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SUBDIFFERENTIAL STABILITY OF THE DISTANCE FUNCTION AND ITS APPLICATIONS TO NONCONVEX ECONOMIES AND EQUILIBRIUM

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Abstract. This paper is devoted to the study of the class of uniform prox-regular sets and their applications. We prove the stability of this class under different operations. We also prove the stability of the subdifferential of the distance function associated with uniform prox-regular sets. These results are used to give an important stability result of the quasi-equilibrium prices in nonconvex economies. Another important application to the equilibrium theory for nonconvex sets is given.

1. Introduction

In [17], Clarke et *al.* introduced and studied the class of uniform prox-regular sets which they called the class of *proximally smooth sets*. This class has been succesfully used in many applications (see for instance [5, 6, 7, 9, 10, 8, 11, 12, 13]. It has been the subject and the main goal of many other papers (see for instance [13, 22]). Our present paper continues the study of this class and their applications. In Section 2, we recall some definitions and results needed in all the paper. In Section 3, we prove the stability of the class of lower C^2 functions under some operations (pointwise maximum, composition, and integral). These results are used to prove some stability results of the class of uniform prox-regular sets. Section 4 is devoted to the stability of the subdifferential of the distance function and of the normal cone associated with uniform prox-regular sets. In Sections 5 and 6 we present two different applications of our abstract results proved in Section 4.

2. Preliminaries

Throughout the paper, H will be a finite dimensional space. Let S be a closed subset of H. We denote by $d_S(.)$ the usual distance function to the subset S, i.e., $d_S(x) := \inf_{u\in S} \|x-u\|$. We need first to recall some notation and definitions that will be used in all the paper.

Let $f : H \longrightarrow I\!R \cup \{+\infty\}$ be a lower semicontinuous (l.s.c.) function and let x be any point where f is finite. We recall that:

- *The Clarke subdifferential* of f at x is defined by (see [23])

$$\partial^C f(x) = \{\xi \in H : \langle \xi, h\rangle \le f^{\uparrow}(x;h), \quad \text{for all } h \in H\},$$

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where $f^{\uparrow}(x;h)$ is *the generalized Rockafellar directional derivative* given by

$$f^{\uparrow}(x;h) := \limsup_{\substack{x' \to_f x \\ t \downarrow 0}} \inf_{h' \to h} t^{-1}\big[f(x'+th') - f(x')\big],$$

where $x' \longrightarrow^f x$ means $x' \longrightarrow x$ and $f(x') \longrightarrow f(x)$.

- *The proximal subdifferential* $\partial^P f(x)$ is the set of all $\xi \in H$ for which there exist $\delta, \sigma > 0$ such that for all $x' \in x + \delta I\!B$

$$\langle \xi, x' - x \rangle \leq f(x') - f(x) + \sigma \|x' - x\|^2.$$

Here $I\!B$ denotes the closed unit ball centered at the origin of H.

Note that one always has $\partial^P f(x) \subset \partial^C f(x)$ and by convention we set $\partial^P f(x) = \partial^C f(x) = \emptyset$ if $f(x)$ is not finite. Note also that $\partial^C f(x)$ is always closed convex and that $\partial^P f(x)$ is always convex but may be non closed.

If f is locally Lipschitz around x, then the generalized Rockafellar directional derivative $f^{\uparrow}(x;h)$ coincides with *the Clarke directional derivative* $f^0(x;h)$ defined by

$$f^0(x;h) := \limsup_{\substack{x' \to x \\ t \downarrow 0}} t^{-1}\big[f(x'+th) - f(x')\big].$$

Let S be a nonempty closed subset of H and x be a point in S. We recall (see [23, 19]) that *the Clarke normal cone* (resp. *the proximal normal cone*) of S at x is defined by $N^C(S;x) := \partial^C \psi_S(x)$ $\Big(\text{resp.} N^P(S;x) := \partial^P \psi_S(x)\Big)$, where ψ_S denotes the indicator function of S, i.e., $\psi_S(x') = 0$ if $x' \in S$ and $+\infty$ otherwise. Note that the proximal normal cone is also given by

$$N^P(S;x) := \{\xi \in H : \exists \alpha > 0 \text{ s.t. } x \in Proj_S(x + \alpha\xi)\}$$

where

$$Proj_S(u) := \{y \in S : d_S(u) = \|u - y\|\}.$$

Recall now that for a given $r \in]0, +\infty]$ a subset S is uniformly r-prox-regular (see [22]) or r-proximally smooth (see [17]) if and only if every nonzero proximal normal to S can be realized by an r-ball, this means that for all $\bar{x} \in S$ and all $0 \neq \xi \in N^P(S;\bar{x})$ one has

$$\Big\langle \frac{\xi}{\|\xi\|}, x - \bar{x} \Big\rangle \leq \frac{1}{2r}\|x - \bar{x}\|^2$$

for all $x \in S$. We make the convention $\frac{1}{r} = 0$ for $r = +\infty$. Recall that for $r = +\infty$ the uniform r-prox-regularity of S is equivalent to the convexity of S. In the following proposition we recall an important characterization proved in Bounkhel and Thibault [13] of the uniform prox-regularity that will be used in the proofs of our main results.

Theorem 2.1 ([13]). *Let S be a nonempty closed subset in H and let $r > 0$. Then the following are equivalent:*

a) *S is uniformly r-prox-regular;*

b) *for all $x \in H$, with $d_S(x) < r$, and all $\xi \in \partial^P d_S(x)$ one has*

$$\begin{cases} \langle \xi, x' - x \rangle \leq \dfrac{8}{r - d_S(x)} \|x' - x\|^2 + d_S(x') - d_S(x), \\ \\ \text{for all } x' \in H \text{ with } d_S(x') \leq r. \end{cases}$$

3. Uniform prox-regularity of level sets and lower-C^2 property

In [24], Rockafellar introduced in the finite dimensional setting an important class of nonsmooth functions which he called "lower-C^2". He showed that such class has favorable properties in optimization. We recall that a function $f : O \to I\!R$ is said to be lower-C^2 on an open subset O of $I\!R^n$ if on some neighbpurhood V of each $\bar{x} \in O$ there is a representation $f(x) = \max_{t \in T} f_t(x)$ in which the functions f_t are of C^2 on V and the index set T is a compact space such that $f_t(x)$ and $\nabla f_t(x)$ depend continously not on just on x but jointly on $(t, x) \in T \times V$. A particular example of lower-C^2 functions one has $f(x) = \max\{f_1(x), \ldots, f_m(x),\}$ when f_i is of class C^2.

Rockafellar [24] proved an important characterization of lower C^2 functions. He showed that a function f is lower-C^2 on an open set $O \subset I\!R^n$ if and only if , relative to some neighboorhoud of each point of O, there is an expression $f = g - \frac{\rho}{2}\| \cdot \|^2$, in which g is finite convex function and $\rho \geq 0$. In what follows we will take such local representation as a definition of lower C^2 functions in general Hilbert spaces.

