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OPTIMUM VALUES SEARCH IN A MARTINGALE FINANCIAL MARKET

JIMBO H. C, BOZEMAN R. E, DJOUNGUELA A, AND SUZUKI T.

Abstract. In this paper, an incomplete financial market model is proposed. This model is an option-pricing model for an incomplete market and is based on the assumption that the price processes have lognormal distribution, the rational prices of option are determined by their optimum values and the class of feasible or acceptable martingale measures is determined by the direct Escher Transform Technique. This model has many good points: firstly, it characterizes *a priori* the price process distribution; secondly, it gives an explicit form for optimum option prices and thirdly, the class of reasonable martingale measures construction is reduced to the computation of Escher Transform. We are going to investigate these problems in the following sections of our paper.

1. Introduction

Financial markets in general possess the stochastic and volatile properties. A lot of models have been proposed to describe the dynamics of financial markets for which, the B-S model proposed by Black and Sholes [1] which had gained reputation among researchers, academics and practioners. Whatever people say about the drawbacks of Black-Scholes (B,S) approach [1] to derivative pricing, it is a standard method and almost any pricing and hedging software in financial institution is based on it. Practioners have got used BS like partial differential equations, martingales, and other mathematical animals. In the field of Option Pricing Theory (OPT), martingale measures play very important roles. For example, in the Black Scholes model, the price of contingent claim is given as the expectation of the return function with respect to the unique risk neutral martingale measure. If the market is complete, then the equivalent martingale measure is unique and the option prices are determined uniquely by the martingale measure. However if the market is incomplete, then there are infinitely many martingale measures and question of how to build, select or get them depends mostly on the method that is used. The problem of selecting suitable martingale measure has been discussed by many authors; for example the minimal entropy martingale measure proposed by Miyahara[15]; minimal martingale measure, variance optimal martingale and utility martingale measure proposed by Delbaen and Schachermayer[3], Follmer and Schweizer[5], Shiryaev[20, 21], and Jimbo[10]. In this paper we propose a new constructive approach based on the Escher Transform Technique for selecting a reasonable class of martingale measures and after we got the class of martingale measures, we will determine the optimum prices of the option. We organize our work as follows: in section two we build our model; in section three we define and

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apply the Escher Transform Technique; in section four we determine the optimum prices of option; in section five we compute the N steps ahead option prices; in section six we end our work with some conclusive remarks.

2. The basic model

There are many works on the distribution of log return of stocks (see Fama and Roll[4], Hurst et al [7], Mitnik[16]). Most researchers have agreed that the distribution of the log return is not normal since the distribution of log return is assymetric and has a fat tail property. For the purpose of our modeling, we assume that the return has a lognormal distribution. We will describe our model and also briefly explain the reason why we adopte this model as the typical model for incomplete market. Our model consisted of two kinds of assets: a riskless asset (bank account) and a risky one (stocks price). Our incomplete model of financial market is functioning at finite time horizon, and according to this model, the bank account is submitted to the recurrent relations:

$$B_N = B_{N-1}e^r, \qquad B = (B_N)_{N>0} \tag{1}$$

$$S_N = S_0 e^{h_1+\cdots h_N}, \qquad S = (S_N)_{N>0} \tag{2}$$

where r is a deterministic interest rate and $h = (h_N)_{N>0}$ is a chaotic sequence with the following characteristics:

$$D(h_N/F_{N-1}) = \sigma_N^2 \tag{3}$$

$$Law\,(h_N/F_{N-1}) = N\left(m, \sigma_N^2\right). \tag{4}$$

We assume that everything occures on a measurable space (Ω, F) with a family $P = \{p\}$ of probability measures and the equivalent class of martingale measure $\tilde{p} \in \tilde{P}$; the sequence of random variables $h = h_N(\omega)$ can be independent as well as dependent, identically distributed with positive probability. Here $F = (F_N)_{N>0}$ where $F_N = \sigma(h_1, \ldots, h_N)$ is the σ-algebra generated by the random variables $h_1, \ldots, h_N$. Let us assume that the option price C_N with strike price K satisfies the equality:

$$C_N = g\,(S_N, K)\,, \tag{5}$$

with $g\,(S_N) = \max(S_N - K, 0)$ and the function $g : R \to R^+$ is differentiable with respect to S_N. We also assume that the price of the option fluctuates in time between its upper and lower values. We assume the set of elementary event to be finite $|\Omega| < \infty$, the trading takes place continuously and there is no transaction costs, taxes, or short scale restrictions. Borrowing and lending take place at a constant rate r with the underlying stock paying no cash dividende. Principally in this work, we will introduce an algorithm for finding: simple, efficient and consistent optimum option prices at each time N. Two main problems are to be solved in this work:

(a) Build a reasonable class of martingale measures by the Escher Transform Technique.
(b) Find the optimum prices for the option.

