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RELAXATION OF SECOND ORDER GEOMETRIC INTEGRALS AND NON-LOCAL EFFECTS

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Abstract. We are concerned with the relaxation of second order geometric integrals, i.e., functionals of the type:

$$C_c^\infty(\mathbb{R}^N) \ni u \mapsto F_\mu(u) := \int_{\mathbb{R}^N} f\big(\nabla^2 u(x)\big) d\mu(x),$$

where $\nabla^2 u$ is the Hessian of u, $f : \mathbb{M}_N^{\rm sym} \to [0,+\infty]$ is a continuous function, and μ is a finite positive Radon measure on $\mathbb{R}^N$. A relaxation problem of this type was studied for the first time by Bouchitté and Fragala in [2] where they pointed out a new phenomenon: *the functional relaxed of F_μ has, in general, a "non-local" representation.* Working on a more formal level than in [2], we develop an alternative method making clear this "strange phenomenon".

1. Main results

Referring to the next section for any unfamiliar notation or definition, in what follows we state the main results of the paper.

Let a real number $p > 1$ and an integer number $N \geq 1$, and let μ be a finite positive Radon measure on $\mathbb{R}^N$. We will make the following connectedness assumption on μ:

(C_0) *for every $u \in C_c^\infty(\mathbb{R}^N)$, if $\nabla_\mu u = 0$ μ-a.e. then $u = 0$ μ-a.e..*

Given a continuous function $f : \mathbb{M}_N^{\rm sym} \to [0,+\infty]$, consider the following three conditions:

(C_1) *there exists $r > 0$ such that $f(\xi) \geq r|\xi|^p$ for all $\xi \in \mathbb{M}_N^{\rm sym}$;*
(C_2) *there exists $R > 0$ such that $f(\xi) \leq R(1 + |\xi|^p)$ for all $\xi \in \mathbb{M}_N^{\rm sym}$;*
(C_3) *for μ-a.e. $x \in \mathbb{R}^N$, the function $T_{2,\mu}(x) \ni \xi \mapsto \inf\big\{f(\xi+\xi') : \xi' \in N_{2,\mu}(x)\big\}$ is convex.*

Consider also the functional $F_\mu : C_c^\infty(\mathbb{R}^N) \to [0,+\infty]$ defined by

$$F_\mu(u) = \int_{\mathbb{R}^N} f\left(\nabla^2 u(x)\right) d\mu(x),$$

where $\nabla^2 u$ is the Hessian of u. The object of this paper is to provide a formula representing the $W^{1,p}_\mu$-functional relaxed of F_μ, i.e., $\mathfrak{F}_{1,\mu} : W^{1,p}_\mu(\mathbb{R}^N) \to [0,+\infty]$ given by

$$\mathfrak{F}_{1,\mu}(u) := \inf\left\{\liminf_{n\to+\infty} F_\mu(u_n) : u_n \in C_c^\infty(\mathbb{R}^N), \overline{u}_n \rightharpoonup u \text{ in } W^{1,p}_\mu(\mathbb{R}^N)\right\},$$

where $W^{1,p}_\mu(\mathbb{R}^N)$ is the first order μ-Sobolev space as defined in [1, Section 7] (cf. §2.2 and §2.3). For this, we introduce a second order μ-Sobolev space that we denote by $W^{2,p}_\mu(\mathbb{R}^N)$ (cf. §2.4 and §2.5), and we prove that the $W^{2,p}_\mu$-functional relaxed of F_μ, i.e., $\mathfrak{F}_{2,\mu} : W^{2,p}_\mu(\mathbb{R}^N) \to [0,+\infty]$ given by

$$\mathfrak{F}_{2,\mu}(u) := \inf\left\{\liminf_{n\to+\infty} F_\mu(u_n) : u_n \in C^\infty_{\rm c}(\mathbb{R}^N), \overline{\overline{u}}_n \rightharpoonup u \text{ in } W^{2,p}_\mu(\mathbb{R}^N)\right\},$$

has an integral representation as follows.

Theorem 1.1. *If* (C_0), (C_2) *and* (C_3) *hold, then*

$$\mathfrak{F}_{2,\mu}(u) = \int_{\mathbb{R}^N} \inf\left\{f\big(\nabla^2_\mu u(x) + \xi\big) : \xi \in N_{2,\mu}(x)\right\} d\mu(x)$$

for all $u \in W^{2,p}_\mu(\mathbb{R}^N)$.

Theorem 1.1 is proved in section 4 by the same method as in [5, 1, 6]. The distinguishing feature here is that $\mathfrak{F}_{1,\mu}$ has, in general, a "non-local" representation which can be computed from $\mathfrak{F}_{2,\mu}$ by using the following result.

Theorem 1.2. *If* (C_0) *and* (C_1) *hold, then*

$$\mathfrak{F}_{1,\mu}(u) = \inf\left\{\mathfrak{F}_{2,\mu}(v) : \Theta_\mu(v) = u\right\} \tag{1}$$

for all $u \in W^{1,p}_\mu(\mathbb{R}^N)$, (*where* $\Theta_\mu : W^{2,p}_\mu(\mathbb{R}^N) \to W^{1,p}_\mu(\mathbb{R}^N)$ *is the bounded operator defined in* §2.6).

We thus have

Corollary 1.3. *If* (C_0), (C_1), (C_2) *and* (C_3) *hold, then*

$$\mathfrak{F}_{1,\mu}(u) = \inf\left\{\int_{\mathbb{R}^N} \inf\left\{f\big(\nabla^2_\mu v(x) + \xi\big) : \xi \in N_{2,\mu}(x)\right\} d\mu(x) : \Theta_\mu(v) = u\right\}$$

for all $u \in W^{1,p}_\mu(\mathbb{R}^N)$.

Note that the "non-local" representation of $\mathfrak{F}_{1,\mu}$ is only due to the fact that Θ_μ is, in general, not injective. More precisely, the formula representing $\mathfrak{F}_{1,\mu}$ in (1) comes from a more general result which can be stated in the setting of Banach spaces (cf. Theorem 3.1). Theorem 1.2, which is proved in section 5, is a consequence of Theorem 3.1.

To complete the paper, in section 6 we show that the same method can be used to prove Theorems 1.4 and 1.5 below. These theorems are consequences of Theorem 3.2 which is the analogue of Theorem 3.1 for compact operators.