Definition 3.1. A function $f : O \to I\!R$ is said to be lower-C^2 on an open subset O of H if relative to some neighbpurhood of each point of O there is a representation $f = g - \frac{\rho}{2}\| \cdot \|^2$, in which g is finite convex function and $\rho \geq 0$.

The following characterization of lower-C^2 functions over convex compact sets will be required in all the paper. Its proof is straightforward.

Proposition 3.1. *Let K be a convex compact subset of H. Then a function f is lower-C^2 on K if and only if there exists a real positive number $\rho \geq 0$ such that $f + \frac{\rho}{2}\| \cdot \|^2$ is a finite convex function on K. In such case we will say that f is ρ-lower-C^2 on K.*

Note that it is easy to see that for a function f that is lower-C^2 over an open set O all the classical subdifferentials included in the Clarke one, coincide at each point x in O. In what follows we will denote $\partial f := \partial^P f = \partial^C f$ for such functions.

Now, we prove a characterization of lower-C^2 functions in terms of their subdifferentials. It will be our main tool to study the stability of the class of lower-C^2 function. Also, it will be used to establish a connection between the lower-C^2 property of a function f and the uniform prox-regularity of its associated level set $[f \leq 0]$. Note that this result has been proved in [17]. We give here a simple and different proof.

Proposition 3.2. *Let K be a convex compact subset of H and let f be a Lipschitz function on K. Let $\rho > 0$. Then the following assertions are equivalent:*

(i) *f is ρ-lower-C^2 on K;*

(ii) *For each* $\bar{x} \in K$ *and* $\xi \in \partial f(\bar{x})$ *one has*

$$\langle \xi, x - \bar{x} \rangle \leq f(x) - f(\bar{x}) + \frac{\rho}{2}\|x - \bar{x}\|^2 \;\; \forall x \in K. \tag{3.1}$$

Proof. (i) $\Rightarrow$ (ii): Let $\bar{x} \in K$ and $\xi \in \partial f(\bar{x})$. Then one has $\xi + \rho\bar{x} \in \partial(f + \frac{\rho}{2}\|\cdot\|^2)(\bar{x})$. By Proposition 3.2 the function $f + \frac{\rho}{2}\|\cdot\|^2$ is finite and convex on K and so for all $x \in K$ one has

$$\langle \xi + \rho\bar{x}, x - \bar{x} \rangle \leq (f + \frac{\rho}{2}\|\cdot\|^2)(x)f(x) - (f + \frac{\rho}{2}\|\cdot\|^2)(\bar{x}).$$

This ensures

$$\begin{aligned}\langle \xi, x - \bar{x} \rangle &\leq f(x) - f(\bar{x}) + \frac{\rho}{2}[\|x\|^2 - \|\bar{x}\|^2 - 2\langle \bar{x}, x - \bar{x} \rangle] \\ &= f(x) - f(\bar{x}) + \frac{\rho}{2}\|x - \bar{x}\|^2,\end{aligned}$$

for all $x \in K$ and then the proof of (3.1) is complete.

(ii) $\Rightarrow$ (i): By (3.1) we have for each $x \in K$

$$f(x) \geq f(\bar{x}) + \langle \xi, x - \bar{x} \rangle - \frac{\rho}{2}\|x - \bar{x}\|^2 \;\text{ for all }\; \bar{x} \in K \;\text{ and }\; \xi \in \partial f(\bar{x}).$$

So, we have

$$f(x) = \max_{(\bar{x},\xi) \in K \times \mathcal{E}} \{f(\bar{x}) + \langle \xi, x - \bar{x} \rangle - \frac{\rho}{2}\|x - \bar{x}\|^2\},$$

where $\mathcal{E} := \bigcup_{x \in K} \partial f(x)$ which is a compact subset in H by Theoreme II-25 in [15]. It follows then that $f = g - \frac{\rho}{2}\|\cdot\|^2$ on K with

$$g(x) = \max_{(\bar{x},\xi) \in K \times \mathcal{E}} \{f(\bar{x}) + \langle \xi, x - \bar{x} \rangle - \frac{\rho}{2}\|\bar{x}\|^2\}$$

which is a finite convex function on K. Thus, Proposition 3.2 completes the proof of this implication. □

The study of the stability under some operations of the class of lower-C^2 functions is very important for applications. Noting that it follows directly from Definition 3.1 that the addition of lower-C^2 functions on an open subset Ω of H is lower-C^2 on Ω. In what follows we prove the stability of that class under the following operations: pointwise maximum, composition, and integral.

Proposition 3.3 (Pointwise maximum of lower-C^2 functions.)**.** *The pointwise maximum of Lipschitz lower-*C^2 *functions over a convex compact set* K *of* H*, is Lipschitz lower-*C^2 *on* K*.*

Proof. Assume that $f(x) := \max_{1 \leq i \leq m} f_i(x)$, for all $x \in K$, where f_i, $1 \leq i \leq m$ is Lipschitz ρ_i-lower-C^2 on K. The Lipschitzness of f is obvious. So we have to show its lower-C^2 property. Taking $\rho := \max_{1 \leq i \leq m} \rho_i$, we get $(f_i)_{1 \leq i \leq m}$ are Lipschitz ρ-lower-C^2 on K. Fix now any $x \in K$ and any $\xi \in \partial^C f(x)$. By subdifferential calculus there exists $\xi_j \in \partial f_j(x)$ and $\alpha_j \geq 0$, $j \in I(x) := \{i \in \{1, \ldots, m\} : \;\; f_i(x) = f(x)\}$

such that $\xi = \sum_{j \in I(x)} \alpha_j \xi_j$ and $\sum_{j \in I(x)} \alpha_j = 1$. By Proposition 3.2 we obtain for all $j \in I(x)$ and all $x' \in K$

$$\langle \xi_j, x' - x \rangle \leq f_j(x') - f_j(x) + \frac{\rho}{2}\|x' - x\|^2.$$

This yields, for all $x' \in K$

$$\begin{aligned}\langle \xi, x' - x \rangle &= \sum_{j \in I(x)} \alpha_j \langle \xi_j, x' - x \rangle \leq \sum_{j \in I(x)} \alpha_j [f_j(x') - f_j(x) + \frac{\rho}{2}\|x' - x\|^2] \\ &\leq \sum_{j \in I(x)} \alpha_j [f(x') - f(x) + \frac{\rho}{2}\|x' - x\|^2] \leq f(x') - f(x) + \frac{\rho}{2}\|x' - x\|^2\end{aligned}$$

and so by Proposition 3.2, the function f is ρ-lower-C^2 on K. □

Proposition 3.4 (Composition)**.** *Let $F : H \to H'$ (H' is another Hilbert space) be a C^2 mapping and K be a convex compact subset in H and let h be a Lipschitz ρ-lower-C^2 function over $F(K)$. Then the function $f = h \circ F$ is ρ'-lower-C^2 on K for some $\rho' > 0$.*

