Journal of Nonlinear and Convex Analysis
Volume 5, Number 3, 2004, 321–329

Yokohama Publishers
ISSN 1880-5221 ONLINE JOURNAL

CONSTANT SELECTIONS FOR CONCAVE-CONVEX MAPS AND MINIMAX THEOREMS

MIRCEA BALAJ

Abstract. In this paper we obtain several intersection theorems for the values of a concave-convex or only concave set-valued mapping. From each of these theorems we derive a Sion type minimax theorem.

1. Introduction and preliminaries

Let X and Y be two sets and $T : X \multimap Y$ be a set-valued mapping (simply, a map), that is, a function that assigns to each $x \in X$ a unique subset $T(x)$ of Y. For each $y \in Y$ the set $T^{-1}(y) = \{x \in X : y \in T(x)\}$ is called the *fiber* of T at the point y and the complement in X of $T^{-1}(y)$ is called the *cofiber* of T at y and is denoted by $T^*(y)$. To a map $T : X \multimap Y$ are therefore associated two maps $T^{-1} : Y \multimap X$, the *inverse* of T, and $T^* : Y \multimap X$, the *dual* of T. We will say that a map T has *finite intersection property* if the family of its values has the finite intersection property.

A map $T : X \multimap Y$ between two convex sets is said to be *convex* if its values are convex, and *concave* if the cofibers $T^*(y)$ are convex. One can readily verify that T is concave if and only if

$$T(\text{co}\{x_1, x_2\}) \subset T(x_1) \cup T(x_2) \text{ for any } x_1, x_2 \in X, \tag{1}$$

where $\text{co}\{x_1, x_2\}$ denotes the convex hull of $\{x_1, x_2\}$.

T is *concave-convex* if it is both concave and convex. The concept of concave-convex map has been introduced by Greco [10] and its motivation comes from minimax theory. Given a function $f : X \times Y \to \overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ we would like to know if

$$\inf_{y \in Y} \sup_{x \in X} f(x, y) = \sup_{x \in X} \inf_{y \in Y} f(x, y).$$

We only need to establish the inequality

$$\inf_{y \in Y} \sup_{x \in X} f(x, y) \leq \sup_{x \in X} \inf_{y \in Y} f(x, y)$$

and we can therefore assume that $\sup_{x \in X} \inf_{y \in Y} f(x, y) < \infty$. In this setting, to a real number λ one can associate a map $T_\lambda : X \multimap Y$ defined as follows

$$T_\lambda(x) = \{y \in Y : f(x, y) \leq \lambda\}. \tag{2}$$

2000 *Mathematics Subject Classification.* 54C60, 49J35, 54H25.

Key words and phrases. concave-convex map, local intersection property, KKM property, fixed point, minimax.

Then, one can see that $\inf_{y\in Y}\sup_{x\in X} f(x,y) = \sup_{x\in X}\inf_{y\in Y} f(x,y)$ if and only if, the map T_λ has a constant selection (that is $\bigcap_{x\in X} T_\lambda(x) \neq \emptyset$), for each $\lambda > \sup_{x\in X}\inf_{y\in Y} f(x,y)$.

If X and Y are convex sets and the function f above is quasi-concave-convex (i.e., for each $x_0 \in X$, $y_0 \in Y$ and $\lambda \in \mathbb{R}$ the sets $\{x \in X : f(x,y_0) \geq \lambda\}$ and $\{y \in Y : f(x_0,y) \leq \lambda\}$ are convex), then for each real λ, T_λ is a concave-convex map. Therefore a theorem on the existence of constant selections for concave-convex maps can be transcribed to yield a minimax theorem for quasi-concave-convex functions.

A most important result, which has achieved the status of a reference point in minimax theory is that of Sion [16] who proved the following:

Theorem 1. *Let X be a convex set in a topological vector space, Y be a compact convex set in a topological vector space and $f : X \times Y \to \overline{\mathbb{R}}$ be an upper-lower-semi-continuous quasi-concave-convex function. Then* $\min_{y\in Y}\sup_{x\in X} f(x,y) = \sup_{x\in X}\min_{y\in Y} f(x,y)$.

