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STRONG CONVERGENCE THEOREMS AND STABILITY PROBLEMS OF MANN AND ISHIKAWA ITERATIVE SEQUENCES FOR STRICTLY HEMI-CONTRACTIVE MAPPINGS

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Abstract. The purpose of this paper is to study the strong convergence theorems of the Mann iterative sequences, necessary and sufficient conditions for the convergence of the Ishikawa iterative sequences and stability of the Ishikawa iterative sequences for strictly hemi-contractive mappings in real Banach spaces with property $(U, \lambda, m+1, m)$.

1. Introduction

Let X be a real normed linear space. Let J denote the normalized duality mapping from X into 2^{X^*} defined by

$$J(x) = \{f \in X^* : \langle x, f\rangle = \|x\|^2 = \|f\|^2\}, \quad x \in X,$$

where X^* is the dual space of X and $\langle \cdot, \cdot\rangle$ denotes the generalized duality pairing. In the sequel, I and $F(T)$ denote the identity mapping and the set of fixed points of an operator T, respectively. $\mathbb{N}$ and $\mathbb{R}$ stand for the set of positive integers and the set of real numbers, respectively.

Definition 1.1 ([2, 4, 14]). Let K be a nonempty subset of a real normed linear space X and let $T : K \to K$ be a mapping. Let $\alpha \geq 0$ and λ, $\beta \in \mathbb{R}$.

(i) T is said to be *strictly pseudo-contractive* if there exists a constant $k \in (0, 1)$ such that for each $x, y \in K$, there exists $j(x - y) \in J(x - y)$ satisfying

$$\langle Tx - Ty, j(x - y)\rangle \leq (1 - k)\|x - y\|^2;$$

(ii) T is said to be *strictly hemi-contractive* if $F(T) \neq \emptyset$ and if there exists a constant $k \in (0, 1)$ such that for each $x \in K$, $q \in F(T)$, there exists $j(x - q) \in J(x - q)$ satisfying

$$\langle Tx - q, j(x - q)\rangle \leq (1 - k)\|x - q\|^2;$$

(iii) X is said to *have property* $(U, \lambda, \alpha, \beta)$ if

$$\|x + y\|^\alpha + \lambda\|x - y\|^\alpha - 2^\beta(\|x\|^\alpha + \|y\|^\alpha) \geq 0, \quad x, y \in X.$$

Chidume-Osilike [4] proved that the class of strictly pseudo-contractive mappings with fixed points is a proper subclass of the class of strictly hemi-contractive mappings. Schu [13] pointed out that each L_p (or l_p) space with $p \geq 2$ possesses property $(U, p - 1, 2, 1)$.

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Let K be a nonempty subset of X and let $T : K \to K$ be a mapping. Assume that $x_0 \in K$ and $x_{n+1} = f(T, x_n)$ defines an iterative process which produces a sequence $\{x_n\}$ in K. Suppose, furthermore, that $F(T) \neq \emptyset$ and $\{x_n\}$ converges strongly to $q \in F(T)$. Let $\{y_n\}$ be an arbitrary sequence in K and define $\{\varepsilon_n\} \subset [0, \infty)$ by $\varepsilon_n = \|y_{n+1} - f(T, y_n)\|$.

Definition 1.2 ([6–8, 11])**.** The sequence $\{x_n\}$ defined by $x_{n+1} = f(T, x_n)$ is said to be *T-stable* on K if $\lim_{n\to\infty} \varepsilon_n = 0$ implies that $\lim_{n\to\infty} y_n = q$.

Definition 1.3 ([9, 10])**.** Let K be a nonempty convex subset of a real normed linear space X and let $T : K \to K$ be a mapping.

(i) The sequence $\{x_n\}$ in K defined by

$$\begin{cases} x_0 \in K, \\ z_n = (1 - \beta_n)x_n + \beta_n T x_n, \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T z_n, \quad n \geq 0, \end{cases}$$

is called the *Ishikawa iterative sequence*, where $\{\alpha_n\}, \{\beta_n\} \subset [0, 1]$.