Theorem 1.4. *Let* $s\text{-}\mathfrak{F}_{1,\mu} : W^{1,p}_\mu(\mathbb{R}^N) \to [0,+\infty]$ *be defined by*

$$s\text{-}\mathfrak{F}_{1,\mu}(u) := \inf\left\{\liminf_{n\to+\infty} F_\mu(u_n) : u_n \in C^\infty_{\rm c}(\mathbb{R}^N), \overline{u}_n \to u \text{ in } W^{1,p}_\mu(\mathbb{R}^N)\right\}.$$

If (C_0) *and* (C_1) *holds and if* Θ_μ *is compact, then*

$$s\text{-}\mathfrak{F}_{1,\mu}(u) = \inf\left\{\mathfrak{F}_{2,\mu}(v) : \Theta_\mu(v) = u\right\} \tag{2}$$

for all $u \in W^{1,p}_\mu(\mathbb{R}^N)$.

Consider the following *p-Poincaré inequality*:

(C$_4$) *there exists $C>0$ such that for every $u \in C_c^\infty(\mathrm{I\!R}^N)$,*

$$\int_{\mathrm{I\!R}^N} |u(x)|^p d\mu(x) \le C \int_{\mathrm{I\!R}^N} |\nabla_\mu u(x)|^p d\mu(x).$$

Note that (C$_4$) implies (C$_0$).

Theorem 1.5. *Let $\mathfrak{F}_\mu : L^p_\mu(\mathrm{I\!R}^N) \to [0,+\infty]$ be defined by*

$$\mathfrak{F}_\mu(u) := \inf \left\{ \liminf_{n\to+\infty} F_\mu(u_n) : u_n \in C_c^\infty(\mathrm{I\!R}^N), \overline{u}_n \to u \text{ in } L^p_\mu(\mathrm{I\!R}^N) \right\}.$$

If (C$_1$) *and* (C$_4$) *hold and if the injection $\mathrm{I}_\mu : W^{1,p}_\mu(\mathrm{I\!R}^N) \to L^p_\mu(\mathrm{I\!R}^N)$ is compact, then*

$$\mathfrak{F}_\mu(u) = \inf \left\{ \mathfrak{F}_{2,\mu}(v) : \big(\mathrm{I}_\mu \text{ o } \Theta_\mu\big)(v) = u \right\} \tag{3}$$

for all $u \in L^p_\mu(\mathrm{I\!R}^N)$.

2. Notation and definitions

2.1. **General notation.** By $\mathrm{I\!M}_N^{\mathrm{sym}}$ we denote the space of symmetric real $N\times N$ matrices. If $x \in \mathrm{I\!R}^N$, then $|x|$ is its Euclidean norm; if $\xi \in \mathrm{I\!M}_N^{\mathrm{sym}}$, $|\xi|$ is the norm of ξ when regarded as a vector in $\mathrm{I\!R}^{N^2}$. For $E = \mathrm{I\!R}$, $\mathrm{I\!R}^N$ or $\mathrm{I\!M}_N^{\mathrm{sym}}$, we write $C_c^\infty(\mathrm{I\!R}^N;E)$ ($C_c^\infty(\mathrm{I\!R}^N)$ if $E=\mathrm{I\!R}$) for the space of smooth functions from $\mathrm{I\!R}^N$ to E with compact support. By $L^p_\mu(\mathrm{I\!R}^N;E)$ ($L^p_\mu(\mathrm{I\!R}^N)$ if $E=\mathrm{I\!R}$) we denote the Banach space of measurable functions $u : \mathrm{I\!R}^N \to E$ such the norm

$$\|u\|_{p,\mu} := \left(\int_{\mathrm{I\!R}^N} |u(x)|^p d\mu(x) \right)^{1/p}$$

is finite. When X is a Banach space, by $u_n \to u$ (resp. $u_n \rightharpoonup u$) in X we mean that u_n converges to u with respect to the strong (resp. weak) topology of X, and for $\mathcal{F} : X \to [0,+\infty]$, $\mathrm{dom}(\mathcal{F}) := \{u \in X : \mathcal{F}(u) < +\infty\}$. Finally, the support of μ is defined by

$$\mathrm{spt}(\mu) := \{x \in \mathrm{I\!R}^N : \mu(B_\rho(x)) > 0 \text{ for all } \rho > 0\},$$

where $B_\rho(x)$ denotes the open ball centered at x with radius ρ.

2.2. **Definition of $N_{1,\mu}(x)$ and $T_{1,\mu}(x)$.** Let $\mathfrak{N}_{1,\mu}$ be the vector subspace of $C_c^\infty(\mathrm{I\!R}^N;\mathrm{I\!R}^N)$ given by

$$\mathfrak{N}_{1,\mu} := \left\{ w \in C_c^\infty(\mathrm{I\!R}^N;\mathrm{I\!R}^N) : \exists v \in \mathcal{N}_{1,\mu} \text{ such that } w = \nabla v \text{ in } \mathrm{spt}(\mu) \right\}$$

with $\mathcal{N}_{1,\mu} := \{v \in C_c^\infty(\mathrm{I\!R}^N) : v = 0 \text{ in } \mathrm{spt}(\mu)\}$. Let $N_{1,\mu} : \mathrm{I\!R}^N \rightrightarrows \mathrm{I\!R}^N$ be the multifunction defined by

$$N_{1,\mu}(x) := \left\{ w(x) : w \in \mathfrak{N}_{1,\mu} \right\}.$$

Then $N_{1,\mu}(x)$ is a vector subspace of $\mathrm{I\!R}^N$, called the *first order normal space to μ at x*. The vector subspace $T_{1,\mu}(x)$ given by the equality

$$\mathrm{I\!R}^N = T_{1,\mu}(x) \overset{\perp}{\oplus} N_{1,\mu}(x)$$

is said to be the *first order tangent space to μ at x*.

2.3. **Definition of $\nabla_\mu u$ and $W^{1,p}_\mu(\mathbb{R}^N)$.** Let $P_{1,\mu}(x) : \mathbb{R}^N \to T_{1,\mu}(x)$ be the *orthogonal projection on* $T_{1,\mu}(x)$. The function $\nabla_\mu u : \mathbb{R}^N \to \mathbb{R}^N$ defined as the orthogonal projection of $\nabla u(x)$ on $T_{1,\mu}(x)$, i.e.,

$$\nabla_\mu u(x) := P_{1,\mu}(x)\big(\nabla u(x)\big), \quad u \in C^\infty_{\rm c}(\mathbb{R}^N),$$

is called the *μ-gradient of u*. In fact,

$$\nabla_\mu u(x) = \operatorname*{argmin}_{\xi \in T_{1,\mu}(x)} \Big|\nabla u(x) - \xi\Big|,$$