In [8] Flåm and Greco introduced what they called the *simplex property.* Let us recall this concept. A map $T : X \to Y$ between two convex sets has the simplex property if for any simplex $S \subset X$ of dimension at least one and for any vertex $v \in S$,

$$\text{if } \bigcap_{x\in S\setminus\{v\}} T(x) \neq \emptyset \text{ then } \bigcap_{x\in S} T(x) \neq \emptyset.$$

The result which we now state forthwith is due to Flåm and Greco [8 Theorem 2.2] and relying on it, the two authors have obtained a generalization of Sion's minimax theorem.

Theorem 2. *Let X and Y be convex subsets of topological vector spaces and $T : X \multimap Y$ be a concave-convex map with nonempty compact values. If T has the simplex property, then $\bigcap_{x\in X} T(x) \neq \emptyset$.*

Unfortunately, the simplex property is rather elusive; there is still no convincing analytic translation of this property.

In this paper we shall see that if either X or Y is paracompact, we can replace the simplex property with other property but of topological nature. This property, introduced by Wu and Shen [18], and called the local intersection property is defined as follows:

Let X and Y be two topological spaces. A map $T : X \multimap Y$ is said to have the *local intersection property* if for each $x \in X$ with $T(x) \neq \emptyset$ there exists an open neighborhood $V(x)$ of x such that $\bigcap_{z\in V(x)} T(z) \neq \emptyset$.

It is not hard to see that each map with open fibers has the local intersection property but the example given in [18 p. 63] shows that the converse is not true.

Further on all topological spaces will be assumed separated.

The following lemma is a particular case of Theorem 1 in [18].

Lemma 3. *Let X be a paracompact topological space and Y be convex subset of a topological vector space. If $T : X \multimap Y$ is a map with nonempty convex values having the local intersection property, then it admits a continuous selection, i.e. there is a continuous function $p : X \to Y$ such that $p(x) \in T(x)$ for each $x \in X$.*

Let X and Y be topological spaces and λ a real number. A function $f : X \times Y \to \overline{\mathbb{R}}$ is said to be *λ-transfer upper semicontinuous* (resp., *weakly λ-transfer upper semicontinuous*) on X if for all $x \in X$ and $y \in Y$ with $f(x, y) < \lambda$ there exist some point $y' \in Y$ and a neighborhood $V(x)$ of x such that $f(z, y') < \lambda$ (resp., $f(z, y') \leq \lambda$) for all $z \in V(x)$.

The concept of λ-transfer upper semicontinuity has been introduced by Tian [17] but that of weakly λ-transfer upper semicontinuity seems to be new. Obviously upper semicontinuity $\Rightarrow$ λ-transfer upper semicontinuity $\Rightarrow$ weakly λ-transfer upper semicontinuity, for any $\lambda \in \mathbb{R}$, and not conversely.

The following example gives a function which is weakly λ-transfer upper semicontinuous but not λ-transfer upper semicontinuous on X, for some $\lambda \in \mathbb{R}$.

Example. Let $X = Y = \mathbb{R}$ and $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by

$$f(x, y) = \begin{cases} 0 & \text{if } x + y \in \mathbb{Q} \\ 1 & \text{if } x + y \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

Then f is weakly 1-transfer upper semicontinuous but not 1-transfer upper semicontinuous in x.

