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STRONG NONLINEAR ERGODIC THEOREMS FOR ASYMPTOTICALLY NONEXPANSIVE SEMIGROUPS IN BANACH SPACES

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Abstract. In this paper the continuous (C, α) method of non-integral order given by the fractional integration is taken as the basic method of summability which generalizes the usual Cesàro method. We investigate the limiting behaviors of the (C, α) mean value processes of arbitrary positive order α for one-parameter semigroups of asymptotically nonexpansive mappings with compact domains in a strictly convex Banach space. The result obtained here is a continuous extension of the author's discrete type result. Moreover, the nonlinear ergodic [Abel→Cesàro] problem is also considered in the Lebesgue space L_p with $1 \leq p < \infty$.

1. Introduction

The purpose of the present paper is to investigate the strong ergodicity of the extended (C, α) mean value processes for one-parameter semigroups of asymptotically nonexpansive mappings with compact domains in Banach spaces, when non-integral orders of summability are considered. In fact, the following results are obtained:

(1) The (C, α) $(0 < \alpha < \infty)$ extension of Atsushiba, Nakajo and Takahashi's $(C, 1)$ result for asymptotically nonexpansive (one-parameter) semigroups.
(2) Hille-type's strong nonlinear ergodic theorems (in the discrete and continuous time cases).
(3) Two nonlinear ergodic extensions of the Littlewood theorem concerning the [Abel→Cesàro] problem.

Let X be a strictly convex Banach space and let C be a nonempty closed convex subset of X. In 1964 Edelstein [5] introduced the strong ω-limit set $\Omega_s(x)$ of $x \in C$ for a nonexpansive self-mapping T of C, which is defined as the set of all strong subsequential limits of $\{T^n x\}$, and established the $(C, 1)$ strong nonlinear ergodic theorem which asserts that if X is reflexive and $x \in C$, then for each $\xi \in \overline{co}\Omega_s(x)$, $\frac{1}{n}\sum_{k=0}^{n-1} T^k \xi$ converges strongly as $n \to \infty$ to a point $\eta \in Fix(T)$, where $Fix(T) = \{z \in C : Tz = z\}$. Recenly Edelstein's strong nonlinear ergodic theorem has extensively improved by Atsushiba and Takahashi [1] to hold for all points of C being compact in such a way that Bruck [3], [4] originally built up his method. In addition, they also obtained the continuous analogue for strongly continuous nonexpansive semigroups with compact domains [2]. A little later, Nakajo and Takahashi [9], [10] showed that Atsushiba and Takahashi's $(C, 1)$ results in both discrete

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and continuous cases remain also valid even for asymptotically nonexpansive self-mappings and strongly continuous semigroups of such mappings respectively.Their proofs of the results just mentioned depend essentially on the technical $(C, 1)$ equality due to Bruck [3]. So it seems to be somewhat interesting to see whether Bruck's equality is also valid for the (C, α) case of arbitrary positive order α. Unfortunately we have no answer to this question as of now. Instead, we will utilize a Tauberian technique in treating the (C, α) case. The following question was raised by Reich [12]: Let X be a uniformly convex Banach space with Fréchet differentiable norm, let C be a nonempty closed convex subset of X, let T be a nonexpansive self-mapping of C with $Fix(T) \neq \varnothing$ and let $\{a_{n,m}\}$ be a strongly regular matrix method. Then the question is whether for each $x \in C$, the mean $\sum_{m=0}^{\infty} a_{n,m} T^m x$ converges strongly as $n \to \infty$ to a point $y \in Fix(T)$ even in the case where T is odd.

Very recently, in connection with this question, the author [17] obtained the discrete (C, α) extension of the $(C, 1)$ result mentioned above to the case of arbitrary real order $\alpha > 0$ with a view to stepping up the question raised by Reich. In this paper we shall be concerned with the continuous extension of our previous discrete result. The result obtained here is the (C, α) extension of the continuous $(C, 1)$ result obtained by Atsushiba, Nakajo and Takahashi as mentioned above. Moreover, we deal with the so-called nonlinear ergodic [Abel→ Cesàro] problem in the positive cone L_p^+ of the Lebesgue space L_p $(1 \leq p < \infty)$ with a view to extending the classical Littlewood theorem in the ergodic theory setting.

2. Strong nonlinear ergodic theorems

In what follows X is a strictly convex Banach space and C is a nonempty closed convex subset of X. Let $G = \{T(t) : t \geq 0\}$ be a strongly continuous one-parameter semigroup of asymptotically nonexpansive self-mappings of C with Lipschitz constants $\{k(t) : t \geq 0\}$. This means that

(i) for each $t > 0$, $T(t)$ is a nonlinear mapping of C into itself;
(ii) $T(0)x = x$, $T(t + s)x = T(t)T(s)x$ for all $s, t \geq 0$ and all $x \in C$;
(iii) for each $x \in C$, $T(t)x$ is continuous in $t \geq 0$ in the sense of the strong topology of X;
(iv) $\| T(t)x - T(t)y \| \leq (1 + k(t)) \|x - y\|$ for all $t \geq 0$ and all $x, y \in C$, where $k(t)$ is a nonnegative continuous function on $[0, \infty)$ with $k(t) \to 0$ as $t \to \infty$ (cf.[6]).