For the next part of our work we will agree that the rate of interest $r(t)$ oscillates around a constant value r.

3. Escher Transform Technique

3.1. **Application.** Let us consider a sequence of random variables $X = (X_t)_{t<N}$, $N \in Z^+$ with independent increment $\Delta X_t = X_{t-1} - X_t$ and particularly formed by sums of independent random variables. Let X be a real value random variable with it Laplace transform $\phi(\lambda) = Ee^{\lambda X} < \infty, \lambda \in R$. Let $P = P(dx)$ be it distribution on $(R, B(R))$. We introduce for the point a a family of probability measures defined by the Escher transform as follows:

$$P^{(a)}(dx) = \frac{e^{ax}}{\phi(a)} P(dx) \tag{6}$$

$$\text{for} \quad Z^{(a)}(dx)\,\tfrac{e^{ax}}{\phi(a)}.$$

We can realize that $Z^{(a)} > 0, EZ^{(a)}(X) = 1$, thus the class of martingale measure is equivalent to the probability measure which means $P^{(a)} \sim P$ and $P^{(a)}(dx) = Z^{(a)}(x)\,P(dx)$. Evidently we read:

$$\phi^{(a)}(\lambda) = E_{P(a)}e^{\lambda S} = \frac{Ee^{(\lambda+a)S}}{\phi(a)} = \frac{\acute{\phi}(a+\lambda)}{\phi(a)} \tag{7}$$

$$\text{Hence} \quad E_{P(a)}S = \left.\tfrac{\delta\phi(a)}{\delta\lambda}\right|_{\lambda=0} = \tfrac{\acute{\phi}(a)}{\phi(a)},$$

where $\acute{\phi}$ denotes the first derivative of ϕ. It is known that if a random variable X is such that $P(X>0)>0$ and $P(X<0)>0$, then the function $\phi(a)$ reaches its minimal value at some point $\tilde{a}$ for which obviously $\acute{\phi}(\tilde{a}) = 0$. Thus for the measure $\tilde{P} = P(\tilde{a})$, the expectation $\tilde{E} = E_{P(\tilde{a})}S = 0$, is expressed by the words that $\tilde{P}$ is *risk neutral probability measure* and the property $\tilde{E}S = 0$ can be regarded as a first stage version of the *martingale properties* which justifies the other name for $\tilde{P}$: *martingale measure.*

Remark 3.2. At the point $a = \tilde{a}$ the family of probability measures $P^{(a)}$ is equivalently transformed into martingale measures and any sequence of normally distributed random variable under the martingale measure $\tilde{P}$ has a null expectation.

Lemma 3.3. *For any sequence of random variables with normal or lognormal distribution, a class of martingale measures can always be established by the Esher transform technique. But in the case of exponentially distributed random variables this procedure is applicable only if the parameter of distribution is shifted to the left.*

Proof. **a. The Gaussian distribution case.** Let us consider a normally distributed random variable X with parameters m and σ^2. We first compute the characteristic function as follows