for all $x \in \mathbb{R}^N$. The closed-valued multifunction $N_{1,\mu}$ being obviously measurable, we deduce that $T_{1,\mu}$ is also measurable, which implies that the mapping $\nabla_\mu u$ is measurable, (see, e.g., [4]). By definition, for every $(x,\xi) \in \mathbb{R}^N \times \mathbb{R}^N$, $|P_{1,\mu}(x)(\xi)| \le |\xi|$, and so $\nabla_\mu u \in L^p_\mu(\mathbb{R}^N;\mathbb{R}^N)$ for all $u \in C^\infty_{\rm c}(\mathbb{R}^N)$. Clearly, if $u = v$ in $\mathrm{spt}(\mu)$ then $\nabla_\mu u = \nabla_\mu v$ in $\mathrm{spt}(\mu)$, which means that μ-gradient of u is compatible with the equality μ-a.e.. In $C^\infty_{\rm c}(\mathbb{R}^N)$, we define an equivalence relation as follows: for all $u, v \in C^\infty_{\rm c}(\mathbb{R}^N)$, we say that $u \sim v$ if $u = v$ μ-a.e., and we denote by $D_{1,\mu}(\mathbb{R}^N)$ the corresponding quotient space. The *first order μ-Sobolev space* $W^{1,p}_\mu(\mathbb{R}^N)$ is then defined as the completion of $D_{1,\mu}(\mathbb{R}^N)$ with respect to the following norm:

$$\|\overline{u}\|_{1,p,\mu} := \left(\int_{\mathbb{R}^N} |u(x)|^p d\mu(x)\right)^{1/p} + \left(\int_{\mathbb{R}^N} |\nabla_\mu u(x)|^p d\mu(x)\right)^{1/p},$$

where $\overline{u} := \big\{v \in C^\infty_{\rm c}(\mathbb{R}^N) : v = u \text{ in } \mathrm{spt}(\mu)\big\}$ denotes the equivalence class of u with respect to $\sim$. Since $\|\nabla_\mu u\|_{p,\mu} \le \|\overline{u}\|_{1,p,\mu}$ for all $\overline{u} \in D_{1,\mu}(\mathbb{R}^N)$ the linear map

$$D_{1,\mu}(\mathbb{R}^N) \ni \overline{u} \mapsto \nabla_\mu u \in L^p_\mu(\mathbb{R}^N;\mathbb{R}^N)$$

has a unique extension to $W^{1,p}_\mu(\mathbb{R}^N)$ which will be still denoted by $\nabla_\mu u$.

2.4. **Definition of $N_{2,\mu}(x)$ and $T_{2,\mu}(x)$.** Let $\mathfrak{N}_{2,\mu}$ be the vector subspace of $C^\infty_{\rm c}(\mathbb{R}^N;\mathbb{M}^{\rm sym}_N)$ given by

$$\mathfrak{N}_{2,\mu} := \Big\{w \in C^\infty_{\rm c}(\mathbb{R}^N;\mathbb{M}^{\rm sym}_N) : \exists v \in \mathcal{N}_{2,\mu} \text{ such that } w = \nabla^2 v \text{ in } \mathrm{spt}(\mu)\Big\}$$

with $\mathcal{N}_{2,\mu} := \{v \in C^\infty_{\rm c}(\mathbb{R}^N) : (v, \nabla v) = (0,0) \text{ in } \mathrm{spt}(\mu)\}$. Let $N_{2,\mu} : \mathbb{R}^N \rightrightarrows \mathbb{M}^{\rm sym}_N$ be the multifunction defined by

$$N_{2,\mu}(x) := \Big\{w(x) : w \in \mathfrak{N}_{2,\mu}\Big\}.$$

Then $N_{2,\mu}(x)$ is a vector subspace of $\mathbb{M}^{\rm sym}_N$, called the *second order normal space to μ at x*. The vector subspace $T_{2,\mu}(x)$ given by

$$\mathbb{M}^{\rm sym}_N = T_{2,\mu}(x) \overset{\perp}{\oplus} N_{2,\mu}(x)$$

is said to be the *second order tangent space to μ at x*.

2.5. **Definition of $\nabla^2_\mu u$ and $W^{2,p}_\mu(\mathbb{R}^N)$.** Let $P_{2,\mu}(x) : \mathbb{R}^N \to T_{2,\mu}(x)$ be the *orthogonal projection on* $T_{2,\mu}(x)$. The function $\nabla^2_\mu u : \mathbb{R}^N \to \mathbb{M}^{\rm sym}_N$ defined as the orthogonal projection of $\nabla u(x)$ on $T_{2,\mu}(x)$, i.e.,

$$\nabla^2_\mu u(x) := P_{2,\mu}(x)\big(\nabla^2 u(x)\big), \quad u \in C^\infty_{\rm c}(\mathbb{R}^N),$$

is called the *μ-Hessian of u*. Analysis similar to that in §2.2 shows that for every $u \in C^\infty_{\rm c}(\mathbb{R}^n)$, $\nabla^2_\mu u \in L^p_\mu(\mathbb{R}^N; \mathbb{M}^{\rm sym}_N)$. The only difference being that $u = v$ in $\mathrm{spt}(\mu)$ does not imply $\nabla^2_\mu u = \nabla^2_\mu v$ in $\mathrm{spt}(\mu)$. In fact, $\nabla^2_\mu u = \nabla^2_\mu v$ in $\mathrm{spt}(\mu)$ whenever $(u, \nabla u) = (v, \nabla v)$ in $\mathrm{spt}(\mu)$, which leads us to introduce the following equivalence relation: for all $u, v \in C^\infty_{\rm c}(\mathbb{R}^N)$, we say that $u \approx v$ if $(u, \nabla u) = (v, \nabla v)$ μ-a.e.. We denote by $D_{2,\mu}(\mathbb{R}^N)$ the corresponding quotient space. For each $u \in C^\infty_{\rm c}(\mathbb{R}^N)$, $\overline{\overline{u}} := \overline{u} \cap \big\{v \in C^\infty_{\rm c}(\mathbb{R}^N) : \nabla v = \nabla u \text{ in } \mathrm{spt}(\mu)\big\}$ denotes the equivalence class of u with respect to $\approx$.

Proposition 2.1. *Under* (C_0)*, the map*

$$D_{2,\mu}(\mathbb{R}^N) \ni \overline{\overline{u}} \mapsto \big\|\overline{\overline{u}}\big\|_{2,p,\mu} := \big\|\overline{u}\big\|_{1,p,\mu} + \left(\int_{\mathbb{R}^N} \big|\nabla^2_\mu u(x)\big|^p d\mu(x)\right)^{1/p} \tag{4}$$

is a norm on $D_{2,\mu}(\mathbb{R}^N)$.