In particular, if $k(t) = 0$ for all $t \geq 0$ then $\{T(t) : t \geq 0\}$ is called a nonexpansive semigroup on C. Let $\alpha > 0$ be any fixed real number and $x \in C$. We are now interested in the (C, α) and the Abel limits, the first being based on the fractional integral (or, the Volterra type integral) of $T(t)x$ of order α, the second on the Laplace transform of $T(t)x$. For each $x \in C$,we consider the (C, α) and the Abel mean value processes $\{C_t^{(\alpha)}[G]x : t > 0\}$ and $\{\lambda R(\lambda; G)x : \lambda > 0\}$ which are defined by

$$C_t^{(\alpha)}[G]x = \frac{\alpha}{t^\alpha} \int_0^t (t - u)^{\alpha - 1} T(u)x du, \quad t > 0,$$

and

$$\lambda R(\lambda; G)x = \lambda \int_0^\infty e^{-\lambda t} T(t)x dt, \quad \lambda > 0,$$

respectively, whenever the integrals on the right exist. We begin by reviewing Atsushiba, Nakajo and Takahashi's result:

Theorem A. *Let X be a strictly convex Banach space, let C be a nonempty compact convex subset of X and let $G = \{T(t) : t \geq 0\}$ be a strongly continuous semigroup of asymptotically nonexpansive self-mappings of C with Lipschitz constants $\{k(t)\}$. Then for each $x \in C$, $C_t^{(1)}[G]T(s)x$ converges strongly as $t \to \infty$ to a point $y \in Fix(G)$ uniformly in $s \geq 0$, where $Fix(G) = \cap_{t \geq 0} Fix(T(t))$.*

We need the following Hille's vector-valued extension [7] of Wiener's general Tauberian theorem [14]:

Theorem B. *Let $K_1(u) \in L_1(-\infty, \infty)$ and assume that for all real t*

$$\text{(I)} \qquad \int_{-\infty}^{\infty} K_1(u) e^{-itu} du \neq 0.$$

Let $z(u)$ be a bounded measurable function on $(-\infty, \infty)$ to a complex Banach space X and assume that for some $w \in X$

$$\text{(II)} \qquad \lim_{\sigma \to \infty} \int_{-\infty}^{\infty} K_1(u - \sigma) z(u) du = w \int_{-\infty}^{\infty} K_1(u) du.$$

Then for any function $K_2(u) \in L_1(-\infty, \infty)$

$$\text{(III)} \qquad \lim_{\sigma \to \infty} \int_{-\infty}^{\infty} K_2(u - \sigma) z(u) du = w \int_{-\infty}^{\infty} K_2(u) du.$$

Here convergence is taken in the sense of the strong topology of X.

According to Wiener's basic theorem [14, Theorem II], condition (I) is satisfied for the function $K_1(u)$ if and only if the set of all linear combinations of the translations of $K_1(u)$ is dense in $L_1(-\infty, \infty)$. The proof of Theorem B depends essentially on this fact and follows from slightly modified Wiener's argument.

We are now in a position to state the (C, α) extension of real order $\alpha > 0$ of Theorem A.

Theorem 1. *Let X be a strictly convex Banach space, let C be a nonempty compact convex subset of X and let $G = \{T(t) : t \geq 0\}$ be a strongly continuous Lipschizian semigroup of asymptotically nonexpansive self-mappings of C. Then the following statements hold:*

(i) *For each $x \in C$, $\lambda R(\lambda; G)T(s)x$ converges strongly as $\lambda \to 0+$ to a point $y \in Fix(G)$ uniformly in $s \geq 0$.*
(ii) *Let $\alpha > 0$ be any fixed real number. Then for each $x \in C$, $C_t^{(\alpha)}[G]T(s)x$ converges strongly as $t \to \infty$ to a point $y \in Fix(G)$ uniformly in $s \geq 0$.*

Proof. Let $x \in C$. It follows from Theorem A and the asymptotic nonexpansiveness of the semigroup in question that $K_x = \sup_{t \geq 0} \|T(t)x\| < \infty$. The (C, α) and the

Abel mean value processes of the semigroup are well-defined in this case. Here we note that for each $\lambda > 0$

$$\lim_{n\to\infty}\left\|\lambda R(\lambda;G)T(s)x - \frac{\int_{\frac{1}{n}}^{n}\lambda e^{-\lambda t}T(t+s)xdt}{\int_{\frac{1}{n}}^{n}\lambda e^{-\lambda t}\,dt}\right\| = 0.$$

It is already known that $C_t^{(1)}[G]T(s)x$ converges strongly as $t\to\infty$ to some point $y \in Fix(G)$ uniformly in $s \geq 0$. Hence statement (i) and the case of $1 < \alpha < \infty$ of statement (ii) follow immediately from

$$\|\lambda R(\lambda;G)T(s)x - y\| \leq \lambda^2\int_0^\infty te^{-\lambda t}\left\|C_t^{(1)}[G]T(s)x - y\right\|dt, \quad \lambda > 0,$$

and

$$\left\|C_t^{(\alpha)}[G]T(s)x - y\right\| \leq \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-1)}t^{-\alpha}\int_0^t u(t-u)^{\alpha-2}\left\|C_u^{(1)}[G]T(s)x - y\right\|du, \quad t > 0$$

respectively. We now prove the case of $0 < \alpha < 1$ of statement (ii). Using the relation between the Abel and the Cesàro averages which is given by

$$\lambda R(\lambda;G)T(s)x = \frac{\lambda^{\alpha+1}}{\Gamma(\alpha+1)}L_\lambda^{(\alpha)}[G]T(s)x, \quad \lambda > 0,$$

where

$$L_\lambda^{(\alpha)}[G]T(s)x = \int_0^\infty t^\alpha e^{-\lambda t}C_t^{(\alpha)}[G]T(s)xdt, \quad \lambda > 0$$

which exists for all values of $\lambda > 0$, it follows that

$$\lim_{\lambda\to 0+}\left\|\lambda^{\alpha+1}L_\lambda^{(\alpha)}[G]T(s)x - y\Gamma(\alpha+1)\right\| = 0$$

uniformly in $s \geq 0$. Modifying Hille's argument, let us set

$$\sigma = -\log\lambda, \quad \xi = \exp[u], \quad K_1(u) = \exp[(\alpha+1)u - e^u].$$