Lemma 4. *Let $f : X \times Y \to \overline{\mathbb{R}}$ be a function and λ a real number such that $\lambda > \sup_{x \in X} \inf_{y \in Y} f(x, y)$. If f is weakly λ-transfer upper semicontinuous on X, then the map T_λ defined by (2) has the local intersection property.*

Proof. Since $\lambda > \sup_{x \in X} \inf_{y \in Y} f(x, y)$, for $x \in X$ arbitrarily fixed there exists $y \in Y$ such that $f(x, y) < \lambda$, hence $y \in T_\lambda(x)$. Since f is weakly λ-transfer upper semicontinuous on X, there exist $y' \in Y$ and a neighborhood $V(x)$ of x such that

$$y' \in \bigcap_{z \in V(x)} \{y \in Y : f(z, y) < \lambda\} = \bigcap_{z \in V(x)} T_\lambda(x). \qquad \square$$

Another necessary concept is that of map with KKM property, introduced by Chang and Yen [3] and defined as follows.

Assume that X is a convex subset of a vector topological space and Y is a topological space. If $S, T : X \multimap Y$ are two maps such that $T(\text{co}A) \subset S(A)$ for each nonempty finite subset A of X, then S is called a *generalized KKM map* w.r.t. T. We say that a map $T : X \multimap Y$ has the *KKM property* if for any map $S : X \multimap Y$ generalized KKM w.r.t. T, the family $\{\overline{S}(x) : x \in X\}$ has the finite intersection property (where $\overline{S}(x)$ denote the closure of $S(x)$).

Let us observe that if $T : X \multimap Y$ is a concave-convex map, then by (1) one can easily prove, by induction, that T is a generalized KKM map w.r.t. itself.

2. Basic results

In this section we establish several existence theorems of constant selections for concave-convex or only concave maps. From each of these theorems we shall derive a Sion type minimax theorem.

Theorem 5. *Let X be a paracompact convex set and Y be a convex set, each in a topological vector space. If $T : X \multimap Y$ is a concave-convex map with nonempty closed values having the local intersection property, then T has the finite intersection property. If $T(x)$ is compact for at least one $x \in X$, then $\bigcap_{x\in X} T(x) \neq \emptyset$.*

Proof. It suffices to proof the first part of the theorem, the last part resulting by a standard topological argument. By Lemma 3, it follows that T has a continuous selection p. Since $T(\text{co}A) \subset T(A)$, it follows that $p(\text{co}A) \subset T(A)$ for each nonempty finite set $A \subset X$, i.e., T is a generalized KKM map w.r.t. p. Since any continuous function has the KKM property (see [15, Theorem 4]), T has the finite intersection property. □

If X is arbitrary set and Y topological space we say that a function $f : X \times Y \to \overline{\mathbb{R}}$ is *inf-compact* on Y (see [8]) if for any $\lambda \in \mathbb{R}$ there exists $x_0 \in X$ such that $\{y \in Y : f(x_0, y) \leq \lambda\}$ is relatively compact.

Corollary 6. *Let X be a paracompact convex set and Y be a convex set, each in a topological vector space. Suppose $f : X \times Y \to \overline{\mathbb{R}}$ is a function:*

(i) *quasi-concave-convex;*
(ii) *lower semicontinuous inf-compact on Y;*
(iii) *weakly λ-transfer upper semicontinuous on X, for any $\lambda > \sup_{x\in X} \inf_{y\in Y} f(x, y)$.*

Then, $\inf_{y\in Y} \sup_{x\in X} f(x, y) = \sup_{x\in X} \inf_{y\in Y} f(x, y)$.

Proof. As we have already seen it suffices to show that for any $\lambda > \sup_{x\in X} \inf_{y\in Y} f(x, y)$ the map $T_\lambda : X \multimap Y$ defined by (2) satisfies $\bigcap_{x\in X} T_\lambda(x) \neq \emptyset$. By (i), T_λ is concave-convex map, by (ii) T_λ has closed values and $T_\lambda(x_0)$ is compact for some $x_0 \in X$, and by (iii), via Lemma 4, T_λ has the local intersection property. The desired conclusion follows now from Theorem 5. □

Theorem 7. *Let X be a convex subset of a locally convex space, Y be a convex subset of a topological vector space and $T : X \multimap Y$ be a concave-convex map with nonempty closed values having the local intersection property. Then, for each compact $K \subset X$, $T|_{\text{co}K}$ has the finite intersection property.*

