, then $\{x_{2n}\}$ converges weakly to a fixed point z of P_2P_1 and $\{x_{2n+1}\}$ converges weakly to P_1z.*
(b) *If $d(C_1, C_2)$ is not attained, then $\{|x_n|\} \to \infty$.*
(c) *If C_1 and C_2 are symmetric with respect to the origin, then $\{x_n\}$ converges strongly to a point in the intersection $C = C_1 \cap C_2$.*

Proof. (a) In this case we already know by Corollary 4.6 of [MR] that $\{x_{2n} : n \in \mathbb{N}\}$ converges weakly to a fixed point z of P_2P_1. Since

$$\begin{aligned} |x_{2n+2} - z| &= |P_2x_{2n+1} - P_2P_1z| \\ &\leq |x_{2n+1} - P_1z| = |P_1x_{2n} - P_1z| \\ &\leq |x_{2n} - z|, \end{aligned}$$

we see that $\lim_{n\to\infty}(|x_{2n+1} - P_1z| - |P_2x_{2n+1} - P_2P_1z|) = 0$ from which it follows (because P_2 is strongly nonexpansive in the sense of [BR]) that the strong $\lim_{n\to\infty}(x_{2n+1} - x_{2n+2}) = P_1z - z$. Thus $\{x_{2n+1}\}$ converges weakly to P_1z, as claimed.
(b) Again we already know that $\{|x_{2n}|\} \to \infty$ for any $x_0 \in H$. But P_2P_1 has a fixed point if and only if P_1P_2 does. Hence $\{|x_{2n+1}|\} \to \infty$ too.
(c) This time we know that $\{x_{2n}\}$ converges in norm to a fixed point z of P_2P_1 and that $C = C_1 \cap C_2$ is not empty. Therefore $z \in C$ and $x_{2n+1} = P_1x_{2n} \to P_1z = z$. □

Applying a Banach space version of Theorem 4.1 to the m-th power E^m of a Banach space E, we can obtain an alternative proof of Theorem 4.8 in [MR]. Here, however, we just illustrate this method of proof by presenting a variant of Corollary 4.10 there. This variant is concerned with the iteration

$$y_{n+1} = a_1P_1y_n + a_2P_2y_n, \tag{8}$$

where $y_0 \in H$, a_1 and a_2 are positive, and $a_1 + a_2 = 1$.

Theorem 4.2. *Let the sequence $\{y_n : n \in \mathbb{N}\}$ be defined by* (8).

(a) *If $d(C_1, C_2)$ is attained, then $\{y_n\}$ converges weakly to a fixed point of $a_1P_1 + a_2P_2$.*
(b) *If $d(C_1, C_2)$ is not attained, then $\{|y_n|\} \to \infty$.*
(c) *If C_1 and C_2 are symmetric, then $\{y_n\}$ converges strongly to a point in the intersection $C = C_1 \cap C_2$.*

We precede the proof of Theorem 4.2 with a few preparatory considerations. Endowing the power $H^2 = H \times H$ with the inner product $\langle (u_1, u_2), (v_1, v_2)\rangle = a_1\langle u_1, v_1\rangle + a_2\langle u_2, v_2\rangle$, we consider this time the closed convex subset $C_1 \times C_2$ and the closed subspace $D = \{(u, u) \in H^2 : u \in H\}$ of H^2 with the corresponding nearest point projection $Q_1 : H^2 \to C_1 \times C_2$ and orthogonal projection $Q_2 : H^2 \to D$. In the setting of Theorem 4.1, the distance $d(C_1, C_2)$ is attained if and only if the fixed point sets of P_2P_1 and P_1P_2 are not empty. We now observe that this is also true for the mappings Q_1Q_2 and Q_2Q_1.

Lemma 4.3. *The mapping Q_1Q_2 has a fixed point if and only if the distance $d(C_1, C_2)$ is attained.*

Proof. If $Q_1Q_2(z_1, z_2) = (z_1, z_2)$, then $P_1(a_1z_1+a_2z_2) = z_1$ and $P_2(a_1z_1+a_2z_2) = z_2$. Since both P_1 and P_2 are sunny retractions, it follows that $P_1z_2 = z_1$ and $P_2z_1 = z_2$. Thus $d(C_1, C_2) = |z_1 - z_2|$. Conversely, if $d(C_1, C_2) = |z_1 - z_2|$ for some $z_1 \in C_1$ and $z_2 \in C_2$, then necessarily $P_1z_2 = z_1$ and $z_2 = P_2z_1$. Hence $P_1(a_1z_1 + a_2z_2) = z_1$ and $P_2(a_1z_1 + a_2z_2) = z_2$. In other words, Q_1Q_2 has a fixed point. □