Then we obtain the following limit of Wiener's type

$$\lim_{\sigma\to\infty}\left\|\int_{-\infty}^\infty K_1(u-\sigma)C_{e^u}^{(\alpha)}[G]T(s)xdu - y\int_{-\infty}^\infty K_1(u)du\right\| = 0$$

uniformly in $s \geq 0$. On the other hand, a simple observation gives

$$\int_{-\infty}^\infty K_1(u)e^{-itu}du = \int_0^\infty \xi^{\alpha-it}e^{-\xi}d\xi = \Gamma(\alpha+1-it) \neq 0.$$

Hence conditions (I) and (II) of Theorem B are satisfied. Now let $\delta > 0$ be fixed arbitrarily small and define the function $K_2(u)$ on $(-\infty,\infty)$ by

$$K_2(u) = \begin{cases} e^u & \text{if } 0 \leq u \leq \log(1+\delta), \\ 0 & \text{otherwise.}\end{cases}$$

Then we can apply Theorem B to the function $K_2(u) \in L_1(-\infty,\infty)$ and in view of condition (III) we have after simplification

$$\lim_{\mu\to\infty}\left\|\frac{1}{\mu\delta}\int_\mu^{\mu(1+\delta)}C_t^{(\alpha)}[G]T(s)x\,dt - y\right\| = 0$$

uniformly in $s \geq 0$. Thus for any given $\varepsilon > 0$ one can find a sufficiently large real number $\mu_0 > 0$ which is independent of s, such that

$$\left\| \frac{1}{\mu\delta} \int_{\mu}^{\mu(1+\delta)} C_t^{(\alpha)}[G]T(s)x \; dt - y \right\| < \varepsilon, \quad \mu > \mu_0.$$

Let us fix a sufficiently large real number μ such that $\mu \geq [\delta^{-1}] + 1$, where $[\delta^{-1}]$ denotes the integer part of δ^{-1}. We estimate the difference $C_t^{(\alpha)}[G]T(s)x - C_\mu^{(\alpha)}[G]T(s)x$ for $\mu < t < \mu(1+\delta)$:

$$\begin{aligned}
&\left\| \frac{\alpha}{t^\alpha} \int_0^t (t-u)^{\alpha-1} T(u+s)x du - \frac{\alpha}{\mu^\alpha} \int_0^\mu (\mu-u)^{\alpha-1} T(u+s)x du \right\| \\
&\qquad \leq \left(\sup_{t\geq 0} \|T(t)x\| \right) \frac{\alpha}{t^\alpha} \int_0^\mu \left| (t-u)^{\alpha-1} - (\mu-u)^{\alpha-1} \right| du \\
&\qquad + \left(\sup_{t\geq 0} \|T(t)x\| \right) \left| \frac{\alpha}{t^\alpha} - \frac{\alpha}{\mu^\alpha} \right| \int_0^\mu (\mu-u)^{\alpha-1} du \\
&\qquad + \left(\sup_{t\geq 0} \|T(t)x\| \right) \frac{\alpha}{t^\alpha} \int_\mu^t (t-u)^{\alpha-1} du
\end{aligned}$$

and

$$\begin{aligned}
\frac{\alpha}{t^\alpha} \int_0^\mu \left| (t-u)^{\alpha-1} - (\mu-u)^{\alpha-1} \right| du &\leq 1 - \left(\frac{\mu}{t}\right)^\alpha + (1 - \frac{\mu}{t})^\alpha \\
&< 1 - \left(\frac{1}{1+\delta}\right)^\alpha + \delta^\alpha \\
\left| \frac{\alpha}{t^\alpha} - \frac{\alpha}{\mu^\alpha} \right| \int_0^\mu (\mu-u)^{\alpha-1} du &\leq 1 - \left(\frac{\mu}{t}\right)^\alpha < 1 - \left(\frac{1}{1+\delta}\right)^\alpha + \delta^\alpha \\
\frac{\alpha}{t^\alpha} \int_\mu^t (t-u)^{\alpha-1} du &\leq \left(1 - \frac{\mu}{t}\right)^\alpha < 1 - \left(\frac{1}{1+\delta}\right)^\alpha + \delta^\alpha
\end{aligned}$$

Hence, summing up these estimates yields

$$\left\| C_t^{(\alpha)}[G]T(s)x - C_\mu^{(\alpha)}[G]T(s)x \right\| < 3K_x\{1 - \left(\frac{1}{1+\delta}\right)^\alpha + \delta^\alpha\}$$

for $\mu < t < \mu(1+\delta)$. We have therefore

$$\begin{aligned}
\left\| C_\mu^{(\alpha)}[G]T(s)x - y \right\| &\leq \left\| \frac{1}{\mu\delta} \int_\mu^{\mu(1+\delta)} C_t^{(\alpha)}[G]T(s)x dt - y \right\| \\
&\quad + \frac{1}{\mu\delta} \int_\mu^{\mu(1+\delta)} \left\| C_t^{(\alpha)}[G]T(s)x - C_\mu^{(\alpha)}[G]T(s)x \right\| dt \\
&< \varepsilon + 3K_x\{1 - \left(\frac{1}{1+\delta}\right)^\alpha + \delta^\alpha\}
\end{aligned}$$

for all $\mu > \max\{\mu_0, [\delta^{-1}] + 1\}$. Consequently, arbitrariness of ε and δ guarantees that $C_t^{(\alpha)}[G]T(s)x$ converges strongly as $t \to \infty$ to the point y uniformly in $s \geq 0$. The proof of the theorem has hereby been completed. □