$$\begin{aligned}
\phi(i\lambda) &= Ee^{i\lambda x} \\
&= \frac{1}{2}\int_{-\infty}^{+\infty} e^{i\lambda - \frac{(x-m)^2}{2\sigma^2}} dx \\
&= \frac{1}{\sqrt{2\pi\sigma}}\int_{-\infty}^{+\infty} e^{(i\lambda x - \frac{x^2}{2\sigma^2} + \frac{xm}{\sigma^2} - \frac{m^2}{2\sigma^2})} dx \\
&= \frac{1}{\sqrt{2\pi\sigma}}\int_{-\infty}^{+\infty} e^{(-\frac{x^2}{2\sigma^2} + \frac{(i\lambda\sigma^2+m)x}{\sigma^2} - \frac{(i\lambda\sigma^2+m)^2}{2\sigma^2} + \frac{(i\lambda\sigma^2+m)^2}{2\sigma^2} - \frac{m^2}{2\sigma^2})} dx \\
&= e^{(i\lambda x - \frac{\lambda^2 x^2}{2})}\frac{1}{\sqrt{2\pi\sigma}}\int_{-\infty}^{+\infty} e^{-\frac{[(i\lambda\sigma^2+m)^2]^2}{2\sigma^2}}
\end{aligned} \tag{8}$$

by the change $y = x - (m + i\lambda\sigma^2)$ we obtain

$$\phi(i\lambda) = Ee^{i\lambda x} = e^{(i\lambda x - \frac{\lambda^2 x^2}{2})}. \tag{9}$$

Let us now compute $P^{(a)}(x)$:

$$\begin{aligned}
P^{(a)}(x) &= \frac{e^{-am\frac{a^2\sigma^2}{2}}}{\sqrt{2\pi\sigma}} e^{-\frac{(x-m)^2}{2\sigma^2} + ax} \\
&= \frac{e^{-am\frac{a^2\sigma^2}{2}}}{\sqrt{2\pi\sigma}} \exp\{-\frac{x^2}{2\sigma^2} + \frac{xm}{\sigma^2} - \frac{m^2}{2\sigma^2} + \frac{(m+\sigma^2 a)}{2\sigma^2} - \frac{(m+\sigma^2 a)}{2\sigma^2}\} \\
&= \frac{e^{-am\frac{a^2\sigma^2}{2}}}{\sqrt{2\pi\sigma}} \exp\{-\frac{[x-(m+\sigma^2 a)]^2}{2\sigma^2}\} \exp\{\frac{(m+\sigma^2 a)^2}{2\sigma^2} - \frac{m^2}{2\sigma^2}\} \\
&= \frac{1}{\sqrt{2\pi\sigma}} \exp\{-\frac{[x-(m+\sigma^2 a)]^2}{2\sigma^2}\},
\end{aligned} \tag{10}$$

it follows that $P^{(a)}(x)$ is normally distributed with parameters $m + \sigma^2 a$ and σ^2. From the measure $P^{(a)} \sim N(m + \sigma^2 a, \sigma^2)$. Let us now construct the measure $P^{(\tilde{a})} \sim N(0, \sigma^2)$ and for that purpose we need the value of $\tilde{a}$ which can be found from the following condition:

$$\left.\frac{\delta ln\phi(\lambda)}{\delta\lambda}\right|_{\lambda=\tilde{a}} = \left.\frac{\delta}{\delta\lambda}\left(m\lambda + \frac{\delta^2\lambda^2}{2}\right)\right|_{\lambda=\tilde{a}} = 0 \tag{11}$$

Thus $m + \sigma^2\tilde{a} = 0$. Hence we obtain $\tilde{a} = -\frac{m}{\sigma^2}$.

Continuity. (i) The martingale measure $P^{(\tilde{a})}(x)$ is defined equivalent to the probability measure $P(x)$. (ii) For any $\tilde{Z}(x) = \frac{e^{\tilde{a}x}}{\phi(\tilde{a})} = e^{\tilde{a}x - m\tilde{a} + \frac{\sigma^2 a^2}{2}} > 0$, the measure $\tilde{P}(x)$ is defined positive everywhere on the right and left sides of the point zero. (iii) $Ee^{\lambda x} < \infty$. These three conditoins imply the continuity of the martingale measure with respect to the original measure $P(x)$.

b. The Exponential distribution case Let consider a random variable X exponentially distributed with parameter λ:

$$P(dx) = \lambda e^{-\lambda x} dx, x > 0 \tag{12}$$

the Laplace transform is given by:

$$\phi_1(a) = E_{P_1^{(a)}} e^{ax} = \lambda \int_{-\infty}^{+\infty} e^{(-\lambda+a)x} dx = \frac{\lambda}{\lambda - a} \tag{13}$$

Thus $ln\phi_1(a) = ln\lambda - ln(\lambda - a)$. It is obvious that $\phi_1(a)$ is a strictly increasing function and nowhere reaches the minimum; therefore we cannot directly construct the class of martingale measures by the Escher Transform procedure.