Proof. Since K is compact in a locally convex space, co$K$ is paracompact by Lemma 1 in [5]. Apply Theorem 5 to the map $T|_{\text{co}K}$. □

Remark. In [10, Proposition 2.3] Greco proves that if X and Y are convex subsets of topological vector spaces and $T : X \multimap Y$ is a concave-convex map with closed values, then $\bigcap_{x\in X} T(x) \neq \emptyset$ provided that there exists a finite open cover $\{G_1, \ldots, G_n\}$ of X such that $\bigcap_{x\in G_i} T(x) \neq \emptyset$ for each $i \in \{1, \ldots, n\}$. It is easy to see that a map T with nonempty values has the local intersection property if and only if there exists an open cover $\mathcal{G}$ of X such that $\bigcap_{x\in G} T(x) \neq \emptyset$ for each $G \in \mathcal{G}$. Therefore, when X is paracompact, Greco's result remains valid if the open cover is infinite.

It woud be of some interest to compare the next corollary with Theorem 4 in [12].

Corollary 8. *Let X be a convex subset of a locally convex space, Y a compact subset of a topological vector space and $f : X \times Y \to \overline{\mathbb{R}}$ a function lower semicontinuous on Y that satisfies conditions (i),(iii) in Corollary 6. Then*

$$\sup_{K \subset X} \inf_{y \in Y} \sup_{x \in coK} f(x, y) = \sup_{x \in X} \inf_{y \in Y} f(x, y),$$

where the supremum on the left-hand side is taken over all compact subsets of X.

Proof. We always have

$$\sup_{x \in X} \inf_{y \in Y} f(x, y) \leq \sup_{K \subset X} \inf_{y \in Y} \sup_{x \in coK} f(x, y).$$

To prove the reverse inequality choose a real number λ such that $\lambda > \sup_{x \in X} \inf_{y \in Y} f(x, y)$. By Theorem 7 and taking into account the compactness of Y, it follows that for each compact $K \subset X$ there exists $y \in \bigcap_{x \in coK} T_\lambda(x)$. Therefore for each compact $K \subset X$ we have $\inf_{y \in Y} \sup_{x \in coK} f(x, y) \leq \lambda$, whence $\sup_{K \subset X} \inf_{y \in Y} \sup_{x \in coK} f(x, y) \leq \lambda$. Clearly this implies the desired inequality. □

A map $T : X \multimap Y$ (X, Y topological spaces) is said to be *transfer closed-valued* [17], if for any $x \in X$, $y \in Y$ with $y \notin T(x)$ there exists $z \in X$ such that $y \notin \overline{T(z)}$. It has be shown in [17] that T is a transfer closed-valued map if and only if $\bigcap_{x \in X} T(x) = \bigcap_{x \in X} \overline{T(x)}$.

The next result is a version of Theorem 5 and the proof is similar.

Theorem 9. *Let X be a paracompact convex set and Y be a convex set, each in a topological vector space. Let $T : X \multimap Y$ be a transfer closed-valued concave-convex map with nonempty values having the local intersection property. If $T(x)$ is relatively compact for at least one $x \in X$, then $\bigcap_{x \in X} T(x) \neq \emptyset$.*

A function $f : X \times Y \to \overline{\mathbb{R}}$ (X, Y topological spaces) is said to be λ-*transfer lower semicontinuous on* Y, for some real λ (see [17]), if for all $x \in X$ and $y \in Y$ with $f(x, y) > \lambda$ there exists some point $z \in X$ and a neighborhood $V(y)$ of y such that $f(z, y') > \lambda$ for all $y' \in V(y)$. It can be easily verified that f is λ-transfer lower semicontinuous on Y if and only if T_λ is transfer closed-valued map.