Proof. We only need to show that $\nabla u = 0$ μ-a.e. whenever $\|\overline{\overline{u}}\|_{2,p,\mu} = 0$. It is clear that if $\|\overline{\overline{u}}\|_{2,p,\mu} = 0$ then $\nabla^2_\mu u = 0$ μ-a.e., and so $\nabla^2 u(x) \in N_{2,\mu}(x)$ for all $x \in \mathrm{spt}(\mu) \setminus N$ with $\mu(N) = 0$. Fix any $x \in \mathrm{spt}(\mu) \setminus N$ and consider $v \in \mathcal{N}_{2,\mu}$ such that $\nabla^2 v(x) = \nabla^2 u(x)$. Thus $\nabla(\partial v/\partial x_i)(x) = \nabla(\partial u/\partial x_i)(x)$ with $\partial v/\partial x_i = 0$ in $\mathrm{spt}(\mu)$ for all $i \in \{1, \cdots, N\}$, hence $\nabla(\partial u/\partial x_i)(x) \in N_{1,\mu}(x)$. It follows that

$$\nabla_\mu(\partial u/\partial x_i) = 0 \ \mu\text{-a.e.}.$$

Since $\partial u/\partial x_i \in C^\infty_{\rm c}(\mathbb{R}^N)$, from (C_0) we deduce that $\partial u/\partial x_i = 0$ μ-a.e., and the proof is complete. □

The *second order μ-Sobolev space* $W^{2,p}_\mu(\mathbb{R}^N)$ is then defined as the completion of $D_{2,\mu}(\mathbb{R}^N)$ with respect to the norm defined in (4). Since $\|\nabla^2_\mu u\|_{p,\mu} \le \|\overline{\overline{u}}\|_{2,p,\mu}$ for all $\overline{\overline{u}} \in D_{2,\mu}(\mathbb{R}^N)$ the linear map

$$D_{2,\mu}(\mathbb{R}^N) \ni \overline{\overline{u}} \mapsto \nabla^2_\mu u \in L^p_\mu(\mathbb{R}^N; \mathbb{M}^{\rm sym}_N)$$

has a unique extension to $W^{2,p}_\mu(\mathbb{R}^N)$ which will be still denoted by $\nabla^2_\mu u$.

2.6. **Definition of Θ_μ.** Let $\Theta_\mu : D_{2,\mu}(\mathbb{R}^N) \to W^{1,p}_\mu(\mathbb{R}^N)$ be defined by

$$\Theta_\mu\big(\overline{\overline{u}}\big) := \overline{u}.$$

Clearly, Θ_μ is a linear map which satisfies

$$\big\|\Theta_\mu\big(\overline{\overline{u}}\big)\big\|_{1,p,\mu} \le \big\|\overline{\overline{u}}\big\|_{2,p,\mu}$$

for all $u \in C^\infty_{\rm c}(\mathbb{R}^N)$, and consequently, it has a unique extension to $W^{2,p}_\mu(\mathbb{R}^N)$ which will be still denoted by Θ_μ.

3. General theorems for non-local relaxation

Let $(X_1, \|\cdot\|_1)$ and $(X_2, \|\cdot\|_2)$ be two Banach spaces, and let $\Psi : X_2 \to X_1$ be a bounded operator. For $i = 1, 2$, denote by w-cl$_i$ the sequential lower semicontinuous envelope with respect to the weak topology of X_i. The following two theorems are more useful versions of [5, Theorem 3.1].

Theorem 3.1. *Let $\mathcal{F} : X_1 \to [0, +\infty]$ be satisfying the following two conditions*:

(A_1) $\mathrm{dom}\big(\mathcal{F}\big) \subset \Psi\big(X_2\big)$;

(A_2) *there exist $\alpha, \beta > 0$ such that $\big[\mathcal{F}(\Psi(v))\big]^\alpha + \|\Psi(v)\|_1 \geq \beta\|v\|_2$ for all $v \in X_2$.*

If X_2 is reflexive, then

$$w\text{-cl}_1\big(\mathcal{F}\big)(u) = \inf\Big\{w\text{-cl}_2\big(\mathcal{F} \text{ o } \Psi\big)(v) : \Psi(v) = u\Big\} \tag{5}$$

for all $u \in X_1$.

Proof. Fix $u \in X_2$ and denote by $\overline{\mathcal{F}}(u)$ the right-hand side of (5). We have to show that

$$\inf\left\{\limsup_{n\to+\infty} \mathcal{F}(u_n) : u_n \rightharpoonup u \text{ in } X_1\right\} \leq \overline{\mathcal{F}}(u) \leq \inf\left\{\liminf_{n\to+\infty} \mathcal{F}(u_n) : u_n \rightharpoonup u \text{ in } X_1\right\}.$$

We first prove the upper bound. Without loss of generality we can assume that $u \in \Psi(X_2)$. Given any $v \in \Psi^{-1}(\{u\})$, there exists $v_n \rightharpoonup v$ in X_2 such that

$$\limsup_{n\to+\infty} \mathcal{F}\big(\Psi(v_n)\big) \leq \text{w-cl}_2\big(\mathcal{F} \text{ o } \Psi\big)(v).$$

As Ψ is bounded, we have $\Psi(v_n) \rightharpoonup u$ in X_1. It follows that

$$\inf\left\{\limsup_{n\to+\infty} \mathcal{F}(u_n) : u_n \rightharpoonup u \text{ in } X_1\right\} \leq \text{cl}_2\big(\mathcal{F} \text{ o } \Psi\big)(v)$$

for all $v \in \Psi^{-1}(\{u\})$, and the upper bound follows.

Consider now $\{u_n\}_{n\geq1} \subset X_1$ such that $u_n \rightharpoonup u$ in X_1. There is no loss of generality in assuming that $\{\mathcal{F}(u_n)\}_{n\geq1}$ is bounded. Thus $\{u_n\}_{n\geq1} \subset \mathrm{dom}(\mathcal{F})$, and from ($A_1$) and ($A_2$), we see that there exists a bounded sequence $\{v_n\}_{n\geq1} \subset X_2$ such that $\Psi(v_n) = u_n$. As X_2 is reflexive, we have $v_n \rightharpoonup v$ in X_2 for some $v \in \Psi^{-1}(\{u\})$. Hence,

$$\overline{\mathcal{F}}(u) \leq \text{w-cl}_2\big(\mathcal{F} \text{ o } \Psi\big)(v) \leq \liminf_{n\to+\infty} \mathcal{F}\big(\Psi(v_n)\big) = \liminf_{n\to+\infty} \mathcal{F}(u_n),$$

and the lower bound follows. □

In the following theorem, s-cl$_1$ denotes the sequential lower semicontinuous envelope with respect to the strong topology of X_1.