The weak nonlinear ergodic theorem has been given a satisfactory formulation in a uniformly convex Banach space with a Fréchet differentiable norm (cf. [13]). In general, the existence of the (C,α) strong (or, weak) nonlinear ergodic limit implies the existence of the Abel strong (or, weak) nonlinear ergodic limit. But the converse does not necessarily hold without any additional condition. Here we need a Tauberian condtion. The following abstract weak form of Wiener's general Tauberian theorem is very useful in treating of the ergodic [Abel→Cesàro] problem (under some Tauberian condition) in general Banach spaces.

Theorem 2. *Let $K_1(u) \in L_1(-\infty,\infty)$ and suppose that condition* (I) *is satisfied for all real t. Let $z(u)$ be a bounded measurable function on $(-\infty,\infty)$ to a complex Banach space X and suppose that condition* (II) *is satisfied in the sense of the weak topology. Then* (III) *holds for any function $K_2(u) \in L_1(-\infty,\infty)$ in the sense of the weak topology of X.*

Proof. Accoding to Wiener's basic theorem [14, Theorem II], condition (I) implies that the set

$$\Lambda = \{\sum_{k=1}^{N} a_k K_1(u+\lambda_k) : a_k \in \mathbf{C},\ \lambda_k \in (-\infty,\infty)\}$$

is dense in $L_1(-\infty,\infty)$. Then it follows from condition (II) that for any $G(u) \in \Lambda$

$$\lim_{\sigma\to\infty} \left(x^*, \int_{-\infty}^{\infty} G(u-\sigma)z(u)du\right) = \left(x^*, w\int -\infty^{\infty} G(u)du\right).$$

Let $K_2(u) \in L_1(-\infty,\infty)$ and for any given $\varepsilon > 0$ choose $G(u) \in \Lambda$ such that $\|K_2 - G\|_1 < \varepsilon$. We then have

$$\begin{aligned}
&\left|\left(x^*, \int_{-\infty}^{\infty} K_2(u-\sigma)z(u)du - w\int_{-\infty}^{\infty} K_2(u)du\right)\right| \\
&\quad \le \left|\left(x^*, \int_{-\infty}^{\infty} [K_2(u-\sigma) - G(u-\sigma)]z(u)du\right)\right| \\
&\qquad + \left|\left(x^*, \int_{-\infty}^{\infty} G(u-\sigma)z(u)du - w\int_{-\infty}^{\infty} G(u)du\right)\right| \\
&\qquad + \left|\left(x^*, w\int_{-\infty}^{\infty} [G(u) - K_2(u)]du\right)\right| \\
&\quad < \varepsilon \|x^*\| (\|z\|_\infty + \|w\|) + \left|\left(x^*, \int_{-\infty}^{\infty} G(u-\sigma)z(u)du - w\int_{-\infty}^{\infty} G(u)du\right)\right|.
\end{aligned}$$

Note here that if $K_3(u) \in L_1(-\infty,\infty)$ and $\|K_2 - K_3\|_1 < \varepsilon$, then

$$\left|(x^*, \int_{-\infty}^{\infty} [K_2(u-\sigma) - K_3(u-\sigma)]z(u)du)\right| < \varepsilon \|x^*\| \|z\|_\infty.$$

Consequently, (III) follows from these estimates and the theorem is proved. □

The existence of the (strong or weak) Abel limit under an appropriate Tauberian condition implies the existence of the (strong or weak) Cesàro (C,α) limit. In fact, using the Tauberian technique (Theorem B and Theorem 2) as in the proof of Theorem 1, we can give an answer to the nonlinear ergodic [Abel→Cesàro] problem.

Theorem 3. *Let X be a complex Banach space, let C be a nonempty bounded closed convex subset of X and let $G = \{T(t) : t \geq 0\}$ be a strongly continuous one-parameter semigroup of nonlinear self-mappings of C. If $x \in C$ and $\lambda G(\lambda; G)T(s)x$ converges strongly (or, weakly) as $\lambda \to 0+$ to some $z \in Fix(G)$ uniformly in $s \geq 0$, then for every $\alpha > 0$, $C_t^{(\alpha)}[G]T(s)x$ converges strongly (or, weakly) as $t \to \infty$ to the point z uniformly in $s \geq 0$.*

The proof of this theorem follows exactly the same line as in the proof of Theorem 1 and we omit the details. The discrete version of Theorem 3 becomes

Theorem 4. *Let X be a complex Banach space, let C be a nonempty bounded closed convex subset of X and let T be a nonlinear self-mapping of C. If $x \in C$ and $(\lambda - 1)\sum_{n=0}^{\infty} \lambda^{-(n+1)} T^{n+m}x$ $(\lambda > 1)$ converges strongly (or, weakly) as $\lambda \to 1+0$ to some $z \in Fix(T)$ uniformly in $m \geq 0$, then for every $\alpha > 0$ $\binom{n+\alpha}{n}^{-1} \sum_{k=0}^{n} \binom{n-k+\alpha-1}{n-k} T^{n+m}x$ converges strongly (or, weakly) as $n \to \infty$ to the point z uniformly in $m \geq 0$.*