Proof. The fact that f is Lipschitz is straightforward. Let $\alpha := \sup_{x \in K} \|\nabla F(x)\|$ and $\beta := \sup_{x \in K} \|\nabla^2 F(x)\|$, and let $\lambda > 0$ be the Lipschitz canstant of h over K. Fix now any $x \in K$ and any $\xi \in \partial^C f(x)$. By subdifferential calculus there exists $\zeta \in \partial h(F(x))$ such that $\xi = \nabla F(x)^* \zeta$. Using Propsotion 3.2 and the fact that h is ρ-lower-C^2 over $F(K)$, we get for all $x' \in K$,

$$\begin{aligned}\langle \zeta, F(x') - F(x) \rangle &\leq f(x') - f(x) + \frac{\rho}{2}\|F(x') - F(x)\|^2 \\ &\leq f(x') - f(x) + \frac{\alpha^2 \rho}{2}\|x' - x\|^2.\end{aligned}$$

On the other hand, as F is C^2 we have

$$F(x') = F(x) + \nabla F(x)(x' - x) + \frac{1}{2}\langle \nabla^2 F(x + \theta(x' - x))(x' - x), x' - x \rangle,$$

with $c = x + \theta(x' - x) \in K$ for some $\theta \in [0, 1]$. Therefore, this equality and the inequality before give

$$\begin{aligned}\langle \xi, x' - x \rangle &= \langle \zeta, \nabla F(x)(x' - x) \rangle \\ &= \langle \zeta, F(x') - F(x) \rangle + \frac{1}{2}\langle \zeta, \langle \nabla^2 F(c)(x' - x), x' - x \rangle \rangle \\ &\leq f(x') - f(x) + \frac{\alpha^2 \rho}{2}\|x' - x\|^2 + \frac{1}{2}\|\zeta\|\|\nabla^2 F(c)\|\|x' - x\|^2 \\ &\leq f(x') - f(x) + \frac{\rho'}{2}\|x' - x\|^2,\end{aligned}$$

where $\rho' := \alpha^2 \rho + \beta\lambda$. The proof then is complete by Proposition 3.2. □

Now we are going to study the stability under the integral operation. Let $I := [0,T]$ with $T > 0$ and let us consider the functional I_f defined from $L^p(I,H)$, with $p \in [2,+\infty[$, to $]-\infty,+\infty]$ by

$$I_f(u) := \int_0^T f(t,u(t))dt, \quad \text{for all } u \in L,$$

where f is a function from $I \times H$ to $]-\infty,+\infty]$.

Before proving our stability result we need the following lemma. Its proof is standard.

Lemma 3.1. *For any $p \in [2,+\infty[$ the function $h : L^p(I,H) \to I\!R$ defined by*

$$h(u) := \frac{1}{2}\int_0^T \|u(t)\|^2 dt, \quad \text{for all } u \in L^p(I,H)$$

is continuously Fréchet differentiable on $L^p(I,H)$ and its Fréchet derivative is given by

$$\langle \nabla h(u), v\rangle = \langle u, v\rangle = \int_0^T \langle u(t), v(t)\rangle dt.$$

Let us consider the following assumptions:

(A_1) $f : I \times H \to I\!R$ is measurable with respect to the σ-field of subsets of $I \times H$ generated by the Lebesgue sets in I and the Borel sets in H.

(A_2) there exist $a \in L^q(I,H)$, $b \in L^1(I)$, and $c \in L^p(I,H)$ such that

$$f(t,c) \in L^1(I) \text{ and } f(t,x) \geq \langle a(t), x\rangle + b(t),$$

for all $t \in I$ and all $x \in H$. Here q satisfies $1/p + 1/q = 1$.

Theorem 3.1. *Let K be a convex compact subset of H and $\rho > 0$. Let f be a continuous fucntion from $I \times H$ to $I\!R$. Assume that f satisfies the assumptions (A_1) and (A_2) and for all $t \in I$ the function $f(t,\cdot)$ is ρ-lower-C^2 on K. Then the functional I_f is ρ-lower-C^2 on the set $\mathcal{K} := \{u \in L^p(I,H) :\ u(t) \in K$ for all $t \in [0,T]\}$. Furthermore, for each $u \in \mathcal{K}$ one has*

$$\partial I_f(u) = \int_0^T \partial f(t,u)dt := \{\xi \in L^q(T,H) :\ \xi(t) \in \partial f(t,u(t)) \text{ a. e. on } [0,T]\}.$$

Proof. By Proposition 3.2 one has $g(t,\cdot) := f(t,\cdot) + \frac{\rho}{2}\|\cdot\|^2$ is finite convex on K for all $t \in I$. Then for any $x \in K$ and any $t \in I$ one has

$$\partial f(t,x) = \partial g(t,x) - \rho x,$$

because the norm $\|\cdot\|$ of H is smooth. Now, since it is easy to see that g also satisfies the assumptions (A_1) and (A_2), then Proposition 2.8 in [3] and the convexity of $g(t,\cdot)$ ensure that the functional I_g is finite convex on $\mathcal{K}$ and its subdifferential at any $u \in \mathcal{K}$ is given by

$$\partial I_g(u) = \int_0^T \partial g(t,u)dt := \{\zeta \in L^q(T,H) :\ \zeta(t) \in \partial g(t,u(t)) \text{ a. e. on } I\}. \tag{3.2}$$

Fix now any $u \in \mathcal{K}$. Then the calculus rules for subdifferentials and Lemma 3.1 yield

$$\partial I_f(u) = \partial(I_g - \rho h)(u) = \partial I_g(u) - \rho \nabla h(u) = \partial I_g(u) - \rho u.$$

Therefore, for any $\xi \in \partial I_f(u)$ one has $\xi + \rho\nabla h(u) \in \partial I_g(u)$ and so by (3.2) we get $\xi(t) \in \partial g(t, u(t)) - \rho\nabla h(u)(t) = \partial f(t, u(t))$ a.e. on I, as claimed. Conversely, it is easy to see that every $\xi \in L^q(I, H)$ satisfying the latter belongs to $\partial I_f(u)$. This completes the proof. □

As a direct application of our subdifferential formula in Theorem 3.1 we give a necessary optimality condition for the following nonconvex variational problem:

$$(\mathcal{P}) \qquad \text{minimize} \int_0^T f(t, u(t))dt$$

over $\mathcal{K} = \{u \in L^2(I, H) : \ u(t) \in K \text{ for all } t \in [0, T]\}$, where K is a convex compact subset of H and f satisfies the hypothesis in Theorem 3.1. We note that $\mathcal{K}$ is a closed convex set in $L^2(I, H)$.