Theorem 3.2. *Let $\mathcal{F} : X_1 \to [0, +\infty]$ be satisfying* (A_1) *and* (A_2). *If X_2 is reflexive and if Ψ is compact, then*

$$s\text{-cl}_1\big(\mathcal{F}\big)(u) = \inf\Big\{w\text{-cl}_2\big(\mathcal{F} \text{ o } \Psi\big)(v) : \Psi(v) = u\Big\}$$

for all $u \in X_1$.

Proof. Replace the argument "Ψ is bounded" by "Ψ is compact" and "$\cdot \rightharpoonup \cdot$ in X_1" by "$\cdot \to \cdot$ in X_1" in the proof of Theorem 3.1. □

4. Proof of Theorem 1.1

Let $F_{2,\mu} : C_c^\infty(\mathbb{R}^N) \to [0, +\infty]$ be defined by

$$F_{2,\mu}(u) := \inf \Big\{ F_\mu(v) : v \in \overline{\overline{u}} \Big\}.$$

The following proposition makes clear the link between $\mathfrak{F}_{2,\mu}$ and $F_{2,\mu}$.

Proposition 4.1. *For every* $u \in C_c^\infty(\mathbb{R}^N)$,

$$\mathfrak{F}_{2,\mu}(u) = \inf \left\{ \liminf_{n\to+\infty} F_{2,\mu}(u_n) : u_n \in C_c^\infty(\mathbb{R}^N), \overline{\overline{u}}_n \rightharpoonup u \text{ in } W_\mu^{2,p}(\mathbb{R}^N) \right\}. \tag{6}$$

Proof. Fix $u \in W_\mu^{2,p}(\mathbb{R}^N)$, and denote by $\hat{\mathfrak{F}}_{2,\mu}(u)$ the right-side of (6). As $F_\mu(v) \geq F_{2,\mu}(v)$ for all $v \in C_c^\infty(\mathbb{R}^N)$, it is clear that $\mathfrak{F}_{2,\mu}(u) \geq \hat{\mathfrak{F}}_{2,\mu}(u)$. Fix any $\varepsilon > 0$, and consider $\{u_n\}_{n\geq 1} \subset C_c^\infty(\mathbb{R}^N)$ with $\overline{\overline{u}}_n \rightharpoonup u$ in $W_\mu^{2,p}(\mathbb{R}^N)$ such that

$$\hat{\mathfrak{F}}_{2,\mu}(u) + \frac{\varepsilon}{2} \geq \liminf_{n\to+\infty} F_{2,\mu}(u_n).$$

To every $n \geq 1$, there corresponds $v_n \in C_c^\infty(\mathbb{R}^N)$ with $\overline{\overline{v}}_n = \overline{\overline{u}}_n$ such that $F_{2,\mu}(u_n) + \frac{\varepsilon}{2} \geq F_\mu(v_n)$. Hence,

$$\hat{\mathfrak{F}}_{2,\mu}(u) + \varepsilon \geq \liminf_{n\to+\infty} F_\mu(v_n)$$

with $\overline{\overline{v}}_n \rightharpoonup u$ in $W_\mu^{2,p}(\mathbb{R}^N)$, and (6) follows by letting $\varepsilon \to 0$. □

4.1. Integral representation of $F_{2,\mu}$.

For every $u \in C_c^\infty(\mathbb{R}^N)$,

$$F_{2,\mu}(u) = \inf_{w\in\mathcal{H}_{2,u}} \int_{\mathbb{R}^N} f(w(x))d\mu(x) \tag{7}$$

with $\mathcal{H}_{2,u} := \big\{ w \in C_c^\infty(\mathbb{R}^N; \mathbb{M}_N^{\rm sym}) : \exists v \in \overline{\overline{u}} \text{ such that } w = \nabla^2 v \text{ in } \operatorname{spt}(\mu) \big\}$. Moreover,

Lemma 4.2. *Every* $\mathcal{H}_{2,u}$ *is* $C_c^\infty(\mathbb{R}^N; [0,1])$*-decomposable (see definition in* §A.3*).*

Proof. Fix $w, \hat{w} \in \mathcal{H}_{2,u}$ and $\phi \in C_c^\infty(\mathbb{R}^N; [0,1])$. Choose $v, \hat{v} \in \overline{\overline{u}}$ such that $w = \nabla^2 v$ (resp. $\hat{w} = \nabla^2 \hat{v}$) in $\operatorname{spt}(\mu)$. Then, $\nabla(\phi v + (1-\phi)\hat{v}) = \phi\nabla v + (1-\phi)\nabla\hat{v} + (v-\hat{v})\nabla\phi$ and $\nabla^2(\phi v + (1-\phi)\hat{v}) = \phi\nabla v + (1-\phi)\nabla\hat{v} + \nabla\phi\otimes(\nabla v - \nabla\hat{v}) + (\nabla v - \nabla\hat{v})\otimes\nabla\phi + (v-\hat{v})\nabla\phi$, and so $\nabla(\phi v + (1-\phi)\hat{v}) = \nabla u$ (resp. $\nabla^2(\phi v + (1-\phi)\hat{v}) = \phi w + (1-\phi)\hat{w}$) in $\operatorname{spt}(\mu)$. As $\phi v + (1-\phi)\hat{v} = u$ in $\operatorname{spt}(\mu)$, we conclude that $\phi w + (1-\phi)\hat{w} \in \mathcal{H}_{2,u}$. □

For $u \in C_c^\infty(\mathbb{R}^N)$, we let $\Lambda_{2,u} : \mathbb{R}^N \rightrightarrows \mathbb{M}_N^{\rm sym}$ be defined by

$$\Lambda_{2,u}(x) := \big\{ w(x) : w \in \mathcal{H}_{2,u} \big\}.$$

According to the notation in §2.3, we have $\Lambda_{2,0} = N_{2,\mu}$. Here is the link between $\Lambda_{2,u}$ and $N_{2,\mu}$.

Lemma 4.3. *For every* $u \in C_c^\infty(\mathbb{R}^N)$ *and every* $x \in \operatorname{spt}(\mu)$,

$$\Lambda_{2,u}(x) = N_{2,\mu}(x) + \big\{ \nabla_\mu^2 u(x) \big\}.$$

Proof. Fix $u \in C_c^\infty(\mathbb{R}^N)$ and $x \in \mathrm{spt}(\mu)$. Given $\xi \in \Lambda_{2,u}(x)$, there exists $v \in \overline{\overline{u}}$ such that $\xi = \nabla^2 v(x)$, and $\xi = \nabla^2 v(x) - P_{2,\mu}(x)(\nabla^2 v(x)) + \nabla^2_\mu u(x)$. Noticing that $\nabla^2 v(x) - P_{2,\mu}(x)(\nabla^2 v(x)) \in N_{2,\mu}(x)$, we deduce that $\Lambda_{2,u}(x) \subset N_{2,\mu}(x) + \big\{\nabla^2_\mu u(x)\big\}$. Consider now $\xi \in N_{2,\mu}(x) + \{\nabla^2_\mu u(x)\}$. As $\nabla^2_\mu u(x) = P_{2,\mu}(x)(\nabla^2 u(x))$ we have $P_{2,\mu}(x)(\xi - \nabla^2 u(x)) = 0$, and so $\xi - \nabla^2 u(x) \in N_{2,\mu}(x)$. Hence, there exists $v \in \mathcal{N}_{2,\mu}$ such that $\xi = \nabla^2(v+u)(x)$, which gives $N_{2,\mu}(x) + \{\nabla^2_\mu u(x)\} \subset \Lambda_{2,u}(x)$. □

Here is our integral representation for $F_{2,\mu}$.