3. The nonlinear ergodic [Abel→Cesàro] problem revisited

The Banach space X of the preceding section will now be replaced by a real Lebesgue space $L_p = L_p(\Omega, \Xi, \mu)$ $(1 \leq p < \infty)$ where (Ω, Ξ, μ) is a $\sigma-$ finite positive measure space. Let $L_p^+ = \{f \in L_p : f \geq 0\}$. Our final consideration concerns the (extended) nonlinear case of the ergodic [Abel→Cesàro] problem in the positive cone L_p^+. The aspect observed in the previous section was the question of relations between different types of summability and corresponding ergodic theorems, in particular (C, α) and Abel summability. By analogy with the previous setting one might expect that the existence of

$$\text{strong} \lim_{\lambda \to 0+} \frac{\lambda^{\alpha}}{\Gamma(\alpha)} \int_0^{\infty} t^{\alpha-1} e^{-\lambda t} T(t) f dt$$

(which we shall call the Abel limit of order $\alpha > 0$) should imply the existence of

$$\text{strong} \lim_{t \to \infty} \frac{\alpha}{t^{\alpha}} \int_0^t u^{\alpha-1} T(t) f dt$$

provided appropriate Tauberian conditions are satisfied. In connection with this question, we have

Theorem 5. *Let $G = \{T(t) : t \geq 0\}$ be a strongly continuous one-parameter semigroup of asymptotically nonexpansive self-mappings of L_p^+ $(1 \leq p < \infty)$ with $Fix(G) \neq \varnothing$. Let $0 < \alpha < \infty$ and $f \in L_p^+$. If $\frac{\lambda^{\alpha}}{\Gamma(\alpha)} \int_0^{\infty} t^{\alpha-1} e^{-\lambda t} T(t+s) f dt$ $(\lambda > 0)$ converges strongly as $\lambda \to 0+$ to a point $f_0 \in L_p^+$ uniformly in $s \geq 0$, then $\frac{\alpha}{t^{\alpha}} \int_0^t u^{\alpha-1} T(u+s) f du$ $(t > 0)$ converges strongly as $t \to \infty$ to the same point f_0 uniformly in $s \geq 0$.*

Proof. Let $0 < \alpha < \infty$ and let $\Phi_{\alpha}(t; f)$ be the function on $(0, \infty)$ to L_p^+ which is defined by

$$\Phi_{\alpha}(t; f) = \frac{1}{\Gamma(\alpha)} t^{\alpha-1} T(t) f, \quad t > 0.$$

Note first that $\{T(t)f : t \geq 0\}$ is norm-bounded. This follows from the asymp- totic nonexpansiveness of the semigroup and $Fix(G) \neq \varnothing$. Now putting $u = e^{-\lambda}$, we have by assumption

$$\lim_{u\to 1-0} \left\| (1-u)^{\alpha} \int_0^{\infty} u^t \ \Phi_{\alpha}(t; T(s)f) dt - f_0 \right\|_p = 0$$

uniformly in $s \geq 0$. Thus, letting $r = \exp[(m+1)^{-1} \log u]$ for some integer $m \geq 0$ gives $u = r^{m+1}$ and

$$\lim_{r\to 1-0} \left\| (1-r)^{\alpha} \int_0^{\infty} r^t \ r^{mt} \ \Phi_{\alpha}(t; T(s)f) \ dt - \frac{f_0}{(m+1)^{\alpha}} \right\|_p = 0$$

uniformly in $s \geq 0$. However, since

$$\frac{1}{\Gamma(\alpha)} \int_0^1 t^{\sigma} (\log \frac{1}{t})^{\alpha-1} dt = \frac{1}{(1+\sigma)^{\alpha}}, \quad \sigma > -1,$$

we see that for every polynomial $P(u)$ on $[0,1]$

$$\lim_{r\to 1-0} \left\| (1-r)^{\alpha} \int_0^{\infty} r^t P(r^t) \Phi_{\alpha}(t; T(s)f) dt - \frac{f_0}{\Gamma(\alpha)} \int_0^1 P(t) (\log \frac{1}{t})^{\alpha-1} dt \right\|_p = 0$$

uniformly in $s \geq 0$. Here we follow modified Karamata's argument for power series (cf. [8], [11]). Let us define the functions $g(t)$, $h_{1,\tau}(t)$, $h_{2,\tau}(t)$ bounded on $[0,1]$ as follows:

$$g(t) = \begin{cases} t^{-1} & \text{if } e^{-1} \leq t \leq 1, \\ 0 & \text{if } 0 \leq t < e^{-1}, \end{cases}$$

$$h_{1,\tau}(t) = \begin{cases} 0 & \text{if } 0 \leq t \leq e^{-1} - \tau, \\ \frac{e}{\tau}(t - e^{-1} + \tau) & \text{if } e^{-1} - \tau \leq t \leq e^{-1}, \\ g(t) & \text{if } e^{-1} \leq t \leq 1, \end{cases}$$

$$h_{2,\tau}(t) = \begin{cases} 0 & \text{if } 0 \leq t \leq e^{-1}, \\ \frac{1}{\tau(e^{-1}+\tau)}(t - e^{-1}) & \text{if } e^{-1} \leq t \leq e^{-1} + \tau, \\ g(t) & \text{if } e^{-1} + \tau \leq t \leq 1, \end{cases}$$