Theorem 3.2. *If u solves the problem $(\mathcal{P})$ then*

$$f(t, u(t)) = \min_{v \in K}\{f(t, v) + \frac{\rho}{2}\|u(t) - v\|^2\}, \text{ for a. e. } t \in [0, T].$$

Proof. Let u be a solution of $(\mathcal{P})$. Then $0 \in \partial I_f(u) + N(\mathcal{K}; u)$ and so there exists $\xi \in \partial I_f(u)$ with $-\xi \in N(\mathcal{K}; u)$. Using Theorem 3.1 we get for a.e. $t \in [0, T]$

$$\xi(t) \in \partial f(t, u(t)) = \partial g(t, u(t)) - \rho u(t),$$

where g is as in Theorem 3.1 (a convex finite function on K). Then for every $x \in K$ and for a.e. $t \in [0, T]$ one gets

$$\big\langle \xi(t) + \rho u(t), x - u(t)\big\rangle \le g(t, x) - g(t, u(t)).$$

Now as $-\xi \in N(\mathcal{K}; u)$ it follows easily that

$$\big\langle -\xi(t), x - u(t)\big\rangle \le 0$$

for every $x \in K$ and for a.e. $t \in [0, T]$. Therefore from the definition of g and from both last inequalities we get

$$\begin{aligned} f(t, u(t)) - f(t, x) &= g(t, u(t)) - g(t, x) - \frac{\rho}{2}(\|u(t)\|^2 - \|x\|^2) \\ &\le \big\langle -\xi(t) - \rho u(t), x - u(t)\big\rangle - \frac{\rho}{2}(\|u(t)\|^2 - \|x\|^2) \\ &\le \rho\big\langle u(t), u(t) - x\big\rangle + \frac{\rho}{2}(\|u(t)\|^2 - \|x\|^2) = \frac{\rho}{2}\|u(t) - x\|^2, \end{aligned}$$

for every $x \in K$ and for a.e. $t \in [0, T]$. Thus completing the proof. □

In [17] the authors proved that a function f is lower-C^2 if and only if its epigraph $\operatorname{epi} f$ is uniformly prox-regular by using the characterization given in Proposition 3.2. In the next theorem, we will establish a sufficient condition of uniform prox-regularity for level sets.

Theorem 3.3. *Let K be any convex compact subset in H and let f be a Lipschitz ρ-lower-C^2 function on K. Let $m := \inf\{\|\xi\| : \xi \in \partial f(x) \text{ with } x \in K\}$. Then either $m = 0$ or the level set $S := \{x \in K : \ f(x) \le 0\}$ is uniformly r-prox-regular for $r := \dfrac{1}{m\rho}$.*

Proof. Put $\mathcal{E} := \bigcup_{x\in K} \partial f(x)$. Since $x \mapsto \partial f(x)$ is compact-valued and upper hemicontinuous and as K is compact then $\mathcal{E}$ is a compact set in H. Hence $m < +\infty$. Assume that $m \neq 0$. Fix any $\bar{x} \in S$ and any $0 \neq \xi \in N(S;\bar{x})$. Without loss of generality, we may assume that $f(\bar{x}) = 0$, because in the other case, i.e., $f(\bar{x}) < 0$ one has $N(S;\bar{x}) = \{0\}$. By Theorem 2.4.7 in [16] there exists $\lambda_\xi > 0$ such that $\frac{\xi}{\lambda_\xi} \in \partial f(\bar{x})$. Further, $\|\frac{\xi}{\lambda_\xi}\| > m > 0$. On the other hand, by Proposition 3.2, we have

$$f(x) \geq -\frac{\rho}{2}\|x-\bar{x}\|^2 + \langle \frac{\xi}{\lambda_\xi}, x-\bar{x}\rangle + f(\bar{x}) \;\; \forall y \in K.$$

Hence

$$\langle \xi, y-\bar{x}\rangle \leq \frac{\lambda_\xi \rho}{2}\|x-\bar{x}\|^2 \;\; \forall x \in S,$$

and hence

$$\begin{aligned}\langle \frac{\xi}{\|\xi\|}, x-\bar{x}\rangle &\leq \frac{\rho\lambda_\xi}{2\|\xi\|}\|x-\bar{x}\|^2\\ &\leq \frac{\rho}{2m}\|x-\bar{x}\|^2,\end{aligned}$$

for all $x \in S$. This ensures that S is uniformly r-prox-regular with $r := \frac{m}{\rho}$. The proof then is complete. □

Using this theorem and our stability resluts of the class of lower-C^2 functions proved above we prove in the following corollary the uniform prox-regularity of some special level sets.

Corollary 3.1. *Let K be a convex compact subset in H.*

1– *If f_1 and f_2 are two Lipschitz lower-C^2 functions on K satisfying for all $x \in K$ $\partial^P f_1(x) \cap \{-\partial^P f_1(x)\} = \emptyset$, then, the set $\{x \in K : \;\; f_1(x) + f_2(x) \leq 0\}$ is uniformly prox-regular.*

2– *If f_i, $i = 1,\ldots,N$ are Lipschitz lower-C^2 functions on K satisfying for all $x \in K$ and all $i = 1,\ldots,N$ $0 \notin \partial^P f_i(x)$, then, the set $\{x \in K : \;\; f_i(x) \leq 0$ for all $i = 1,\ldots,N\}$ is uniformly prox-regular.*

3– *If $F : H \to H'$ (H' is another Hilbert space) is a C^2 mapping and h is a Lipschitz lower-C^2 function on $F(K)$ satisfying for all $y \in F(K)$ $0 \notin \partial h(y)$, then, the set $\{x \in\in K : \;\; h \circ F(x) \leq 0\}$ is uniformly prox-regular.*

4– *If f is a continuous fucntion from $I \times H$ to $I\!R$ satisfying the assumptions (A_1) and (A_2). Assume that for all $t \in I$ the function $f(\cdot,t)$ is Lipschitz ρ-lower-C^2 on K and for all $x \in K$ and all $t \in I$ $0 \notin \partial f(t,x)$. Then, the set $\{u \in L^p(I,H) : \;\; u(t) \in K$ and $I_f(u) \leq 0\}$ is uniformly prox-regular.*

4. Subdifferential and co-normal stability

Our purpose in this section is to study the stability of normal cones and of the subdifferential of the distance functions to uniformly prox-regular sets. That property is very useful for applications. Our motivations come from some applications in economies and equilibrium theory (see the next sections). We start with the following definitions.

Definition 4.1. Let $\{S_k\}_k$ be any sequence of nonempty closed sets in H. We will say that a nonempty closed set S is *the Painlevé-Kuratowski PK-lower limit* (resp. PK-upper limit) of S_k provided that

$$S \subset \liminf_k S_k := \{x \in H : \text{ there exists } x_k \to x \text{ such that } x_k \in S_k\},$$

$$(\text{resp.} \quad \{x \in H : \text{ there exists } x_k \to x \text{ such that } x_k \in S_{s(k)}\} =: \limsup_k S_k \subset S.)$$

Here $S_{s(k)}$ is a subsequence of S_k.

We will say that S_k PK-converges to S or S is the PK-limit of S_k provided that S is both the PK-upper limit and the PK-lower limit of S_k.