(ii) If $\beta_n = 0$ for each $n \geq 0$ in (i), then the sequence $\{x_n\}$ in K defined by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T x_n, \quad n \geq 0,$$

is called the *Mann iterative sequence*, where $\{\alpha_n\} \subset [0, 1]$.

Chidume [2] proved that if $X = L_p$ (or l_p), $p \geq 2$, K is a nonempty bounded closed convex subset of X and $T : X \to X$ is a Lipschitz strictly pseudo-contractive mapping, then the Mann iterative sequence converges strongly to a fixed point of T under suitable conditions. Afterwards, Chidume [3] and Deng [5] extended the result in [2] in the same spaces. Schu [13] established an inequality in Banach spaces with property $(U, \lambda, m+1, m)$ (see, Lemma 2.1) and proved that the Ishikawa iterative sequence can be used to approximate fixed points of Lipschitz strictly pseudo-contractive mappings by means of the inequality.

Many stability results for certain classes of nonlinear mappings have been established by several authors (see, [6–8, 11]). Osilike [11] obtained stability result for certain Ishikawa iterative sequence for fixed points of Lipschitz strictly pseudo-contractive mappings in real q-uniformly smooth Banach spaces.

Motivated and inspired by the above results, the purpose of this paper is to study, under certain conditions, the strong convergence theorems for the Mann iterative sequences, necessary and sufficient conditions for the convergence of the Ishikawa iterative sequences and stability of the Ishikawa iterative sequences for strictly hemi-contractive mappings in real Banach spaces with property $(U, \lambda, m + 1, m)$. The results in this paper extend, improve and unify the results due to Chidume [2] and Schu [13].

2. Lemmas

Lemma 2.1 ([13])**.** *Let $(X, \|\cdot\|)$ be a real Banach space with property $(U, \lambda, m + 1, m)$, $\lambda \in \mathbb{R}$, $m \in \mathbb{N}$. Then*

$$\|x + y\|^{m+1} \leq \|x\|^{m+1} + \frac{\lambda}{2^m - 1}\|y\|^{m+1} + (m + 1)\|x\|^{m-1}\langle y, j(x)\rangle,$$

for all $x, y \in X$ *and* $j(x) \in J(x)$.

Lemma 2.2 ([13]). *Let* $m \in \mathbb{N}$ *and* $k, t \in [0,1]$. *Then*

$$(1-t)^{m+1} + (m+1)t(1-k)(1-t)^m \leq (1-tk)^{m+1}.$$

Lemma 2.3 ([1]). *Let* $\{a_n\}$, $\{b_n\}$ *and* $\{c_n\}$ *be nonnegative sequences such that*

$$a_{n+1} \leq (1-b_n)a_n + b_n c_n, \qquad n \geq 0,$$

where $\sum_{n=0}^{\infty} b_n = \infty$, $b_n \in [0,1]$ *and* $\lim_{n\to\infty} c_n = 0$. *Then* $\lim_{n\to\infty} a_n = 0$.

3. Main Results

Theorem 3.1. *Let* X *be a real Banach space with property* $(U, \lambda, m+1, m)$, $\lambda \geq 0$, $m \in \mathbb{N}$. *Let* K *be a nonempty convex subset of* X *and let* $T : K \to K$ *be strictly hemi-contractive. Suppose that* $\{\alpha_n\}$ *is a sequence in* $[0,1)$ *satisfying*

$$\lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=0}^{\infty} \alpha_n = \infty. \tag{3.1}$$

Then the Mann iterative sequence $\{x_n\}$ *defined by*

$$\begin{cases} x_0 \in K, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n, \quad n \geq 0, \end{cases} \tag{3.2}$$

converges strongly to the unique fixed point of T *provided that* $\{Tx_n\}$ *is bounded.*