To overcome this difficulty we make a shift to $-b$ in the distribution:

$$P_2(dx) = \lambda e^{-\lambda(x+b)} dx, x \geq 0, x \geq -b \tag{14}$$

then

$$\phi_2(a) = E_{P_3(a)} e^{ax} = \lambda \int_{-\infty}^{+\infty} e^{-\lambda x - \lambda b + ax} dx \tag{15}$$

$$= \lambda e^{-\lambda b} \int_{-\infty}^{+\infty} e^{(-\lambda+a)x} dx = \frac{\lambda}{\lambda - a} e^{ab}, \tag{16}$$

we lastly compute the logarithmic derivative to find the optimum point and from there it becomes easy to represent our martingale measure:

$$\left.\frac{\delta ln\phi_2(a)}{\delta a}\right|_{a=\tilde{a}} = \left.\frac{\delta(ln\lambda - ln\lambda - a) - ab}{\delta a}\right|_{a=\tilde{a}} = \frac{1}{\lambda - \tilde{a}} e^{ab} \tag{17}$$

Hence $\tilde{a} = \lambda - \frac{1}{b}$. In this case we see that the point $\tilde{a}$ depends both on the scale and shift parameters.

c. The Levy process case. Let $X = (X_n)_{n \leq N}$ be a levy process with characteristic function $Ee^{i\theta X_n} = e^{t\psi(\theta)}$
where the cumulant Ψ is defined by:

$$\psi(\theta) = i\theta b - \frac{\theta^2}{2} C + \int_R (e^{i\theta x} - 1 - i\theta h(x))\nu(dx), \tag{18}$$

and $h(x)$ is the truncating function (for example, $h(x) = xI(|x| \leq 1)$). By putting formally in these formula $\theta = -i\lambda$, we get $Ee^{\lambda X_n} = e^{t\phi(\lambda)}$ where

$$\phi(\lambda) = \lambda b + \frac{\lambda^2}{2} C + \int_R \left(e^{\lambda x} - 1 - \lambda h(x)\right) \nu(dx). \tag{19}$$

A rigourous proof of this representation for the Laplace transform can be obtain most easily if based on the remark that the process $Z^{(\lambda)} = (Z_n^{(\lambda)})_{n \geq 0}$ with $Z_n^{(\lambda)} = \exp\{\lambda X_n - n\phi(\lambda)\}$ is a martingale.

By analogy to (1) for each $a \in R$, let us introduce the class of equivalent probability measures $P_N^{(a)}$ defined by the Escher transform technique: $dP_N^{(a)} = Z_N^{(a)} dP_N$ where P_N is the original probability measure defined on (Ω, F_N) for which $X = (X_n)_{n \leq N}$ is a sequence of random variables.

Theorem 3.4. *For the measure $P_N, a \in R$, the sequence $X = (X_n)_{n \in N}$ is an independent increment with*

$$E^{(a)} e^{\lambda X_n} = e^{n\phi^{(a)}(\lambda)} \quad where \quad \phi^{(a)}(\lambda) = \phi(a + \lambda) - \phi(a).$$

Proof. The proof follows immediately from the formula of measure change in conditional expectations by which ($P_N^{(a)} - a.s$):

$$E^{(a)}\left(e^{\lambda(X_n-X_s)}\middle|F_s\right) = E^{(a)}\left(e^{\lambda(X_n-X_s)}\frac{Z_n^{(a)}}{Z_s^{(a)}}\middle|F_s\right) \tag{20}$$

$$= E^{(a)}\left(e^{(a+\lambda)(X_n-X_s)-\phi(a)(n-s)}\middle|F_s\right) \tag{21}$$

$$= E^{(a)}\left(e^{(a+\lambda)(X_n-X_s)-\phi(a)(n-s)}\right) \tag{22}$$

$$= e^{(a+\lambda)(X_n-X_s)-\phi(a)(n-s)}. \tag{23}$$

which ends our proof.