Definition 4.2. Let $\{S_k\}_k$ be a sequence of nonempty closed sets in H that converges in some sense to a closed set S in H. We will say that the sequence $\{S_k\}_k$ is *subdifferentially stable* if one has

$$\limsup_{x_k \longrightarrow \bar{x}} \partial^P d_{S_k}(x_k) \subset \partial^P d_S(\bar{x}),$$

that is, for any sequence x_k (not necessarily in S_k) and such that $x_k \to x$ and any $\xi \in \partial^P d_{S_k}(x_k)$ weakly converging to some $\xi \in H$, one has $\xi \in \partial^P d_S(\bar{x})$. In the same way, we will say that $\{S_k\}_k$ is *co-normally stable* provided that

$$\limsup_k N^P(S_k, x_k) \subset N^P(S, x),$$

that is, for any sequence x_k such that $x_k \in S_k$ and $x_k \to x$ and any $\xi \in N^P(S_k, x_k)$ weakly converging to some $\xi \in H$, one has $\xi \in N^P(S, x)$.

We recall the following lemma needed in the proof of Theorem 4.1. It gives a characterization of the PK-lower limit in terms of the distance function to sets. For its proof we refer the reader to [25].

Lemma 4.1. *Let $\{S_k\}_k$ and S be nonempty closed sets in H. Then S is the PK-lower limit of the sequence $\{S_k\}_k$ if and only if there exists for each $\rho > 0$ and $\epsilon > 0$ an integer $k_0 \in I\!N$ such that for all $x \in \rho I\!B$ and all $k \geq k_0$ one has*

$$d(x, S_k) \leq d(x, S) + \epsilon.$$

Now we are ready to prove the subdifferential and co-normal stability for general uniformly prox-regular sets under an additional hypthesis on their distance functions.

Theorem 4.1. *Let $\{S_k\}_{k \in I\!N}$ be a sequence of nonempty closed subsets in H and let S be a nonempty closed in H. Let $r > 0$ and $\bar{x} \in S$. Assume that S is the PK-lower limit of $\{S_k\}_{k \in I\!N}$ and that all the subsets $\{S_k\}_{k \in I\!N}$ are uniformly r-prox-regular. Then*

(i) *the sequence d_{S_k} is subdifferentially stable, that is,*

$$\limsup_{x_k \xrightarrow{S_k} \bar{x}} \partial^P d_{S_k}(x_k) \subset \partial^P d_S(\bar{x}),$$

where $x_k \xrightarrow{S_k} \bar{x}$ *means that* x_k *converging to* $\bar{x}$ *and* $x_k \in S_k$ *for all* $k \in I\!N$.

(ii) *If, in addition,* H *is a finite dimensional space, then the sequence* S_k *is co-normally stable.*

Proof. (i) Let $x_k \xrightarrow{S_k} \bar{x}$ and $\xi_k \longrightarrow \bar{\xi}$ with $\xi_k \in \partial^P d_{S_k}(x_k)$ for all $k \in I\!N$. As the subsets $\{S_k\}_{k \in I\!N}$ are uniformly r-prox-regular, Theorem 3.1 ensures that for all $k \in I\!N$ one has

$$(2.1) \qquad \begin{cases} \langle \xi_k, x - x_k \rangle \leq \dfrac{2}{r}\|x - x_k\|^2 + d_{S_k}(x), \\ \\ \text{for all } x \in H \text{ with } d_{S_k}(x) \leq r. \end{cases}$$

Fix any $y \in \bar{x} + \frac{r}{2}I\!B$. Then, by Lemma 4.1 there exists $k_0 \in I\!N$ such that

$$(2.2) \qquad d_{S_k}(y) \leq d_S(y) + \frac{1}{1+k} \text{ for all } k \geq k_0.$$

One may choose k_0 large enough so that $\dfrac{1}{1+k} \leq \dfrac{r}{2}$ for all $k \geq k_0$. Thus, one gets $d_{S_k}(y) \leq d_S(y) + \frac{r}{2} \leq \|y - \bar{x}\| + \frac{r}{2} \leq \frac{r}{2} + \frac{r}{2} \leq r$ and so one can apply (2.1) with $x := y$ to get for all $k \geq k_0$

$$\langle \xi_k, y - x_k \rangle \leq \frac{2}{r}\|y - x_k\|^2 + d_{S_k}(y)$$

and by (2.2) one obtains for all $k \geq k_0$

$$\langle \xi_k, y - x_k \rangle \leq \frac{2}{r}\|y - x_k\|^2 + d_S(y) + \frac{1}{1+k}.$$

By letting $k \to +\infty$ in the last inequality one gets

$$\langle \bar{\xi}, y - \bar{x} \rangle \leq \frac{2}{r}\|y - \bar{x}\|^2 + d_S(y) - d_S(\bar{x}),$$

for all $y \in \bar{x} + \frac{r}{4}I\!B$. This ensures that $\bar{\xi} \in \partial^P d_S(\bar{x})$.

Assume that $dim\, H < +\infty$. Let $x_k \xrightarrow{S_k} \bar{x}$ and $\xi_k \longrightarrow \bar{\xi}$ with $\xi_k \in N^P(S_k, x_k)$ for all $k \in I\!N$. Put $\zeta_k := \frac{\xi_k}{1+\|\xi_k\|}$. Then $\zeta_k \in N^P(S_k, x_k) \cap I\!B$ and hence by Theorem 4.1 in [14] one gets $\zeta_k \in \partial^P d_{S_k}(x_k)$. Now as $dim\, H < +\infty$ the sequence ζ_k converges to $\frac{\bar{\xi}}{1+\|\bar{\xi}\|}$ and so we get by (i) that $\frac{\bar{\xi}}{1+\|\bar{\xi}\|} \in \partial^P d_S(\bar{x}) \subset N^P(S, \bar{x})$, which ensures that $\bar{\xi} \in N^P(S, \bar{x})$. This completes the proof. □

Now we proceed to prove a similar result for level sets. First, we recall the following definition

Definition 4.3. Let $\{f_k\}_k$ be any sequence of functions on H and let $x \in H$. We will say that f_k *upper-epi-converges* to some function f at x provided that

$$\text{epi}-\limsup_n f_n(x) \leq f(x),$$

or equivalently there exists $x_k \to x$ such that

$$\limsup_k f_k(x_k) \leq f(x).$$

Recall now the following lemma needed in the proof. Its proof can be found in [25].

Lemma 4.2. *Let $\{f_k\}_k$ be a sequence of functions on H. Assume that $\{f_k\}_k$ upper-epi-converges to some function f over H. Then one has*

$$[f \leq 0] \subset \liminf_k \{[f_k \leq \alpha_k]\},$$

for some sequence $\alpha_k \downarrow 0$. In other words the level set $[f \leq 0]$ is the PK-lower limit of the sequence of the level sets $\{[f_k \leq \alpha_k]\}_{k\in\mathbb{N}}$.

Now we are able to prove a subdifferential and co-normal stability result for level sets.