Proof. Since T is strictly hemi-contractive, it follows that $F(T)$ is a singleton. Let $F(T) = \{q\}$ for some $q \in K$. Put

$$D = \frac{\lambda}{2^m - 1} \sup\{\|Tx_n - q\|^{m+1} : n \geq 0\}.$$

Using Lemma 2.1, Lemma 2.2 and (3.2), we have

$$\begin{aligned} \|x_{n+1} - q\|^{m+1} &= (1-\alpha_n)^{m+1} \|(x_n - q) + \frac{\alpha_n}{1-\alpha_n}(Tx_n - q)\|^{m+1} \\ &\leq (1-\alpha_n)^{m+1}\|x_n - q\|^{m+1} + \frac{\lambda}{2^m - 1}\alpha_n^{m+1}\|Tx_n - q\|^{m+1} \\ &\quad + (m+1)\|x_n - q\|^{m-1}\alpha_n(1-\alpha_n)^m \langle Tx_n - q, j(x_n - q)\rangle \\ &\leq \big\{(1-\alpha_n)^{m+1} + (1-k)(m+1)\alpha_n(1-\alpha_n)^m\big\} \\ &\quad \times \|x_n - q\|^{m+1} + D\alpha_n^{m+1} \\ &\leq (1-k\alpha_n)^{m+1}\|x_n - q\|^{m+1} + D\alpha_n^{m+1} \\ &\leq (1-k\alpha_n)\|x_n - q\|^{m+1} + D\alpha_n^{m+1}, \qquad n \geq 0. \end{aligned} \tag{3.3}$$

Let

$$a_n = \|x_n - q\|^{m+1}, \quad b_n = k\alpha_n \quad \text{and} \quad c_n = k^{-1}D\alpha_n^m$$

for each $n \geq 0$. Then (3.3) can be rewritten as

$$a_{n+1} \leq (1-b_n)a_n + b_n c_n, \quad n \geq 0.$$

It follows from Lemma 2.3 and (3.1) that $\lim_{n\to\infty} a_n = 0$. That is, $\lim_{n\to\infty} x_n = q$. This completes the proof. □

Remark 3.1. Theorem 3.1 extends the Theorem of Chidume [2] and Theorem 1 of Schu [13] in the following sense:

(i) The continuous strongly pseudo-contractive operators in [2, 13] are replaced by the more general strictly hemi-contractive operators;
(ii) Condition (3.1) is weaker than $\sum_{n=0}^{\infty} \alpha_n = \infty$, $\sum_{n=0}^{\infty} \alpha_n^2 < \infty$ in [2, 13];
(iii) The closedness assumption of K in [2, 13] and the Lipschitzian continuity of T in [2] are removed.

Theorem 3.2. *Let X, T and K be as in Theorem 3.1. Suppose that $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$ satisfying*

$$\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \beta_n = 0; \tag{3.4}$$

and

$$\sum_{n=0}^{\infty} \alpha_n = \infty. \tag{3.5}$$

Assume that T is uniformly continuous. Then the following conditions are equivalent:

(i) *The Ishikawa iterative sequence $\{x_n\}$ defined by*

$$\begin{cases} x_0 \in K, \\ z_n = (1-\beta_n)x_n + \beta_n T x_n, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T z_n, \qquad n \geq 0, \end{cases} \tag{3.6}$$

converges strongly to the unique fixed point q of T;
(ii) $\lim_{n\to\infty} Tz_n = q$;
(iii) *$\{Tz_n\}$ is bounded.*

Proof. As in the proof of Theorem 3.1, we know that $F(T) = \{q\}$ for some $q \in K$. Suppose that (i) holds. Then the uniform continuity of T, (3.4) and (3.6) yield that

$$\lim_{n\to\infty} Tz_n = T \lim_{n\to\infty} ((1-\beta_n)x_n + \beta_n T x_n) = Tq = q,$$

which implies that $\{Tz_n\}$ is bounded.