Proof of Theorem 4.2. This theorem is now seen to follow from Theorem 4.1 and Lemma 4.3 because $(y_n, y_n) = (Q_2Q_1)^n(y_0, y_0)$ for all $n \in \mathbb{N}$ and the mapping Q_2Q_1 has a fixed point if and only if Q_1Q_2 does. □

Remark 4.4. Denote the limits obtained in parts (a) and (c) of Theorem 4.2 by z. Then we also see that in part (a), the sequence $\{P_1y_n\}$ converges weakly to P_1z and the sequence $\{P_2y_n\}$ converges weakly to P_2z. In part (c), both $\{P_1y_n\}$ and $\{P_2y_n\}$ converge strongly to z. In part (b), $\max\{|P_1y_n|, |P_2y_n|\} \to \infty$.

Applying the product space method to the iteration defined in Theorem 5.1 of [BMR], we observe that the weak convergence asserted in part (a) of Theorem 4.1 cannot, in general, be replaced with convergence in norm, even when $C = C_1 \cap C_2 \neq \emptyset$ and one of these closed convex subsets is, in fact, a closed subspace (cf. [Hu]). Thus part (c) of Theorem 4.1 seems to be a good nonlinear analog of von Neumann's Theorem 1.1. We remark in passing that such results can be used to derive certain product formulas for nonlinear semigroups in the spirit of [Re1] and [Re3].

References

[AA] I. Amemiya and T. Ando, *Convergence of random products of contractions in Hilbert space*, Acta. Sci. Math. (Szeged) 26 (1965), 239-244.

[Ar] N. Aronszajn, *Theory of reproducing kernels*, Trans. Amer. Math. Soc. 68 (1950), 337-403.

[BB1] H. H. Bauschke and J. M. Borwein, *On the convergence of von Neumann's alternating projection algorithm for two sets*, Set-Valued Anal. 1 (1993), 185-212.

[BB2] H. H. Bauschke and J. M. Borwein, *Dykstra's alternating projection algorithm for two sets*, J. Approximation Theory 79 (1994), 418-443.

[BMR] H. H. Bauschke, E. Matoušková and S. Reich, *Projection and proximal point methods: convergence results and counterexamples*, Nonlinear Anal. 56 (2004), 715-738.

[BR] R. E. Bruck and S. Reich, *Nonexpansive projections and resolvents of accretive operators in Banach spaces*, Houston J. Math. 3 (1977), 459-470.

[De] F. Deutsch, *Best Approximation in Inner Product Spaces*, Springer, New York, NY, 2001.

[DR] J. M. Dye and S. Reich, *On the unrestricted iteration of projections in Hilbert space*, J. Math. Anal. Appl. 156 (1991), 101-119.

[Ha] I. Halperin, *The product of projection operators*, Acta. Sci. Math. (Szeged) 23 (1962), 96-99.

[Hu] H. S. Hundal, *An alternating projection that does not converge in norm*, Nonlinear Anal. 57 (2004), 35-61.

[La] M. L. Lapidus, *Generalization of the Trotter-Lie formula*, Integral Equations Operator Theory 4 (1981), 365-415.

[MR] E. Matoušková and S. Reich, *The Hundal example revisited*, J. Nonlinear Convex Anal. 4 (2003), 411-427.

[Na] H. Nakano, *Spectral Theory in Hilbert Space*, Japan Soc. Promotion Sci., Tokyo, 1953.
[Ne] J. von Neumann, *On rings of operators. Reduction theory*, Ann. of Math. 50 (1949), 401-485.
[Pi] G. Pierra, *Decomposition through formalization in a product space*, Math. Programming 28 (1984), 96-115.
[Re1] S. Reich, *A complement to Trotter's product formula for nonlinear semigroups generated by the subdifferentials of convex functionals*, Proc. Japan Acad. 58 (1982), 193-195.
[Re2] S. Reich, *A limit theorem for projections*, Linear and Multilinear Algebra 13 (1983), 281-290.
[Re3] S. Reich, *Solutions of two problems of H. Brezis*, J. Math. Anal. Appl. 95 (1983), 243-250.
[Sa] M. Sakai, *Strong convergence of infinite products of orthogonal projections in Hilbert space*, Applicable Anal. 59 (1995), 109-120.
[Wi] N. Wiener, *On the factorization of matrices*, Commentarii Math. Helv. 29 (1955), 97-111.

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