Theorem 4.2. *Let K be a convex compact subset of H and let $\{f_k\}_k$ be a sequence of functions on H that upper-epi-converges to some function f over H. Let $\sigma > 0$ and $\bar{x} \in K$ with $f(\bar{x}) = 0$. Assume that $\{f_k\}_k$ are σ-lower-C^2 on K with $0 \notin \partial f_k(x)$ for all $x \in K$. Then there exists $\alpha_k \downarrow 0$ such that the distance function d_{S_k} is subdifferentially stable, that is,*

$$\limsup_{x_k \xrightarrow{S_k} \bar{x}} \partial^P d_{S_k}(x_k) \subset \partial^P d_S(\bar{x}),$$

where $S_k := [f_k \leq \alpha_k]$ and $S := [f \leq 0]$. If, in addition, H is a finite dimensional space then the sequence S_k is co-normally stable.

Proof. By Lemma 4.2 there exists $\alpha_k \downarrow 0$ such that $S := [f \leq 0]$ is the PK-lower limit of the sequence $S_k := [f_k \leq \alpha_k]$. On the other hand as the functions $\{f_k\}_k$ are σ-lower-C^2 on K with $0 \notin \partial f_k(x)$ for all $x \in K$ we get by Theorem 3.3 that the subsets $S_k := [f_k \leq \alpha_k]$ are uniformly r-prox-regular for some $r > 0$. Thus Theorem 4.1 completes the proof. □

5. Applications to economies

In this section we consider the following economic model established by Arrow and Debreu (1959). In this model there are a finite number of goods l, consumers m, and producers n. Each consumer has a preference set-valued mapping $P_i : \prod_k X_k \rightrightarrows X_i$, where $X_i \subset \mathbb{R}^l$ is a set of consumptions for the consumer i. For a given $(x_1, \ldots, x_m) \in \prod_k X_k$, the set $clP_{i_0}(x_1, \ldots, x_m)$ (resp. $P_{i_0}(x_1, \ldots, x_m)$) represents all those elements in X_{i_0} that are preferred (resp. strictly preferred) to $(x_1, \ldots, x_m)$ for the consumer i_0. Each producer j has a production set $Y_j \subset \mathbb{R}^l$. Thus an economy $\mathcal{E}$ is defined as $\mathcal{E} = ((X_i), (P_i), (Y_j), e)$, where $e \in \mathbb{R}^l$ is the total initial endowment for the economy, that is, $e = \sum_{i=1}^m e_i$ with e_i is the initial endowment for the consumer i. A fundamental result of this theory is the second welfare theorem which gives a price decentralization of a Pareto optimum allocation. A recent result proves an extension of this welfare's theorem to general nonconvex economies was proved in [21]. Before stating it we need to recall the definitions of feasible allocation, Pareto optimum, and the *Asymptotic Included Condition* (**A.I.C**) for the economy $\mathcal{E}$.

Definition 5.1. • We will say that $((x_i^*), (y_j^*)) \in \mathbb{R}^{\ell m} \times \mathbb{R}^{\ell n}$ is a *feasible allocation* for the economy $\mathcal{E}$ if the following conditions are satisfied:

(a) for each $i = 1, ..., m$ and $j = 1, ..., n$, $x_i^* \in X_i$ and $y_j^* \in Y_j$.
(b) $\sum_i x_i^* - \sum_j y_j^* = e$.

• A feasible allocation $((x_i^*), (y_j^*)) \in I\!R^{\ell m} \times I\!R^{\ell n}$ is a *Pareto optimum* for the economy $\mathcal{E}$ if there is no feasible allocation $((x_i'), (y_j')) \in \prod X_i \times \prod Y_j$ such that

(i) for each $i \in \{1, ..., m\}$, $x_i' \in clP_i(x_1^*, ..., x_m^*)$
(ii) for some $i_0 \in \{1, ..., m\}$, $x_{i_0}' \in P_{i_0}(x_1^*, ..., x_m^*)$.

• We will say that $\mathcal{E}$ satisfies the *Asymptotic Included Condition* at a point $((x_i^*), (y_j^*)) \in \prod X_i \times \prod Y_j$ if there exists $i_0 \in \{1, ..., m\}$, $\epsilon > 0$ and a sequence $h_k \to 0$ such that for k sufficiently large we have

$$-h_k + \sum_i [clP_i^* \cap B(x_i^*, \epsilon)] - \sum_j [Y_j \cap B(y_j^*, \epsilon)] \subseteq P_{i_0}^* + \sum_{i \neq i_0} [clP_i^*] - \sum_j Y_j,$$

where $P_i^* := P_i(x_1^*, ..., x_m^*)$.

Theorem 5.1. [21] *Let $((x_i^*), (y_j^*)) \in I\!R^{\ell m} \times I\!R^{\ell n}$ be a Pareto optimum point for the economy $\mathcal{E} = ((X_i), (P_i), (Y_j), e)$ which satisfies the A.I.C. on it. If $x_i^* \in clP_i^*$, $i \in \{1, ..., m\}$ and Y_j is closed, then there exists a price vector $p^* \in I\!R^\ell$, $\|p^*\| \geq \frac{1}{n+m}$, such that*

$$p^* \in \bigcap_j \partial d_{Y_j}(y_j^*) \qquad\qquad -p^* \in \bigcap_i \partial d_{clP_i^*}(x_i^*).$$

In [21] the authors studied the stability of the quasi-equilibrium prices, that is, if we are given a sequence e^k converging to some $e \in I\!R^l$ and we assume that each economy $\mathcal{E}_k := ((X_i), (P_i), (Y_j), e_k)$ satisfies A.I.C. at a Pareto optimum point $((x_{i,k}^*), (y_{j,k}^*))$ and that this Pareto optimum sequence converges to some $((x_i^*), (y_j^*))$, is it possible to get the conclusion of Theorem 5.1 for the limit economy $\mathcal{E} := ((X_i), (P_i), (Y_j), e)$ at $((x_i^*), (y_j^*))$? They gave a positive answer under some hypothesis on the subdifferential of the distance function to the producers and preferences sets. Our main result in this section is in this vein. We will use our abstract results proved in Sections 3 and 4 to prove that stability for general nonconvex economies.

Let $\mathcal{E}^k := ((X_{i,k}), (P_{i,k}), (Y_{j,k}), e_k)$ be a sequence of nonconvex economies, with $e_k \to e \in I\!R^l$, $(X_{i,k})$ and $(Y_{j,k})$ lower-converge to (X_i) and (Y_j) respectively in $I\!R^l$, and the sequence of set-valued mappings $(P_{i,k})$ admits a PK-lower limit set-valued mappings (P_i) in the following sense: for each $i = 1, \ldots, m$ and for any $(x_{1,k}, \ldots, x_{m,k}) \to (x_1, \ldots, x_m)$ one has

$$(A_1) \qquad \liminf_k cl\, P_{i,k}(x_{1,k}, \ldots, x_{m,k}) \subset cl\, P_i(x_1, \ldots, x_m).$$

Assume that each economy $\mathcal{E}_k$ satisfies A.I.C. at some Pareto optimum point $((x_{i,k}^*), (y_{j,k}^*))$ and that this Pareto optimum sequence converges to some $((x_i^*), (y_j^*))$. Then we can state the main result.