From Theorem 9 one obtains

Corollary 10. *Let X be a paracompact convex set and Y be a convex set, each in a topological vector space. Suppose that $f : X \times Y \to \overline{\mathbb{R}}$ is a function*

(i) *quasi-concave-convex;*
(ii) *weakly λ-transfer upper semicontinuous on X and λ-transfer lower semicontinuous on Y, for any $\lambda > \sup_{x \in X} \inf_{y \in Y} f(x, y)$;*
(iii) *inf-compact on Y.*

Then $\inf_{y \in Y} \sup_{x \in X} f(x, y) = \sup_{x \in X} \inf_{y \in Y} f(x, y)$.

Next theorem generalizes under many aspects Proposition 5 (Intersection property for multifunctions with marginally closed values) in [11].

Theorem 11. *Let X be a finite dimensional convex set and Y a convex set. Suppose that either (a) X is compact and Y is a paracompact subset of a topological vector space, or (b) Y is a compact subset of a locally convex space. Let $T : X \multimap Y$ be a concave map satisfying the following conditions:*

(i) *$\bigcap_{x\in U} T(x)$ is closed for every open subset U of X;*
(ii) *there exists a map $\Omega : X \multimap Y$ with nonempty convex values and with the local intersection property such that $\Omega \subset T$.*

Then $\bigcap_{x\in X} T(x) \neq \emptyset$.

Proof. By Lemma 3 there exists a continuous selection of $\Omega, p : X \to Y$. Obviously $p(x) \in T(x)$ for each $x \in X$.

Let us suppose that $\bigcap_{x\in X} T(x) = \emptyset$. This means that the map T^* has nonempty values. By (i), T^* is lower semicontinuous. Since T is a concave map, T^* has convex values. Hence, since the values of T^* are finite dimensional , by Michael's selection theorem [14, Theorem 3.1"'], there is a continuous selection $p^* : Y \to X$ of T^*.

In case (a), by Brouwer's fixed point theorem, the continuous function $p^* \circ p : X \to X$ has a fixed point, while in case (b), by Tychonoff's fixed point theorem, $p \circ p^* : Y \to Y$ has a fixed point. Thus, in both cases, there exist $x_0 \in X$ and $y_0 \in Y$ such that $y_0 = p(x_0)$ and $x_0 = p^*(y_0)$. On the one hand we have $y_0 = p(x_0) \in T(x_0)$. On the other hand we have $x_0 = p^*(y_0) \in T^*(y_0)$, hence $y_0 \notin T(x_0)$. This contradiction completes the proof. □

A function $f : X \times Y \to \overline{\mathbb{R}}$ is said to be *marginally lower semicontinuous* on Y (see [11]) if for every open subset U of X the function $y \to \sup\limits_{x\in U} f(x, y)$ is lower semicontinuous on Y. It is clear that any function lower semicontinuous on Y is marginally lower semicontinuous on Y but the example given in [1, p.249] shows that the converse is not true.

Corollary 12. *Let X and Y be as in Theorem 11. Let $f, g : X \times Y \to \overline{\mathbb{R}}$ be two functions such that:*

(i) *$f \leq g$;*
(ii) *f is quasiconcave on X;*
(iii) *f is marginally lower semicontinuous on Y;*
(iv) *g is quasiconvex on Y;*
(v) *g is weakly λ-transfer upper semicontinuous on X, for any $\lambda > \sup\limits_{x\in X} \inf\limits_{y\in Y} g(x, y)$.*

Then $\inf\limits_{y\in Y} \sup\limits_{x\in X} f(x, y) \leq \sup\limits_{x\in X} \inf\limits_{y\in Y} g(x, y)$.