Now we prove that (iii) $\Longrightarrow$ (i) holds. Let

$$M = 1 + \|x_0 - q\| + \sup\{\|Tz_n - q\| : n \geq 0\} \quad \text{and} \quad t_n = \|Tz_n - Tx_n\|$$

for $n \geq 0$. It is easy to verify that $\|x_n - q\| \leq M$ for each $n \geq 0$. Note that T is uniformly continuous. Hence $\{Tx_n\}$ is bounded. Thus there exists $D > M$ such that $\|Tx_n - q\| \leq D$ for each $n \geq 0$. It follows from (3.4) and (3.6) that

$$\|z_n - x_n\| = \beta_n \|x_n - Tx_n\| \leq 2D\beta_n \to 0,$$

as $n \to \infty$. The uniform continuity of T ensures that $\lim_{n\to\infty} t_n = 0$. (3.4) implies that there exists $n_0 \in \mathbb{N}$ such that $\alpha_n < 1$ for each $n \geq n_0$. Using Lemma 2.1, Lemma 2.2 and (3.6), we obtain that

$$\begin{aligned} \|x_{n+1} - q\|^{m+1} &= (1-\alpha_n)^{m+1} \|(x_n - q) + \frac{\alpha_n}{1-\alpha_n}(Tz_n - q)\|^{m+1} \\ &\leq (1-\alpha_n)^{m+1}\|x_n - q\|^{m+1} + \frac{\lambda}{2^m - 1}\alpha_n^{m+1}\|Tz_n - q\|^{m+1} \end{aligned} \tag{3.7}$$

$$
\begin{aligned}
&+ (m+1)(1-\alpha_n)^m \alpha_n \|x_n - q\|^{m-1} \langle Tz_n - q, j(x_n - q)\rangle \\
&\le (1-\alpha_n)^{m+1}\|x_n - q\|^{m+1} + \frac{\lambda}{2^m - 1} M^{m+1}\alpha_n^{m+1} \\
&\quad + (m+1)(1-\alpha_n)^m \alpha_n \|x_n - q\|^{m-1} \\
&\quad \times \left\{ t_n \|x_n - q\| + (1-k)\|x_n - q\|^2 \right\} \\
&\le \left\{ (1-\alpha_n)^{m+1} + (1-k)(m+1)(1-\alpha_n)^m \alpha_n \right\} \\
&\quad \times \|x_n - q\|^{m+1} + \frac{\lambda}{2^m - 1} M^{m+1}\alpha_n^{m+1} \\
&\quad + (m+1)(1-\alpha_n)^m \alpha_n t_n \|x_n - q\|^m \\
&\le (1-k\alpha_n)^{m+1}\|x_n - q\|^{m+1} + \frac{\lambda}{2^m - 1} M^{m+1}\alpha_n^{m+1} \\
&\quad + (m+1)M^m \alpha_n t_n \\
&\le (1-k\alpha_n)\|x_n - q\|^{m+1} + \alpha_n s_n
\end{aligned}
$$

for all $n \ge n_0$, where $s_n = \frac{\lambda}{2^m - 1} M^{m+1}\alpha_n^m + (m+1)M^m t_n \to 0$ as $n \to 0$. Thus (3.7) and Lemma 2.3 guarantee that $\lim_{n\to\infty} \|x_n - q\| = 0$. This completes the proof. □

Remark 3.2. Theorem 3.2 extends the Theorem of Chidume [2] and Theorem 2 of Schu [13] in the following directions.

- (i) The Mann iterative sequence in [2] is replaced by the more general Ishikawa iterative sequence;
- (ii) The Lipschitzian continuity in [2, 13] is replaced by the uniform continuity;
- (iii) Conditions (3.4) and (3.5) are weaker than $\sum_{n=0}^{\infty} \alpha_n = \infty$, $\sum_{n=0}^{\infty} \alpha_n^2 < \infty$ in [2].

Theorem 3.3. *Let X, T and K be as in Theorem 3.1. Suppose that T is a Lipschitzian mapping with Lipschitz constant $L \ge 1$. Assume that $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$ and r is a constant in $(0,1]$ satisfying (3.1) and*

$$
\beta_n < \min\left\{ \frac{rk}{L(1+L)}, \quad 1 \right\}, \quad n \ge n_1, \tag{3.8}
$$

where n_1 is some positive integer. Then the Ishikawa iterative sequence $\{x_n\}$ defined by (3.6) converges strongly to the unique fixed point q of T.