Theorem 5.2. *Asuume that (A_1) is satisfied, $x_{i,k}^* \in clP_{i,k}^*$, $(Y_{j,k})$ are closed, and that both sequences $(Y_{j,k})$ and $(clP_{i,k})$ are uniformly prox-regular. Then there exists* *a price vector* $0 \neq p^* \in I\!R^\ell$ *such that*

$$p^* \in \bigcap_j \partial d_{Y_j}(y_j^*) \quad \textit{and} \quad -p^* \in \bigcap_i \partial d_{clP_i^*}(x_i^*).$$

Proof. By Theorem 5.1 there exists a sequence of prices $p_k^* \in I\!R^l$ with $1 \geq \|p_k^*\| \geq \frac{1}{n+m} > 0$, satisfying

$$p_k^* \in \bigcap_j \partial d_{Y_{j,k}}(y_{j,k}^*) \quad \text{and} \quad -p_k^* \in \bigcap_i \partial d_{clP_{i,k}^*}(x_{i,k}^*),$$

where $clP_{i,k}^* = clP_{i,k}(x_{i,k}^*)$. By our assumption (A_1) the set $clP_i^* := clP_i(x_i^*)$ is the PK-lower limit of the sequence $clP_{i,k}^*$. Now as all the sets clP_i^* are uniformly prox-regular we get by Theorem 4.1 that the limit $p^* \neq 0$ of p_k^* will belong to $-\bigcap_i \partial d_{clP_i^*}(x_i^*)$. Applying Theorem 4.1 once again with the sequences $Y_{j,k}$ and their PK-lower limits Y_j we get $p^* \in \bigcap_i \partial d_{Y_j}(y_j^*)$. Thus completing the proof. □

Many corollaries can be obtained directly from this theorem. We give only the two followings. First we begin with the case when the preference $(P_{i,k})$ defining the economie $\mathcal{E}_k$ is not perturbed, that is, $\mathcal{E}_k = ((X_i), (P_i), (Y_{j,k}), e_k)$.

Corollary 5.1. *Assume that the set-valued mapping* $cl\, P_i$ *is l.s.c. at* $((x_i^*), (y_i^*))$ *with uniformly prox-regular values, for each* $i \in \{1, \ldots, m\}$, $x_i^* \in clP_i^*$, *and that the sequence* $(Y_{j,k})$ *is uniformly prox-regular. Then there exists a price vector* $0 \neq p^* \in I\!R^\ell$ *such that*

$$p^* \in \bigcap_j \partial d_{Y_j}(y_j^*) \quad \textit{and} \quad -p^* \in \bigcap_i \partial d_{clP_i^*}(x_i^*).$$

Now, we assume that the preferences $P_{i,k}$ are defined by Lipschitz utility functions $f_{i,k} : I\!R^l \to I\!R$, that is,

$$P_{i,k}(x_1, \ldots, x_m) = \{x \in X : \ f_{i,k}(x) > f_{i,k}(x_{i,k})\}.$$

Corollary 5.2. *Assume that the following assumptions are satisfied:*

(i) $X_{i,k}$ *are convex compact in* $I\!R^l$ *and* $x_{i,k}^* \in clP_{i,k}^*$;
(ii) $(Y_{j,k})$ *are closed uniformly prox-regular sets.*
(iii) $-f_{i,k}$ *are* σ*-lower-*C^2 *on* $X_{i,k}$ *and upper-epi-converges to some function* f_i *over* $I\!R^l$ *with* $0 \notin \partial f_{i,k}(x_{i,k})$.

Then there exists a price vector $0 \neq p^* \in I\!R^\ell$ *such that*

$$p^* \in \bigcap_j \partial d_{Y_j}(y_j^*) \quad \textit{and} \quad -p^* \in \bigcap_i \partial d_{clP_i^*}(x_i^*),$$

where P_i *is the limit preference defined by the limit utility function* f_i.

6. Applications to existence of equilibrium

In this last section we are going to give an application of our co-normal stability result to the equilibrium theory for nonconvex sets in the infinite dimensional setting. We start with the following definition of generalized equilibrium.

Definition 6.1. For a closed set $S \subset H$ and a set-valued mapping $F : S \rightrightarrows H$, we will say that $\bar{x} \in S$ is a *generalized equilibrium* of F over S if one has

$$0 \in F(\bar{x}) - N(S, \bar{x}),$$

where $N(S; x)$ is a prescribed normal cone.

This concept of equilibrium has been considered in [25] and studied later by [20] in the finite dimensional setting. We recall now the classical definition of equilibrium.

Definition 6.2. For a closed set $S \subset H$ and a set-valued mapping $F : S \rightrightarrows H$, we will say that $\bar{x} \in S$ is an *equilibrium* of F over S if one has $0 \in F(\bar{x})$.

The existence of equilibrium has been the subject of many works in the finite (see for example [20, 18] and the refrences therein) and inifinte dimensional setting (see for example [4, 18] and the refrences therein). The best known equilibrium result in the Hilbert (infinite dimensional) setting is the following theorem by Ben-El-Mechaiekh and Kryszewski [4].

Theorem 6.1 ([4])**.** *Let S be a compact $\mathcal{L}$-retract in H with $\chi(S) \neq 0$. If $F : S \rightrightarrows H$ is an upper hemicontinuous map with closed convex values satisfying* for all $x \in S$ and all $p \in ret^{-1}(x)$*:*

$$\inf_{y \in F(x)} \langle p - x, y \rangle \leq 0,$$

then F has an equilibrium over S.

Here $\chi(S)$ stands the Euler characteristic of S. Recall that (see [4]) a closed subset $S \subset H$ is said to be $\mathcal{L}$-retract if there exist an open neighbourhood O of S, a continuous retraction $ret : O \to S$, and a constant $L \geq 0$ such that

$$\|x - ret(x)\| \leq L d_S(x), \quad \text{for all } x \in O.$$

This definition was introduced by [4] for metric spaces. To prove our main theorem in this section we need to prove some preliminary results.

Lemma 6.1. *Every uniformly prox-regular set is $\mathcal{L}$-retract.*

Proof. Let $r > 0$ be the constant of the uniform prox-regularity of S and put $U(r') := \{x \in H : \ 0 < d_S(x) < r'\}$ and $S(r') := \{x \in H : \ 0 \leq d_S(x) < r'\}$. It suffices to take $ret := Proj_S$, $O := S(r')$, and $L := 1$. Indeed, by Theorem 4.2 in [17], the projection $Proj_S$ is singl-valued Lipschitz mapping of rank $\frac{r}{r-r'}$ on $U(r')$ for all $r' \in]0, r[$. In particular, it is continuous on the open set $S(r')$. Finally, as $\|x - Proj_S(x)\| = d_S(x)$ for all $x \in S(r')$, the proof then is complete. □

Remark 6.1. Note that in Proposition 5.1 in [20] the authors proved in the finite dimensional setting that every uniformly prox-regular (more general every proximal nondegenerate (see [20] for the definition)) and compact set is $\mathcal{L}$-retract. In our result in Lemma 6.1 we dont need the compactness of S. So, it generalizes Proposition 5.1 in [20] to uniformly prox-regular sets not necessarily compact and to the Hilbert space setting.