where $\tau > 0$ is fixed arbitrarily small such that $0 < \tau < e^{-1}$. Now, for the functions $g_1(t) = h_{1,\tau}(t) + \tau$, $g_2(t) = h_{2,\tau}(t) - \tau$ continuous on $[0,1]$, there exist by the Weierstrass approximation theorem two polynomials $P(t)$, $Q(t)$ on $[0,1]$, which may depend on τ, such that

$$\begin{aligned} &|g_1(t) - Q(t)| < \tau, \quad g(t) \leq h_{1,\tau}(t) \leq Q(t), && 0 \leq t \leq 1, \\ &|g_2(t) - P(t)| < \tau, \quad P(t) \leq h_{2,\tau}(t) \leq g(t), && 0 \leq t \leq 1, \\ &Q(t) - P(t) \leq \{h_{1,\tau}(t) - h_{2,\tau}(t)\} + 2\tau, && 0 \leq t \leq 1. \end{aligned}$$

Let $\varepsilon > 0$ be any given number, however small. Then choosing a sufficiently small τ, depending on ε, such that

$$0 < \tau < \min\{\frac{\varepsilon}{2(e+1)}, \frac{\varepsilon}{4\Gamma(\alpha)}\}, \quad \int_{e^{-1}-\tau}^{e^{-1}+\tau} \left(\log \frac{1}{t}\right)^{\alpha-1} dt < \frac{\varepsilon}{2e},$$

we have

$$\int_0^1 \{Q(t) - P(t)\}dt < 2e\tau + 2\tau < \varepsilon$$

and

$$\begin{aligned}\int_0^1 \{Q(t) - P(t)\}(\log \frac{1}{t})^{\alpha-1}dt &< 2\tau\Gamma(\alpha) + \int_{e^{-1}-\tau}^{e^{-1}} h_{1,\tau}(t)\left(\log \frac{1}{t}\right)^{\alpha-1} dt \\ &\quad + \int_{e^{-1}}^{e^{-1}+\tau} \{g(t) - h_{2,\tau}(t)\}\left(\log \frac{1}{t}\right)^{\alpha-1} dt \\ &< 2\tau\Gamma(\alpha) + e\int_{e^{-1}-\tau}^{e^{-1}+\tau} \left(\log \frac{1}{t}\right)^{\alpha-1} dt < \varepsilon.\end{aligned}$$

Furthermore there exists a small $\delta > 0$ such that if $0 < 1 - r < \delta$, then

$$\| (1-r)^\alpha \int_0^\infty r^t P(r^t)\Phi_\alpha(t;T(s)f)dt - \frac{f_0}{\Gamma(\alpha)}\int_0^1 P(t)(\log \frac{1}{t})^{\alpha-1}dt \|_p < \varepsilon$$

and

$$\| (1-r)^\alpha \int_0^\infty r^t Q(r^t)\Phi_\alpha(t;T(s)f)dt - \frac{f_0}{\Gamma(\alpha)}\int_0^1 Q(t)(\log \frac{1}{t})^{\alpha-1}dt \|_p < \varepsilon$$

On the other hand, one has, a priori,

$$\begin{aligned}(1-r)^\alpha \int_0^\infty r^t P(r^t)\Phi_\alpha(t;T(s)f)dt &\le (1-r)^\alpha \int_0^\infty r^t g(r^t)\Phi_\alpha(t;T(s)f)dt \\ &\le (1-r)^\alpha \int_0^\infty r^t Q(r^t)\Phi_\alpha(t;T(s)f)dt.\end{aligned}$$

Thus for $0 < 1 - r < \delta$ we have

$$\begin{aligned}&\left\|(1-r)^\alpha \int_0^\infty r^t g(r^t)\Phi_\alpha(t;T(s)f)dt - \frac{f_0}{\Gamma(\alpha)}\int_0^1 g(t)\left(\log \frac{1}{t}\right)^{\alpha-1} dt\right\|_p \\ &\le \left\|(1-r)^\alpha \int_0^\infty r^t Q(r^t)\Phi_\alpha(t;T(s)f)dt - (1-r)^\alpha \int_0^\infty r^t P(r^t)\Phi_\alpha(t;T(s)f)dt\right\|_p \\ &\quad + \left\|(1-r)^\alpha \int_0^\infty r^t Q(r^t)\Phi_\alpha(t;T(s)f)dt - \frac{f_0}{\Gamma(\alpha)}\int_0^1 Q(t)(\log \frac{1}{t})^{\alpha-1}dt\right\|_p \\ &\quad + \left\|\frac{f_0}{\Gamma(\alpha)}\int_0^1 Q(t)(\log \frac{1}{t})^{\alpha-1}dt - \frac{f_0}{\Gamma(\alpha)}\int_0^1 g(t)(\log \frac{1}{t})^{\alpha-1}dt\right\|_p \\ &\le \left\|(1-r)^\alpha \int_0^\infty r^t P(r^t)\Phi_\alpha(t;T(s)f)dt - \frac{f_0}{\Gamma(\alpha)}\int_0^1 P(t)(\log \frac{1}{t})^{\alpha-1}dt\right\|_p \\ &\quad + 2\left\|(1-r)^\alpha \int_0^\infty r^t Q(r^t)\Phi_\alpha(t;T(s)f)dt - \frac{f_0}{\Gamma(\alpha)}\int_0^1 Q(t)(\log \frac{1}{t})^{\alpha-1}dt\right\|_p \\ &\quad + 2\left\|\frac{f_0}{\Gamma(\alpha)}\int_0^1 Q(t)(\log \frac{1}{t})^{\alpha-1}dt - \frac{f_0}{\Gamma(\alpha)}\int P(t)(\log \frac{1}{t})^{\alpha-1}dt\right\|_p\end{aligned}$$