Proof. Since T is strictly hemi-contractive, $F(T) = \{q\}$ for some $q \in K$. It follows from (3.6), (3.8), Lemma 2.1 and Lemma 2.2 that

$$
\begin{aligned}
\|z_n - q\|^{m+1} &= (1-\beta_n)^{m+1} \|(x_n - q) + \frac{\beta_n}{1-\beta_n}(Tx_n - q)\|^{m+1} \\
&\le (1-\beta_n)^{m+1}\|x_n - q\|^{m+1} + \frac{\lambda}{2^m - 1}\beta_n^{m+1} \\
&\quad \times \|Tx_n - q\|^{m+1} + (m+1)(1-\beta_n)^m \beta_n \\
&\quad \times \|x_n - q\|^{m-1}\langle Tx_n - q, j(x_n - q)\rangle \\
&\le (1-\beta_n)^{m+1}\|x_n - q\|^{m+1}
\end{aligned} \tag{3.9}
$$

$$
\begin{aligned}
&+ \frac{\lambda}{2^m-1}(L\beta_n)^{m+1}\|x_n - q\|^{m+1} \\
&+ (m+1)(1-\beta_n)^m\beta_n(1-k)\|x_n - q\|^{m+1} \\
\le& \left\{(1-k\beta_n)^{m+1} + \frac{\lambda}{2^m-1}(L\beta_n)^{m+1}\right\}\|x_n - q\|^{m+1} \\
\le& \left(1 + \frac{\lambda}{2^m-1}(L\beta_n)^{m+1}\right)\|x_n - q\|^{m+1}
\end{aligned}
$$

for any $n \ge n_1$. (3.1) means that there exists $n_0 \ge n_1$ such that

$$
\alpha_n < \min\left\{1, \quad \left\{\frac{(1-r)k}{2}\left(\frac{\lambda}{2^m-1}L^{m+1} + (\frac{\lambda}{2^m-1}L^{m+1})^2\right)^{-1}\right\}^{\frac{1}{m}}\right\} \tag{3.10}
$$

for all $n \ge n_0$. In view of (3.6), (3.8)∼(3.10), Lemma 2.1 and Lemma 2.2, we get that

$$
\begin{aligned}
&\|x_{n+1} - q\|^{m+1} \\
&= (1-\alpha_n)^{m+1}\|(x_n - q) + \frac{\alpha_n}{1-\alpha_n}(Tz_n - q)\|^{m+1} \\
&\le (1-\alpha_n)^{m+1}\|x_n - q\|^{m+1} + \frac{\lambda}{2^m-1}\alpha_n^{m+1}\|Tz_n - q\|^{m+1} \\
&\quad + (m+1)(1-\alpha_n)^m\alpha_n\|x_n - q\|^{m-1}\langle Tz_n - q, j(x_n - q)\rangle \\
&\le (1-\alpha_n)^{m+1}\|x_n - q\|^{m+1} + \frac{\lambda}{2^m-1}(L\alpha_n)^{m+1}\|z_n - q\|^{m+1} \\
&\quad + (m+1)(1-\alpha_n)^m\alpha_n\|x_n - q\|^{m-1} \\
&\quad \times \{\langle Tz_n - Tx_n, j(x_n - q)\rangle + \langle Tx_n - q, j(x_n - q)\rangle\} \\
&\le \left[(1-\alpha_n)^{m+1} + \frac{\lambda}{2^m-1}(L\alpha_n)^{m+1}\left\{1 + \frac{\lambda}{2^m-1}(L\beta_n)^{m+1}\right\}\right] \\
&\quad \times \|x_n - q\|^{m+1} + (m+1)(1-\alpha_n)^m\alpha_n\|x_n - q\|^{m-1} \\
&\quad \times \left\{L\|z_n - x_n\|\|x_n - q\| + (1-k)\|x_n - q\|^2\right\} \\
&\le \left[(1-\alpha_n)^{m+1} + \frac{\lambda}{2^m-1}(L\alpha_n)^{m+1}\left\{1 + \frac{\lambda}{2^m-1}(L\beta_n)^{m+1}\right\}\right] \\
&\quad \times \|x_n - q\|^{m+1} + (m+1)(1-\alpha_n)^m\alpha_n \\
&\quad \times \{1 - k + \beta_n L(1+L)\}\|x_n - q\|^{m+1} \\
&\le \left[(1-\alpha_n)^{m+1} + \frac{\lambda}{2^m-1}(L\alpha_n)^{m+1}\left\{1 + \frac{\lambda}{2^m-1}(L\beta_n)^{m+1}\right\}\right] \\
&\quad \times \|x_n - q\|^{m+1} + (m+1)(1-\alpha_n)^m\alpha_n\{1 - (1-r)k\}\|x_n - q\|^{m+1} \\
&\le \left[\{1 - (1-r)k\alpha_n\}^{m+1} + \frac{\lambda}{2^m-1}(L\alpha_n)^{m+1}\right. \\
&\quad \left. + \left(\frac{\lambda}{2^m-1}L^{m+1}\right)^2(\alpha_n\beta_n)^{m+1}\right]\|x_n - q\|^{m+1}
\end{aligned} \tag{3.11}
$$