Lemma 6.2. *Let S be a uniformly r-prox-regular subset in H and $F : S \rightrightarrows H$ be any set-valued mapping. Then the following assertions are equivalent:*

1– *for all* $x \in S$ *and all* $p \in Proj_S^{-1}(x)$ *one has*

$$\inf_{\xi \in F(x) - \partial d_S(x)} \langle p - x, \xi \rangle \le 0;$$

2– *for all* $x \in S$ *and all* $p \in Proj_S^{-1}(x)$ *one has*

$$\inf_{\xi \in F(x)} \langle p - x, \xi \rangle \le \|p - x\|;$$

3– *for all* $x \in S$ *and all* $p \in Proj_S^{-1}(x)$ *with* $p \neq x$ *one has*

$$\inf_{\xi \in F(x)} \langle \frac{p - x}{\|p - x\|}, \xi \rangle \le 1;$$

Proof. Assume that (1) holds. Then for any $x \in S$ and $p \in Proj_S^{-1}(x)$, there exists $\xi_1 \in F(x)$ and $\xi_2 \in \partial d_S(x)$ such that $\langle p - x, \xi_1 \rangle \le \langle p - x, \xi_2 \rangle$. So $\langle p - x, \xi_1 \rangle \le \|\xi_2\| \|p - x\| \le \|p - x\|$, because one always has $\partial d_S(x) \subset I\!B$. Therfore, (2) holds.

As the equivalence between (2) and (3)is obvious, we have to show (3) $\Rightarrow$ (1). Assume that (3) holds. Fix any $x \in S$ and $p \in Proj_S^{-1}(x)$ with $p \neq x$. Then by (3), there exists $\xi \in F(x)$ such that

$$\langle \frac{p - x}{\|p - x\|}, \xi \rangle \le 1. \tag{6.1}$$

As $p \in Proj_S^{-1}(x)$, we have $\dfrac{p - x}{\|p - x\|} \in \partial d_S(x)$. Put $\widetilde{\xi} := \xi - \dfrac{p - x}{\|p - x\|} \in F(x) - \partial d_S(x)$. Then (6.1) yields

$$\langle p - x, \widetilde{\xi} \rangle = \langle p - x, \xi \rangle - \langle p - x, \frac{p - x}{\|p - x\|} \rangle = \langle p - x, \xi \rangle - \|p - x\| \le 0.$$

Thus (1) holds and so the proof is complete. □

Now we are in position to prove our main result.

Theorem 6.2. *Let* S_k *be a sequence of compact uniformly* r*-prox-regular subsets in* H *with* $\chi(S_k) \neq 0$ *and let* $F : S_k \rightrightarrows H$ *be an upper hemicontinuous map with closed convex values. Assume that* S_k PK*-converges to some compact subset* S. *Assume also that for all* $x_k \in S_k$ *and all* $p_k \in Proj_{S_k}^{-1}(x_k)$ *one has*

$$\inf_{\xi_k \in F(x_k)} \langle p_k - x_k, \xi_k \rangle \le \|p_k - x_k\|. \tag{6.2}$$

Then F *has a generalized equilibrium over* S *with respect to the proximal normal cone, i.e., there exists* $\bar{x} \in S$ *such that* $0 \in F(\bar{x}) - N^P(S; \bar{x})$.

Proof. For every $k \ge 1$ we define the set-valued mapping $\widetilde{F}_k := F - \partial d_{S_k}$. By Lemma 6.2 our hypothesis (6.2) is equivalent to

$$\inf_{\xi_k \in \widetilde{F}_k(x)} \langle p_k - x_k, \xi_k \rangle \le 0;$$

for all $x_k \in S_k$ and all $p_k \in Proj_{S_k}^{-1}(x_k)$. On the other, by Lemma 6.1 the set S_k is $\mathcal{L}$-retract with $ret := Proj_{S_k}$. Then as it is easily to see that the set-valued mapping

$\widetilde{F}_k$ is upper hemicontinuous with closed convex values, we can apply Theorem 6.1. to get an equilibrium of $\widetilde{F}_k$ over S_k, i.e., there exists $\bar{x}_k \in S_k$ such that

$$0 \in \widetilde{F}_k(\bar{x}_k) = F(\bar{x}_k) - \partial d_{S_k}(\bar{x}_k). \tag{6.3}$$

Now, using the fact that S_k PK-converges to S, we get that $d_S(\bar{x}_k) \to 0$ as $k \to \infty$, which ensures the relative compactness of the sequence $\bar{x}_k$ because S is a compact set in H. There exists then some subsequence of $\bar{x}_k$ that converges to some point $\bar{x} \in S$. On the other hand, by the relation (6.3) there exists $\xi_k \in \partial d_{S_k}(\bar{x}_k) \cap F(\bar{x}_k) \subset I\!B$. Then, a subsequence of ξ_k may be extracted converging weakly to some $\bar{\xi}$. Finally, by our subdifferential stability result in Theorem 4.1, we conclude that $\bar{\xi} \in \partial^P d_S(\bar{x})$ and by the upper hemicontinuity of F we also have $\bar{\xi} \in F(\bar{x})$. Therefore,

$$0 \in F(\bar{x}) - \partial^P d_S(\bar{x}) \subset F(\bar{x}) - N^P(S; \bar{x}).$$

This ends the proof. □

Remark 6.2. 1– In the statement of Theorem 6.2, we specify the normal cone of S with which we work, because the limit set S is not necesarily uniformly prox-regular and so the classical subdifferentials a priori do not coincide with the proximal one. Therefore our result in Theorem 6.2 proves the existence of generalized equilibrium for nonconvex sets that are not necessarily uniformly prox-regular.

2– From the part (1) of Remark 6.2, our Theorem 6.2 cannot be covered by the main result in [20] even in the finite dimensional setting, because the limit set S in Theorem 6.2 is not necessarily proximally nondegenerate in the sense of [20].

3– Another approch, but with less importance relatively to our Theorem 6.2, that can be used to prove the existence of generalized equilibrium for uniformly prox-regular sets not necessarily convex, is to approximate a set S with uniformly prox-regular sets S_k satisfying (6.2) and all the other hypothesis of Theorem 6.2. Then we use our subdifferential stability result in Theorem 4.1 to get the condition (6.2) for the set S and then we follow the same argument in the proof of Theorem 6.2 to obtain a generalized equilibrium of the set-valued mapping F over S.

References

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