Proof. Let $\lambda > \sup\limits_{x\in X} \inf\limits_{y\in Y} g(x, y)$ be arbitrary fixed. Define $T_\lambda, \Omega_\lambda : X \multimap Y$ by

$$T_\lambda(x) = \{y \in Y : f(x, y) \leq \lambda\}, \ \ \Omega_\lambda(x) = \{y \in Y : g(x, y) \leq \lambda\}.$$

From (ii), T_λ is concave map. Since f is marginally lower semicontinuous on Y, for each open $U \subset X$ the set

$$\bigcap_{x\in U} T_\lambda(x) = \{y \in Y : \sup_{x\in U} f(x, y) \leq \lambda\}$$

is closed, hence T_λ satisfies condition (i) in Theorem 11. By (i), $\Omega_\lambda \subset T_\lambda$, by (iv), Ω_λ has nonempty convex values, and by (v), Ω_λ has the local intersection property. Applying Theorem 11 we get $\bigcap_{x\in X} T_\lambda(x) \neq \emptyset$, i.e., $\inf\limits_{y\in Y} \sup\limits_{x\in X} \leq \lambda$, and the proof is complete. □

The previous corollary could be compared with earlier two function minimax inequalities due to Fan [7] and Liu [13].

Theorem 13. *Let X be a compact convex set in a locally convex space, Y a convex set in a topological vector space and $T : X \multimap Y$ a concave map satisfying the following conditions:*

(i) *$T^{-1}(y)$ is open in X for each $y \in Y$;*
(ii) *$\bigcap_{x\in F} T(x)$ is open for every closed subset F of X;*
(iii) *there exists a map $\Omega : X \multimap Y$ with nonempty convex values and with the local intersection property such that $\Omega \subset T$.*

Then $\bigcap_{x\in X} T(x) \neq \emptyset$.

Proof. By Lemma 3, there exists a continuous selection of Ω, $p : X \to Y$. We have $p(x) \in T(x)$ for each $x \in X$.

By way of contradiction suppose that $\bigcap_{x\in X} T(x) = \emptyset$. Then T^* has nonempty values. By (ii), T^* is upper semicontinuous. Moreover the values of T^* are closed (by (i)) and convex (since T is concave map). Thus the map $T^* \circ p$ is upper semicontinuous with nonempty closed convex values. From Fan-Glicksberg fixed point theorem (see [6],[9]), there exists $x_0 \in X$ such that $x_0 \in T^*(p(x_0))$. Put $y_0 = p(x_0)$ and obtain the following contradiction

$$y_0 = p(x_0) \in T(x_0),$$
$$x_0 \in T^*(y_0) \Rightarrow y_0 \notin T(x_0).$$ □

Corollary 14. *Let X be a compact convex set in a locally convex space and Y be a convex set in a topological vector space. Let $f, g : X \times Y \to \overline{\mathbb{R}}$ be two functions such that:*

(i) *$f \leq g$;*
(ii) *f is quasiconcave on X;*
(iii) *f is upper semicontinuous on $X \times Y$;*
(iv) *g is quasiconvex on Y;*
(v) *g is weakly λ-transfer upper semicontinuous on X, for any $\lambda > \sup\limits_{x\in X} \inf\limits_{y\in Y} g(x, y)$.*

Then $\inf\limits_{y\in Y} \max\limits_{x\in X} f(x, y) \leq \sup\limits_{x\in X} \inf\limits_{y\in Y} g(x, y)$.

Proof. First, let us observe that if f is upper semicontinuous, then for each $y \in Y$, $f(\cdot, y)$ is also an upper semicontinuous function of x on X and therefore its maximum, $\max\limits_{x\in X} f(x, y)$ on the compact set X exists.

Let $\lambda > \sup\limits_{x\in X} \inf\limits_{y\in Y} g(x, y)$ be arbitrarily fixed. Define $T_\lambda, \Omega_\lambda : X \multimap Y$ by

$$T_\lambda(x) = \{y \in Y : f(x, y) < \lambda\}, \ \ \Omega_\lambda(x) = \{y \in Y : g(x, y) < \lambda\}.$$

From the hypotheses it readily follows that T_λ is concave map with open fibers, and Ω_λ is a map with nonempty convex values having the local intersection property.