Proposition 4.4. *If* (C$_2$) *holds, then*

$$F_{2,\mu}(u) = \int_{\mathbb{R}^N} \inf\Big\{ f\big(\nabla^2_\mu u(x) + \xi\big) : \xi \in N_{2,\mu}(x) \Big\} d\mu(x) \tag{8}$$

for all $u \in C_c^\infty(\mathbb{R}^N)$.

Proof. Fix $u \in C_c^\infty(\mathbb{R}^N)$ and denote by $\Gamma_{2,u} : \mathbb{R}^N \rightrightarrows \mathbb{M}_N^{\mathrm{sym}}$ the μ-essential supremum of $\mathcal{H}_{2,u}$ (see definition in §A.2). Taking (7), Lemma 4.2 and (C$_2$) into account, from Theorem A.2 we obtain

$$F_{2,\mu}(u) = \int_{\mathbb{R}^N} \inf_{\xi \in \Gamma_{2,u}(x)} f(\xi)\, d\mu(x).$$

Since $\mathcal{H}_{2,u} \subset C_c^\infty(\mathbb{R}^N)$, $\Gamma_{2,u}(x) = \mathrm{cl}\{w(x) : w \in \mathcal{H}_{2,u}\}$ μ-a.e. from Lemma A.1(ii). Hence $\Gamma_{2,u}(x) = N_{2,\mu}(x) + \{\nabla^2_\mu u(x)\}$ μ-a.e. by Lemma 4.3, and (8) follows. □

4.2. **Proof of Theorem 1.1.** Since μ is finite and (C$_2$) holds, Vitali's theorem shows that the functional

$$W^{2,p}_\mu(\mathbb{R}^N) \ni u \mapsto \hat{\mathfrak{F}}_{2,\mu}(u) := \int_{\mathbb{R}^N} \inf\Big\{ f\big(\nabla^2_\mu u(x) + \xi\big) : \xi \in N_{2,\mu}(x) \Big\} d\mu(x)$$

is strongly continuous. By Proposition 4.4, (8) is satisfied for all $u \in C_c^\infty(\mathbb{R}^N)$, and taking Proposition 4.1 into account, we deduce that $\hat{\mathfrak{F}}_{2,\mu} \geq \mathfrak{F}_{2,\mu}$. From (C$_3$) we see that $\hat{\mathfrak{F}}_{2,\mu}$ is convex, which implies that $\hat{\mathfrak{F}}_{2,\mu}$ is weakly lower semicontinuous. It follows that $\hat{\mathfrak{F}}_{2,\mu} \leq \mathfrak{F}_{2,\mu}$, and the proof is complete. □

5. Proof of Theorem 1.2

Let $F_{1,\mu} : C_c^\infty(\mathbb{R}^N) \to [0, +\infty]$ be defined by

$$F_{1,\mu}(u) := \inf\Big\{ F_\mu(v) : v \in \overline{u} \Big\}. \tag{9}$$

Similarly to Proposition 4.1, we have

Proposition 5.1. *For every* $u \in C_c^\infty(\mathbb{R}^N)$,

$$\mathfrak{F}_{1,\mu}(u) = \inf\left\{ \liminf_{n\to+\infty} F_{1,\mu}(u_n) : u_n \in C_c^\infty(\mathbb{R}^N), \overline{u}_n \rightharpoonup u \textit{ in } W^{1,p}_\mu(\mathbb{R}^N) \right\}.$$

5.1. **Representation of $F_{1,\mu}$.** The functional $F_{1,\mu}$ has, in general, a "nonlocal" representation which can be computed from $F_{2,\mu}$ by using the following result.

Proposition 5.2. *For every $u \in C_c^\infty(\mathbb{R}^N)$,*

$$F_{1,\mu}(u) = \inf\Big\{F_{2,\mu}(v) : v \in \overline{u}\Big\}. \tag{10}$$

Proof. Fix $u \in C_c^\infty(\mathbb{R}^N)$ and denote by $\hat{F}_{1,\mu}(u)$ the right-hand side of (10). Of course, $F_\mu(v) \geq F_{2,\mu}(v)$ for all $v \in C_c^\infty(\mathbb{R}^N)$, and so $F_{1,\mu}(u) \geq \hat{F}_{1,\mu}(u)$. Given any $v \in \overline{u}$, it is clear that if $\varphi \in \overline{\overline{v}}$ then $\varphi \in \overline{u}$, hence $F_\mu(\varphi) \geq F_{1,\mu}(u)$ for all $\varphi \in \overline{\overline{v}}$. Thus, for every $v \in \overline{u}$, $F_{2,\mu}(v) \geq F_{1,\mu}(u)$, and (10) follows. □

The following lemma is a direct consequence of Proposition 5.2.

Lemma 5.3. *For $i = 1, 2$, define $\mathcal{F}_{i,\mu} : W_\mu^{i,p}(\mathbb{R}^N) \to [0, +\infty]$ by*

$$\mathcal{F}_{i,\mu}(u) := \begin{cases} F_{i,\mu}(v) & \text{with } v \in u \text{ if } u \in D_{i,\mu}(\mathbb{R}^N) \\ +\infty & \text{otherwise.} \end{cases} \tag{11}$$

Then:

(i) $\mathcal{F}_{1,\mu}(u) = \inf\{\mathcal{F}_{2,\mu}(v) : \Theta_\mu(v) = u\}$ *for all $u \in W_\mu^{1,p}(\mathbb{R}^N)$;*
(ii) $\mathcal{F}_{2,\mu} = \mathcal{F}_{1,\mu} \circ \Theta_\mu$.