4. An algorithm for computing the upper and lower option prices

In this section we will compute the optimum prices (the upper and lower prices) for our opinion. By the definition, the option price or rational price is the conditional expectation of the sequence $\left\{\frac{g(S_N)}{e^{rN}}\right\}$ with respect to the $\sigma-$algebra F_n and the martingale measure $\tilde{P}$ that is:

$$C_N = E_{\tilde{p}}\left(\frac{g(S_N)}{e^{rN}}\middle|F_n\right) = \frac{g(S_N)}{e^{rN}} \quad \text{or more simply} \quad C_N = E_{\tilde{p}}\left\{\frac{g(S_N)}{e^{rN}}\right\}$$

In general the rational price C_N fluctuates between it lower and upper values or bounds which we call optimum values. We will find these values by the following formula:

$$C_{*N} \leq C_N \leq C_N^*, \quad C_{*N} = \inf_{\tilde{p}} E_p\left\{\frac{g(S_N)}{e^{rN}}\right\}, \quad C_N^* = \sup_{\tilde{p}} E_p\left\{\frac{g(S_N)}{e^{rN}}\right\} \tag{24}$$

Therefore to make our calculation easier we will use the Jensen inequality.

4.1. **The Jensen inequality.** Let $g(S_N)$ be a convex function, S_N and r are respectively as before the stocks price at the time N and the constant interest rate. If the expectations $E[g(S_N)]$ and $E[S_N]$ are finite, then the following inequality holds:

$$E_{(\cdot)}[g(S_N)] \geq g[E_{(\cdot)}(S_N)]. \tag{25}$$

Let us now use the above inequality to find the one step ahead price.

4.2. **One step ahead prices (C_{*1} and C^{*1}).** From the equation (2) we have: $S_1 = S_0e^{h_1}$ and $g(S_N) = (S_N - K)^+$ which gives

$$g(S_N) = \begin{cases} 0, & S_N \leq k \\ S_N - k, & S_N > k \end{cases} \tag{26}$$

Applying the Jensen inequality to the starting value of our recursion we have

$$E_{\tilde{p}}[S_0e^{h_1}] = S_0E[e^{h_1}] \tag{27}$$

we now express this expectation in terms of integral as follows

$$E_{\tilde{p}}[e^{h_1}] = \int_{-\infty}^{+\infty} e^x p(x)dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^x e^{\frac{-x^2}{2\sigma^2}}dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{x - \frac{-x^2}{2\sigma^2} - \frac{\sigma^2}{2} + \frac{\sigma^2}{2}} dx \tag{28}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\frac{(x-\sigma^2)}{2\sigma^2}} e^{\frac{\sigma^2}{2}} dx = \frac{e^{\frac{\sigma^2}{2}}}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\frac{(x-\sigma^2)}{2\sigma^2}} e^{\frac{\sigma^2}{2}} dx = \sigma e^{\frac{\sigma^2}{2}} \tag{29}$$

where $y = x - \sigma^2$. Using the probability neutral risk assumption for given upper (S_h) and lower (S_0) stocks prices we get

$$p = \frac{S_0 e^{\frac{\sigma^2}{2}}}{S_h} \quad \text{and} \quad 1 - p = \frac{S_h - S_0 e^{\frac{\sigma^2}{2}}}{S_h}$$

For the supremum we write

$$\sup_{\tilde{p}\sim p} E_{\tilde{p}} \frac{g(S_0)}{e^r} = \frac{g(S_h)}{e^r} \frac{S_0 e^{\frac{\sigma^2}{2}}}{S_h} + \frac{g(0)}{e^r} \frac{S_h - S_0 e^{\frac{\sigma^2}{2}}}{S_h} \tag{30}$$

And finally the one step rational price for the supremum is computed as follow:

$$C_1^* = \overline{\lim_{S_h\to\infty}} \left[\frac{g(S_h)}{e^r} \frac{S_0 e^{\frac{\sigma^2}{2}}}{S_h} + \frac{g(0)}{e^r} \frac{S_h - S_0 e^{\frac{\sigma^2}{2}}}{S_h} \right] \tag{31}$$

$$= S_0 e^{\frac{\sigma^2}{2}} \overline{\lim_{S_h\to\infty}} \left[\frac{g(S_h)}{S_h} \right] + g(0) e^{-r} = S_0 e^{\frac{\sigma^2}{2} - r}$$