We show that T_λ satisfies condition (ii) in Theorem 13, or equaivalently that T_λ^* is upper semicontinuous. Since f is upper semicontinuous on $X \times Y$ the graph of T_λ^*, that is the set $\{(y,x) \in Y \times X : f(x,y) \geq \lambda\}$ is closed in $Y \times X$. Hence $T_\lambda^* : Y \to X$ is a map with closed values and with closed graph. Since X is compact, it follows that T_λ^* is upper semicontinuous (see [2, p.112]).

Applying Theorem 13 we get $\bigcap_{x\in X} T_\lambda(x) \neq \emptyset$, i.e., $\inf\limits_{y\in Y} \max\limits_{x\in X} f(x,y) \leq \lambda$, and the proof is complete. □

References

[1] R. C. Bassanezi and G. H. Greco, *A minimax theorem for marginally upper/lower semicontinuous functions*, Topol. Methods Nonlinear Anal. 5 (1995), 249–253.

[2] C. Berge, *Topological Spaces*, Edinburgh, London, Oliver and Boyel, 1963.

[3] T. H. Chang and C. L. Yen, *KKM property and fixed point theorems*, J. Math. Anal. Appl. 203 (1996), 224–235.

[4] X. P. Ding, *New H-KKM theorems and their applications to geometric property, coincidence theorems, minimax inequalities and maximal elements*, Indian J. Pure Appl. Math. 26 (1995), 1–19.

[5] X. P. Ding, W. K. Kim and K. K. Tan, *A selection theorem and its applications*, Bull. Austral. Math. Soc. 46 (1992), 205–212.

[6] K. Fan, *Fixed points and minimax theorems in locally convex topological linear spaces*, Proc. Nat. Acad. Sci. U.S.A. 38 (1952), 131–136.

[7] K. Fan, *Sur un théorème minimax*, C. R. Acad. Sci. Paris Ser. I. Math. 259 (1964), 3925-3928.

[8] S. D. Flåm and G. H. Greco, *Minimax and intersection theorems*, in: Fixed Point Theory and Applications (M. A. Théra, J. B. Baillon, eds.), Longman Scientific and Technical, 1990, pp. 123–140.

[9] I. L. Glicksberg, *A further generalization of the Kakutani fixed point theorem with applications to Nash equilibrium points*, Proc. Amer. Math. Soc. 3 (1952), 170–174.

[10] G. H. Greco, *Minimax theorems and saddling transformations*, J. Math. Anal. Appl. 147 (1990), 180–197.

[11] G. H. Greco and M. P. Moschen, *A minimax inequality for marginally semicontinuous functions*, in Minimax Theory and Applications (B. Ricceri, S. Simons eds.), Kluwer Academic Publishers, Dordrecht, 1998, pp. 41-50.

[12] C. W. Ha, *Minimax and fixed point theorems*, Math. Ann. 248 (1980), 73–77.

[13] F. C. Liu, *A note on the von Neumann-Sion minimax principle*, Bull. Inst. Math. Acad. Sinica 6 (1978), 517–524.

[14] E. Michael, *Continuous selections I*, Ann. Math. 63 (1956), 361–382.

[15] S. Park, *Generalizations of Ky Fan's matching theorems and their applications*, J. Math. Anal. Appl. 141 (1989), 164–176.

[16] M. Sion, *On general minimax theorems*, Pacific J. Math. 8 (1958), 171–176.

[17] G. Q. Tian, *Generalization of the FKKM theorem and the Ky Fan minimax inequality, with applications to maximal elements, price equilibrium, and complementarity*, J. Math. Anal. Appl. 170 (1992), 457–471.

[18] X. Wu and S. Shen, *A further generalization of Yannelis-Prabhakar's continuous selection and its applications*, J. Math. Anal. Appl. 197 (1996), 61–74.

Manuscript received December 11, 2003
revised April 14, 2004

Mircea Balaj
Department of Mathematics, University of Oradea, 3700, Oradea, ROMANIA
E-mail address: mbalaj@uoradea.ro