5.2. **Proof of Theorem 1.2.** According to Proposition 5.1, it is easy to see that for $i = 1, 2$, $\mathfrak{F}_{i,\mu}$ is the sequential lower semicontinuous envelope with respect to the weak topology of $W_\mu^{i,p}(\mathbb{R}^N)$ of the functional $\mathcal{F}_{i,\mu}$ defined in (11). From (C_1) we deduce that for every $u \in C_c^\infty(\mathbb{R}^N)$,

$$\big(\mathcal{F}_{1,\mu}(\overline{u})\big)^{1/p} + \|\overline{u}\|_{1,p,\mu} \geq \min\{1, r^{1/p}\}\|\overline{\overline{u}}\|_{2,p,\mu}.$$

Since $\mathrm{dom}(\mathcal{F}_{1,\mu}) = \Theta_\mu(D_{2,\mu}(\mathbb{R}^N)) = D_{1,\mu}(\mathbb{R}^N)$, ($A_1$) and ($A_2$) are satisfied with $X_1 = W_\mu^{1,p}(\mathbb{R}^N)$, $X_2 = W_\mu^{2,p}(\mathbb{R}^N)$, $\mathcal{F} = \mathcal{F}_{1,\mu}$ and $\Psi = \Theta_\mu$. Denote by w-cl$_{2,p,\mu}$ the sequential lower semicontinuous envelope with respect to the weak topology of $W_\mu^{2,p}(\mathbb{R}^N)$. As $W_\mu^{2,p}(\mathbb{R}^N)$ is reflexive (see §A.1), from Theorem 3.1 we have

$$\mathfrak{F}_{1,\mu}(u) = \inf\Big\{\text{w-cl}_{2,p,\mu}\big(\mathcal{F}_{1,\mu} \circ \Theta_\mu\big)(v) : \Theta_\mu(v) = u\Big\} \tag{12}$$

for all $u \in W_\mu^{1,p}(\mathbb{R}^N)$, and (1) follows by Lemma 5.3(ii). □

6. Proof of Theorems 1.4 and 1.5

6.1. **Proof of Theorem 1.4.** Similarly to Proposition 5.1, we have

$$\text{s-}\mathfrak{F}_{1,\mu}(u) = \inf\left\{\liminf_{n\to+\infty} F_{1,\mu}(u_n) : u_n \in C_c^\infty(\mathbb{R}^N), \overline{u}_n \to u \text{ in } W_\mu^{1,p}(\mathbb{R}^N)\right\}$$

for all $u \in W_\mu^{1,p}(\mathbb{R}^N)$, with $F_{1,\mu} : C_c^\infty(\mathbb{R}^N) \to [0, +\infty]$ given by (9). Thus, s-$\mathfrak{F}_{1,\mu}$ is the sequential lower semicontinuous envelope with respect to the strong topology of $W_\mu^{1,p}(\mathbb{R}^N)$ of the functional $\mathcal{F}_{1,\mu}$ defined in (11). Analysis similar to that in the proof of Theorem 1.2 shows that (A_1) and (A_2) are satisfied with $X_1 = W_\mu^{1,p}(\mathbb{R}^N)$, $X_2 = W_\mu^{2,p}(\mathbb{R}^N)$, $\mathcal{F} = \mathcal{F}_{1,\mu}$ and $\Psi = \Theta_\mu$. As $W_\mu^{2,p}(\mathbb{R}^N)$ is reflexive and Θ_μ is compact, from Theorem 3.2 we deduce that s-$\mathfrak{F}_{1,\mu}(u)$ is equal to the right side of (12) for all $u \in W_\mu^{1,p}(\mathbb{R}^N)$, and (2) follows by Lemma 5.3(ii). □

6.2. **Proof of Theorem 1.5.** Again, we have

$$\mathfrak{F}_\mu(u) = \inf\left\{\liminf_{n\to+\infty} F_{1,\mu}(u_n) : u_n \in C_{\rm c}^\infty(\mathbb{R}^N), \overline{u}_n \to u \text{ in } L^p_\mu(\mathbb{R}^N)\right\}$$

for all $u \in W^{1,p}_\mu(\mathbb{R}^N)$, with $F_{1,\mu} : C_{\rm c}^\infty(\mathbb{R}^N) \to [0,+\infty]$ given by (9). Thus, $\mathfrak{F}_\mu$ is the sequential lower semicontinuous envelope with respect to the strong topology of $L^p_\mu(\mathbb{R}^N)$ of the functional $\mathcal{F}_\mu : L^p_\mu(\mathbb{R}^N) \to [0,+\infty]$ defined by

$$\mathcal{F}_\mu(u) := \begin{cases} F_{1,\mu}(v) & \text{with } v \in u \text{ if } u \in D_{1,\mu}(\mathbb{R}^N) \\ +\infty & \text{otherwise.}\end{cases}$$

We claim that (A_1) and (A_2) are satisfied with $X_1 = L^p_\mu(\mathbb{R}^N)$, $X_2 = W^{2,p}_\mu(\mathbb{R}^N)$, $\mathcal{F} = \mathcal{F}_\mu$ and $\Psi = \mathrm{I}_\mu \text{ o } \Theta_\mu$. Indeed, By ($C_4$) there exists $C > 0$ such that for every $u \in C_{\rm c}^\infty(\mathbb{R}^N)$ and every $i \in \{1, \cdots, N\}$,

$$\int_{\mathbb{R}^N} \big|(\partial u/\partial x_i)(x)\big|^p d\mu(x) \le C \int_{\mathbb{R}^N} \big|\nabla_\mu(\partial u/\partial x_i)(x)\big|^p d\mu(x). \tag{13}$$

As $\gamma|\xi|^p \le \sum_{i=1}^N |\xi_i|^p$ for all $\xi \in \mathbb{R}^N$ and some $\gamma > 0$ (which does not depend on ξ), and as $|\nabla_\mu u(x)| \le |\nabla u(x)|$ and $|\nabla_\mu(\partial u/\partial x_i)(x)| \le |\nabla(\partial u/\partial x_i)(x)| \le |\nabla^2 u(x)|$, from (13) we have

$$\int_{\mathbb{R}^N} \big|\nabla^2 u(x)\big|^p d\mu(x) \ge \big(\gamma/NC\big) \int_{\mathbb{R}^N} \big|\nabla_\mu u(x)\big|^p d\mu(x).$$

Using (C_1), we deduce that for every $u \in C_{\rm c}^\infty(\mathbb{R}^N)$,

$$\big(\mathcal{F}_\mu(\overline{u})\big)^{1/p} + \big\|\overline{u}\big\|_{p,\mu} \ge \min\left\{1, \big(r\gamma/2^p NC\big)^{1/p}, \big(r/2^p\big)^{1/p}\right\}\big\|\overline{\overline{u}}\big\|_{2,p,\mu}$$

and the claim follows because $\mathrm{dom}\big(\mathcal{F}_\mu\big) = (\mathrm{I}_\mu \text{ o } \Theta_\mu)\big(D_{2,\mu}(\mathbb{R}^N)\big) = D_{1,\mu}(\mathbb{R}^N)$.