The one step rational price for the infimum is given by the expression

$$C_{*1} = \inf_{\tilde{p}\sim p} E_{\tilde{p}} \left[\frac{g(S_1)}{e^r} \right] \tag{32}$$

Applying again the Jensen inequality we obtain

$$C_{*1} = \inf_{\tilde{p}\sim p} E_{\tilde{p}} \left[\frac{g(S_1)}{e^r} \right] > \inf_{\tilde{p}\sim p} g \left[E_{\tilde{p}} \left[\frac{S_1}{e^r} \right] \right] \tag{33}$$

$$= \inf_{\tilde{p}\sim p} g \left[S_0 E_{\tilde{p}} e^{\frac{\sigma^2}{2}} \right] e^{-r} = g \left[S_0 e^{\frac{\sigma^2}{2}} \right] e^{-r} \tag{34}$$

$$= \max \left[0, S_0 e^{\frac{\sigma^2}{2}} - K \right] e^{-r} = \begin{cases} \left[S_0 e^{\frac{\sigma^2}{2}} - K \right]; & S_0 e^{\frac{\sigma^2}{2}} > K \\ 0; & \text{otherwise} \end{cases} \tag{35}$$

which gives

$$\left[0, S_0 e^{\frac{\sigma^2}{2}} - K \right] e^{-r} \leq C_1 \leq S_0 e^{\frac{\sigma^2}{2}}, \tag{36}$$

and means $C_1^* = C_{*1} + K e^{-r}, K > 0$.

Let us now look for the N-steps ahead rational prices (infimum and the supremum) for the option. We will separate the cases that the random variables h_i are independent from which they are not.

5. N-steps ahead option prices

5.1. **For h_i iid random variables.** We first consider the random variable $h_i, i = 1, \cdots, N$ to be iid and $N(0, \sigma^2)$ and we define the stocks price as before: $S_n = S_0 e^{h_1+\cdots+h_n}, B_n = B_{n-1}e^r$ Since $Ee^{h_1+\cdots+h_n} = e^{\frac{n\sigma^2}{2}}$, we compute C_N^* and C_{*N} by the recurrent formula taking into account the fact that S_n is a *martingale* under the measure $\tilde{p}$, with respect to the filtration F_{N-1} which, equivalently means

$$E_{\tilde{p}}\left[\frac{S_n}{S_{n-1}}\right] = E_{\tilde{p}}\left[\frac{S_n}{S_0, \cdots, S_{n-1}}\right] \tag{37}$$

For $n = 1$ we compute the one step ahead optimum prices C_{*1} and C_1^* as above. For $n = 2$ we compute the two steps ahead optimum prices as follows.

$$\begin{aligned}
C_2^* &= e^{-2r} \sup_{\tilde{p}\sim p} E_{\tilde{p}} g\left(\frac{S_2}{S_1}\right) = \left(S_1 e^{\frac{\sigma^2}{2}} \lim_{x\to\infty} \frac{g(S_h)}{S_h} + g(0)\right) e^{-2r} \\
&= \left(S_0 e^{\sigma^2} \lim_{x\to\infty} \frac{g(S_h)}{S_h} + g(0)\right) e^{-2r} \\
C_{*2} &= e^{-2r} \inf_{\tilde{p}\sim p} E_{\tilde{p}} g\left(\frac{S_2}{S_1}\right) = g\left(S_1 e^{\frac{\sigma^2}{2}-2r}\right) = g\left(S_0 e^{\sigma^2 - r}\right)
\end{aligned}$$

For $n = 3$ we compute the three steps ahead optimum prices as follows.

$$\begin{aligned}
C_3^* &= e^{-3r} \sup_{\tilde{p}\sim p} E_{\tilde{p}} g\left(\frac{S_3}{S_2}\right) = \left(S_2 e^{\frac{\sigma^2}{2}} \lim_{x\to\infty} \frac{g(S_h)}{S_h} + g(0)\right) e^{-2r} \\
&= \left(S_2 e^{\frac{3\sigma^2}{2}} \lim_{x\to\infty} \frac{g(S_h)}{S_h} + g(0)\right) e^{-3r} \\
C_{*3} &= e^{-r} \inf_{\tilde{p}\sim p} E_{\tilde{p}} g\left(\frac{S_3}{S_2}\right) = g\left(S_2 e^{\frac{\sigma^2}{2}}\right) e^{-r} = g\left(S_0 e^{\frac{3\sigma^2}{2}-3r}\right) \\
&\vdots
\end{aligned}$$