$$< \varepsilon + 2\varepsilon + \frac{2\left\|f_0\right\|_p}{\Gamma(\alpha)}\varepsilon = (3 + \frac{2\left\|f_0\right\|_p}{\Gamma(\alpha)})\varepsilon,$$

and arbitrariness of ε yields

$$\lim_{r\to 1-0}\left\|(1-r)^\alpha \int_0^\infty r^t g(r^t)\Phi_\alpha(t;T(s)f)dt - \frac{f_0}{\Gamma(\alpha)}\int_0^1 g(t)(\log\frac{1}{t})^{\alpha-1}dt\right\|_p = 0$$

uniformly in $s \geq 0$. Thus it follows that

$$\lim_{r\to 1-0}\left\|(1-r)^\alpha \int_0^\infty r^t\ g(r^t)\ \Phi_\alpha(t;T(s)f)dt - \frac{f_0}{\Gamma(\alpha+1)}\right\|_p = 0$$

uniformly in $s \geq 0$. Again, letting $r = \exp[-\frac{1}{\tau}]$, we get

$$\lim_{\tau\to\infty} \|\ (1-e^{-\frac{1}{\tau}})^\alpha \int_0^\tau \Phi_\alpha(t;T(s)f)dt - \frac{f_0}{\Gamma(\alpha+1)}\ \|_p = 0$$

and hence

$$\lim_{\tau\to\infty} \|\ \frac{1}{\tau^\alpha}\int_0^\tau \Phi_\alpha(t;T(s)f)dt - \frac{f_0}{\Gamma(\alpha+1)}\ \|_p = 0$$

uniformly in $s \geq 0$. This immediately gives

$$\lim_{t\to\infty} \|\ \frac{\alpha}{t^\alpha}\int_0^t u^{\alpha-1}T(u+s)f du - f_0\ \|_p = 0$$

uniformlyin $s \geq 0$, as required. □

This theorem has the following discrete analogue.

Theorem 6. *Let T be an asymptotically nonexpansive self-mapping of L_p^+ $(1 \leq p < \infty)$ with $Fix(T) \neq \varnothing$. Let $f \in L_p^+$ and $0 < \alpha < \infty$. If $\lambda^\alpha \int_0^\infty e^{-\lambda t}\ \Psi_\alpha(t;T^m f)dt$ $(\lambda > 0)$ converges strongly as $\lambda \to 0+$ to a point $f_0 \in L_p^+$ uniformly in $m \geq 0$, where*

$$\Psi_\alpha(t;f) = \binom{n+\alpha-1}{n}T^n f, \quad n \leq t < n+1,\ n \geq 0,$$

then $\binom{n+\alpha}{n}^{-1}\sum_{k=0}^n \binom{k+\alpha-1}{k}T^{k+m}f$ converges strongly as $n \to \infty$ to the point f_0 uniformly in $m \geq 0$.

Sketch of proof. As in the proof of Theorem 5, we have with $\Phi_\alpha(t;f)$ replaced by $\Psi_\alpha(t;f)$

$$\lim_{\tau\to\infty}\left\|\frac{\Gamma(\alpha+1)}{\tau^\alpha}\int_0^\tau \Psi_\alpha(t;T^m f)\ dt - f_0\right\|_p = 0$$

uniformly in $m \geq 0$. If τ is sufficiently large, then

$$\frac{\Gamma(\alpha+1)}{\tau^\alpha}\int_0^\tau \Psi_\alpha(t;T^m f)dt = \frac{[\tau]^\alpha}{\tau^\alpha}\frac{\Gamma(\alpha+1)}{[\tau]^\alpha}\sum_{k=0}^{[\tau]}\binom{k+\alpha-1}{k}T^{k+m}f,$$

where $[\tau]$ denotes the integer part of τ. Note further that $\frac{[\tau]^\alpha}{\tau^\alpha} \to 1$ as $\tau \to \infty$ and $\binom{n+\alpha-1}{n} = p_{n,\alpha} \times \frac{n^\alpha}{\Gamma(\alpha+1)}$ with $p_{n,\alpha}$ such that $p_{n,\alpha} \to 1$ as $n \to \infty$. Then by analogy with the argument used in the proof of Theorem 5 we obtain

$$\lim_{\tau\to\infty}\left\|\frac{\Gamma(\alpha+1)}{[\tau]^\alpha}\sum_{k=0}^{[\tau]}\binom{k+\alpha-1}{k}T^{k+m}f-f_0\right\|_p=0$$

and hence

$$\lim_{n\to\infty}\left\|\frac{1}{\binom{n+\alpha}{n}}\sum_{k=0}^{n}\binom{k+\alpha-1}{k}T^{k+m}f-f_0\right\|_p=0$$

uniformly in $m\geq 0$. This is the very thing for our purpose. □

It seems to be natural to ask when (under what conditions) the Abel limits of order $\alpha>0$ in Theorems 5 and 6 do exist. We consider the case that X is a reflexive Banach space. Let $G=\{T(t):t\geq 0\}$ be a strongly continuous semigroup of linear operators on X with $\|T(t)\|\leq M$ for all $t\geq 0$ and let T be a linear operator on X with $\|T^n\|\leq M$ for all $n\geq 0$. Then it follows that for some α with $0<\alpha\leq 1$ and all $x\in X$