As I_μ is compact, so is I_μ o Θ_μ. Since $W^{2,p}_\mu(\mathbb{R}^N)$ is reflexive (see §A.1), from Theorem 3.2 we obtain

$$\mathcal{F}_\mu(u) = \inf\left\{\text{w-cl}_{2,p,\mu}\big(\mathcal{F}_\mu \text{ o } I_\mu \text{ o } \Theta_\mu\big)(v) : \big(I_\mu \text{ o } \Theta_\mu\big)(v) = u\right\}$$

for all $u \in L^p_\mu(\mathbb{R}^N)$, where $\text{w-cl}_{2,p,\mu}$ denotes the sequential lower semicontinuous envelope with respect to the weak topology of $W^{2,p}_\mu(\mathbb{R}^N)$. Using Proposition 5.2, it is easily seen that $\mathcal{F}_\mu$ o I_μ o $\Theta_\mu = \mathcal{F}_{2,\mu}$ with $\mathcal{F}_{2,\mu} : W^{2,p}_\mu(\mathbb{R}^N) \to [0,+\infty]$ given by (11), and (3) follows because $\mathfrak{F}_{2,\mu} = \text{w-cl}_{2,p,\mu}\big(\mathcal{F}_{2,\mu}\big)$ by Proposition 4.1. □

Appendix A. Auxiliary results

A.1. **Reflexivity of $W^{2,p}_\mu(\mathbb{R}^N)$.** Since the linear map

$$D_{2,\mu}(\mathbb{R}^N) \ni \overline{\overline{u}} \mapsto (u, \nabla_\mu u, \nabla^2_\mu u) \in X := L^p_\mu(\mathbb{R}^N) \times L^p_\mu(\mathbb{R}^N; \mathbb{R}^N) \times L^p_\mu(\mathbb{R}^N; \mathbb{M}^{\rm sym}_N)$$

is an isometry, it has an extension to $W^{2,p}_\mu(\mathbb{R}^N)$ which is still an isometry. We denote it by Φ. Thus $\Phi\big(W^{2,p}_\mu(\mathbb{R}^N)\big)$ is a closed vector subspace of X. As $p > 1$, the product space X is reflexive, and so is $\Phi\big(W^{2,p}_\mu(\mathbb{R}^N)\big)$. It follows that $W^{2,p}_\mu(\mathbb{R}^N)$ is reflexive.

A.2. **The μ-essential supremum of a set of measurable functions.** Denote by $\mathcal{M}$ the class of all closed-valued measurable multifunctions from $\mathbb{R}^N$ to $\mathbb{M}_N^{\rm sym}$, and set $\mathcal{M}^* := \big\{\Gamma \in \mathcal{M} : \forall x \in \mathbb{R}^N,\ \Gamma(x) \neq \emptyset\big\}$.

Following Valadier [7, Proposition 14], if $\mathcal{F} \subset \mathcal{M}^*$ is nonempty, then there exists $\Gamma \in \mathcal{M}^*$ satisfying the following two properties:

- $\forall \Lambda \in \mathcal{F}$, $\Lambda(x) \subset \Gamma(x)$ μ-a.e.;
- $\Gamma' \in \mathcal{M}$ and $\forall \Lambda \in \mathcal{F}$, $\Lambda(x) \subset \Gamma'(x)$ μ-a.e. $\Rightarrow \Gamma(x) \subset \Gamma'(x)$ μ-a.e..

Note that Γ is unique with respect to the equality μ-a.e. Valadier called it the μ-essential upper bound of $\mathcal{F}$.

In this paper, by *the μ-essential supremum* of a set $\mathcal{H}$ of measurable functions from $\mathbb{R}^N$ to $\mathbb{M}_N^{\rm sym}$, we mean the μ-essential upper bound of $\{\{w\} : w \in \mathcal{H}\}$, where $\{w\} : \mathbb{R}^N \rightrightarrows \mathbb{M}_N^{\rm sym}$ is defined by $\{w\}(x) = \{w(x)\}$. Thus, if we denote by Γ the μ-essential supremum of $\mathcal{H}$, we have:

- $\forall w \in \mathcal{H}$, $w(x) \in \Gamma(x)$ μ-a.e.;
- $\Gamma' \in \mathcal{M}$ and $\forall w \in \mathcal{H}$, $w(x) \in \Gamma'(x)$ μ-a.e. $\Rightarrow \Gamma(x) \subset \Gamma'(x)$ μ-a.e..

The following lemma gives (classical) representations of the μ-essential supremum. For a proof we refer the reader to [3, §2.2].

Lemma A.1. *Let Γ denote the μ-essential supremum of $\mathcal{H} \subset L^p_\mu(\mathbb{R}^N; \mathbb{M}_N^{\rm sym})$. Then*:

(i) *There exists a countable subset $\mathcal{D} \subset \mathcal{H}$ such that $\Gamma(x) = {\rm cl}\{w(x) : w \in \mathcal{D}\}$ μ-a.e., where* cl *denotes the closure in $\mathbb{M}_N^{\rm sym}$.*
(ii) *If $\mathcal{H} \subset C_c^\infty(\mathbb{R}^N; \mathbb{M}_N^{\rm sym})$, then $\Gamma(x) = {\rm cl}\{w(x) : w \in \mathcal{H}\}$ μ-a.e..*

A.3. **Interchange of infimum and integral.** A set $\mathcal{H} \subset L^p_\mu(\mathbb{R}^N; \mathbb{M}_N^{\rm sym})$ is said to be $C_c^\infty(\mathbb{R}^N; [0,1])$*-decomposable* if for all $w, \hat{w} \in \mathcal{H}$ and all $\phi \in C_c^\infty(\mathbb{R}^N; [0,1])$, $\phi w + (1-\phi)\hat{w} \in \mathcal{H}$. The following result is a consequence of [1, Theorem 1.1] where we refer the reader for a proof.

Theorem A.2. *Let $\mathcal{H} \subset L^p_\mu(\mathbb{R}^N; \mathbb{M}_N^{\rm sym})$ be a $C_c^\infty(\mathbb{R}^N; [0,1])$-decomposable set. If* ($C_2$) *holds, then*

$$\inf_{w \in \mathcal{H}} \int_{\mathbb{R}^N} f(w(x))d\mu(x) = \int_{\mathbb{R}^N} \inf_{\xi \in \Gamma(x)} f(\xi)\ d\mu(x)$$

with $\Gamma : \mathbb{R}^N \rightrightarrows \mathbb{M}_N^{\rm sym}$ given by the μ-essential supremum of $\mathcal{H}$.

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