For $n = N$ we compute the N-steps ahead prices as follows.

$$\begin{aligned}
C_N^* &= e^{-Nr} \sup_{\tilde{p}\sim p} E_{\tilde{p}} g\left(S_N\right) = \left(S_0 e^{\frac{N\sigma^2}{2}} \lim_{x\to\infty} \frac{g(S_h)}{S_h} + g(0)\right) e^{-Nr} \\
&= \left(S_0 e^{\frac{N\sigma^2}{2}} \lim_{x\to\infty} \frac{g(S_h)}{S_h} + g(0)\right) e^{-Nr} \\
C_{*N} &= e^{-Nr} \inf_{\tilde{p}\sim p} E_{\tilde{p}} g\left(S_N\right) = g\left(S_0 e^{\frac{N\sigma^2}{2}}\right) e^{-Nr} \\
&= \begin{cases} S_0 e^{N(\frac{\sigma^2}{2}-r)} - Ke^{Nr}, & S_0 e^{\frac{N\sigma^2}{2}} > K \\ 0, & \text{otherwise} \end{cases}
\end{aligned}$$

We can easily realize that $C_{*N} < C_N < C_N^*$ for every non zero N. Now let us investigate the case of dependently distributed h_i.

5.2. h_i **dependently distributed random variables (ddrv).** From the probability point of view, it is specified that if random variables are dependently distributed ,there exist a non zero correlation among them. We will define those correlations explicitely below.

As before we consider our model defined by:

$$S_n = S_0 e^{h_1+\cdots+h_n}, B_n = B_{n-1}e^r, h_i - ddrv \tag{38}$$

and write

$$\log\left(\frac{S_n}{S_{n-1}}\right) = log(e^{h_n}) = h_n \quad \text{and} \quad \rho(k) = Eh_n h_{n+k}.$$

Now we compute $\rho(k)$ for different values of k:

$$\text{For } k=0: \quad \rho(0) = Eh_n^2 = 1 \tag{39}$$

$$\text{For } k=1: \quad \rho(1) = Eh_n h_{n+1} \tag{40}$$

$$\text{For } k=2: \quad \rho(2) = Eh_n h_{n+2} \tag{41}$$

$$\vdots \tag{42}$$

$$\text{For } k=N: \quad \rho(N) = Eh_N h_{2N} \tag{43}$$

and build the following covariation matrix:

$$\begin{bmatrix} \rho(0) & \rho(1) & \rho(2) & \cdots & \rho(n-1) \\ \rho(1) & \rho(0) & \rho(1) & \cdots & \rho(n-2) \\ \vdots & & & & \\ \vdots & & & & \\ \rho(N-1) & & & & \rho(0) \end{bmatrix} \tag{44}$$

Theorem 5.3 (on the analytic continuation)**.** *Let $h_i \sim N(m, \sigma^2)$ be a sequence of random variables and $m = (m_1, \cdots, m_N)$ be a vector of the expectation. Let also $E(h_i - m_i)(h_j - m_j) = b_{ij}$ be the covariation matrix, then for any complex vector $t = (t_1, \cdots, t_N)$ we have*

$$Ee^{i\sum_{k=1}^n t_k h_k} = \exp\left(iTm - \frac{1}{2}TBt\right). \tag{45}$$

where T is the transpose of t and B is a non zero matrix.