$$\text{strong}\lim_{\lambda\to 0+}\lambda^\alpha\int_0^\infty e^{-\lambda t}\Phi_\alpha(t;x)dt$$

and

$$\text{strong}\lim_{\lambda\to 0+}\lambda^\alpha\int_0^\infty e^{-\lambda t}\Psi_\alpha(t;x)dt$$

exist, where $\Phi_\alpha(t;x)$ and $\Psi_\alpha(t;x)$ are given as in Theorems 5 and 6, respectively. For example, to avoid complexity, we show the case for the operator T when $0<\alpha\leq 1$. By the mean ergodic theorem we have the decomposition $X=N(I-T)\oplus\overline{R(I-T)}$, where $N(I-T)$ and $R(I-T)$ denote the null space and range of $I-T$, respectively. If $x\in N(I-T)$, then

$$\begin{aligned}
&\text{strong}\lim_{\lambda\to 0+}\lambda^\alpha\int_0^\infty e^{-\lambda t}\Psi_\alpha(t;x)dt\\
&\qquad=\text{strong}\lim_{\lambda\to 0+}\lambda^\alpha\sum_{n=0}^{\infty}\binom{n+\alpha-1}{n}T^n x\int_n^{n+1}e^{-\lambda t}dt\\
&\qquad=\text{strong}\lim_{\lambda\to 0+}\frac{\lambda^\alpha}{\lambda}\left(1-\frac{1}{e^\lambda}\right)x\sum_{n=0}^{\infty}\binom{n+\alpha-1}{n}e^{-\lambda n}\\
&\qquad=\text{strong}\lim_{\lambda\to 0+}\left(\frac{\lambda e^\lambda}{e^\lambda-1}\right)^{\alpha-1}x=x.
\end{aligned}$$

Next let $x\in\overline{R(I-T)}$. For any given $\varepsilon>0$ there exist $y,z\in X$ such that $x=(I-T)y+z$ and $\|z\|<\varepsilon$. Then we obtain

$$\left\|\frac{\lambda^\alpha}{\lambda}(1-\frac{1}{e^\lambda})\sum_{n=0}^{\infty}e^{-\lambda n}\binom{n+\alpha-1}{n}T^n(I-T)y\right\|$$

$$= \left\| \frac{\lambda^\alpha}{\lambda}(1 - \frac{1}{e^\lambda})[y + \sum_{n=1}^{\infty}\{e^{-\lambda n}\binom{n+\alpha-1}{n} - e^{-\lambda(n-1)}\binom{n-1+\alpha-1}{n-1}\}T^n y] \right\|$$

$$\leq M\|y\| \frac{\lambda^\alpha}{\lambda}(1 - \frac{1}{e^\lambda})[1 + \sum_{n=1}^{\infty}\{e^{-\lambda(n-1)}\binom{n-1+\alpha-1}{n-1} - e^{-\lambda n}\tbinom{n+\alpha-1}{n}\}]$$

$$\leq 2M\|y\| \lambda^\alpha \frac{e^\lambda - 1}{\lambda e^\lambda} \to 0 \quad (\lambda \to 0+)$$

and

$$\left\| \frac{\lambda^\alpha}{\lambda}(1 - \frac{1}{e^\lambda}) \sum_{n=0}^{\infty} e^{-\lambda n}\tbinom{n+\alpha-1}{n} T^n z \right\| \leq M\|z\| \left(\frac{\lambda e^\lambda}{e^\lambda - 1}\right)^{\alpha-1} \leq 2M\varepsilon \quad (\lambda \to 0+).$$

Hence

$$\text{strong} \lim_{\lambda \to 0+} \lambda^\alpha \int_0^\infty e^{-\lambda t}\Psi_\alpha(t;x)dt = 0.$$

Remark. The original [Abel→Cesàro] problem was studied by Littlewood (1911) who proved that a series is $(C,1)$ summable if its partial sums are bounded and it is Abel summable. That such a series is actually (C,α) summable for every $\alpha > 0$ was first proved by Andersen (1921). Hille (1945) extended the latter to Banach spaces and applied the result to give some ergodic theorems for bounded linear operators on complex Banach spaces (see Hille [7]). Hille's ergodic theorems has extensively been improved by the author [15], [16]. Theorems 3 and 4 may be regarded as the nonlinear versions of Hille's ergodic theorems. Theorems 5 and 6 may be regarded as the nonlinear ergodic extensions of the Littlewood theorem. It should be noticed that in Theorem 5, the strong convergence of $\lambda^\alpha \int_0^\infty t^{\alpha-1}e^{-\lambda t}T(t+s)fdt$ does not necessarily imply the strong convergence of $\frac{\alpha}{t^\alpha}\int_0^t u^{\alpha-1}T(u+s)fdu$ without assuming any additional condition (cf.[17]). To justify the implication, it is needed that the semigroup in question satisfies an appropriate Tauberian condition. The Tauberian condition for the semigroup in Theorem 5 is necessarily fulfilled because $\lim_{t\to\infty}\left\|\frac{T(t)f}{t^\alpha}\right\|_1 = 0$ for all $\alpha > 0$ and all $f \in L_1^+$. A similar question can also arise in Theorem 6. Finally, it is worthwhile to note that the Tauberian technique used in Theorem 5 (or Theorem 6) can also be applicable to the study of the mean law of large numbers for (not necessarily independent) random processes in probability theory.

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