6. Application of the Theorem

In the context of our problem we have $m = 0$ and $h_i \sim N(0, \sigma^2)$. We know that the normal distribution decreases faster to the infinity and consequently its characteristic function is an analystic function i.e:

$$Ee^{i\sum_{k=1}^n t_k h_k} = \exp\left(-\frac{1}{2}TBt\right).$$

Putting $t_k = i, k = 1, \cdots, N$ and applying the theorem on *analytic continuation* we obtain:

$$Ee^{h_1+\cdots+h_N} = \exp\left(-\frac{1}{2}TBt\right) \tag{46}$$

which is equivalent to

$$-\frac{1}{2}TBt = -\frac{1}{2}(i^2)(1,\cdots,1)B\begin{pmatrix}1\\ \vdots\\ 1\end{pmatrix} \tag{47}$$

$$= \frac{1}{2}\sum_{ij}^{n} b_{ij}$$

$$= \frac{1}{2}(n\rho(0) + 2(n-1)\rho(1) + 2(n-2)\rho(2) + \cdots + 2\rho(n-1)). \tag{48}$$

Finally we got $Ee^{h_1+\cdots+h_N} = \exp\left(\frac{n}{2} + \sum_{k=1}^{n-1}(n-k)\rho(k)\right)$. From the above result we can compute the N-steps ahead option price values; which are C_{*N} and C_N^* for dependent random variables h_i.

$$C_N^* = \sup_{\tilde{p}\sim p} E_{\tilde{p}} g\left(\frac{S_N}{e^{Nr}}\right) = \left(S_0 \exp\left(\frac{n}{2} + \sum_{k=1}^{n-1}(n-k)\right)\lim_{x\to\infty}\frac{g(S_h)}{S_h} + g(0)\right)e^{-Nr}$$
$$= S_0\exp\left(\frac{n}{2} + \sum_{k=1}^{n-1}(n-k)\rho(k)\right)e^{-Nr}$$

$$C_{*N} = e^{-Nr}\inf_{\tilde{p}\sim p} E_{\tilde{p}} g(S_N)$$
$$= g\left(S_0\exp\left(\frac{n}{2} + \sum_{k=1}^{n-1}(n-k)\rho(k)\right)e^{-Nr}e^{\frac{N\sigma^2}{2}}\right)e^{-Nr}$$
$$= \begin{cases} S_0\exp\left(\frac{n}{2} + \sum_{k=1}^{n-1}(n-k)\rho(k) - N_r\right) \\ \qquad\qquad - ke^{-Nr}e^{\frac{N\sigma^2}{2}}, & S_0\exp\left(\frac{n}{2} + \sum_{k=1}^{n-1}(n-k)\rho(k)\right) > k \\ 0, & \text{otherwise} \end{cases}$$

Theorem 5.5. *Let g be a convex function and $C_N = E_{\tilde{p}}\left[\frac{g(S_N)}{e^{Nr}}\right]$ be the rational price of option. Let the stock price $S_N, N > 0$ be stochastic process (unbounded in the sence that can be infinitely large for a given market) where h_i is the sequence of dependent random variables. Then the rational price C_N always reaches it optimum values which are upper and lower bounds.*

Proof. The proof is obtained from our algorithm.

7. Conclusion

In this paper we have investigated an optimal value problem in an incomplete financial market given by its distribution and the representation problem of martingale measure by the Escher transform technique. We have proved that knowing the bound values of the stocks price, we can exactly compute the optimal option price by a recursive algorithm in both cases when the stock prices are independently identically distributed random variables and the case there are not. Our results are efficient in terms of measure (martingale measures) and easy to compute as an exponential forms. Finally our results extend the work of Rose-Anne Dane and Monique Jean blanc Picque where the stock price is given in a much more simple form i.e. $S_n = S_{n-1}(1+r)$; $r=$ constant; $S_N \sim N(0,\sigma^2)$. Also as we can realize all our results are obtained without the assumption of the square integrability of the Random Nikodym derivative and can be applied to many other complex cases.

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Jimbo H. C
Institute of Statistical Mathematics 4-6-7 Minami Azabu, Minato-ku, Tokyo 106-8569, Japan
and
Advanced Research Center for Science & Engineering, 3-4-1, Ohkubo, Shinjuku-ku, Room 59-416B
Waseda University, Tokyo, Japan
E-mail address: jimbo_maths@yahoo.com

Bozeman R. E
Dept of Mathematics Morehouse College, 830 west view Drive, Atlanta, Georgia, USA

Djounguela A.
Von Carl Ossiestzky Universitat, Dept of Applied Physics, Germany

Suzuki T.
Advanced Research Center for Science & Engineering, 3-4-1, Ohkubo, Shinjuku-ku, Room 59-416B
Waseda University, Tokyo-Japan