$$\leq \left\{1-(1-r)k\alpha_n+\frac{1}{2}(1-r)k\alpha_n\right\}\|x_n-q\|^{m+1}$$
$$=\left\{1-\frac{1}{2}(1-r)k\alpha_n\right\}\|x_n-q\|^{m+1}$$

for any $n \geq n_0$. Using Lemma 2.3, (3.1) and (3.11), we conclude that $\lim_{n\to\infty}\|x_n-q\|=0$. This completes the proof. □

Remark 3.3. Theorem 3.3 generalizes the Theorem of Chidume [2] and Theorem 2 of Schu [13] in the following directions:

(i) The strictly pseudo-contractive mapping in [2, 13] is replaced by the strictly hemi-contractive mapping;
(ii) The boundedness assumptions of $\{Tx_n\}$ and $\{Tz_n\}$ in [13] and the closedness condition of K in [2, 13] are removed.

Theorem 3.4. *Let X, T, K and L be as in Theorem 3.3. Assume that $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$ and r, t, s are constants satisfying*

$$\frac{\lambda L^{m+1}}{2^m-1}\alpha_n^m+\left(\frac{\lambda L^{m+1}}{2^m-1}\right)^2\alpha_n^m\beta_n^{m+1}\leq(1-r)k-t, \qquad n\geq 0; \tag{3.12}$$

$$1>\alpha_n\geq s>0, \qquad n\geq 0, \tag{3.13}$$

$$\beta_n<\min\left\{1, \quad \frac{rk}{L(1+L)}\right\}, \qquad n\geq 0, \tag{3.14}$$

where $t\in(0,(1-r)k)$. For arbitrary $x_0\in K$, the Ishikawa iterative sequence $\{x_n\}$ is defined as in (3.6). *Let $\{y_n\}$ be an arbitrary sequence in K and define $\{\varepsilon_n\}\subset[0,\infty)$ as follows*:

$$\begin{cases} w_n=(1-\beta_n)y_n+\beta_nTy_n, \\ \varepsilon_n=\|y_{n+1}-p_n\|, \qquad n\geq 0, \end{cases} \tag{3.15}$$

where $p_n=(1-\alpha_n)y_n+\alpha_nTw_n$. Then

(i) *the sequence $\{x_n\}$ converges strongly to the unique fixed point q of T. Moreover,*
$$\|x_{n+1}-q\|\leq(1-ts)^{\frac{n+1}{m+1}}\|x_0-q\|, \quad n\geq 0.$$
(ii) $\|y_{n+1}-q\|\leq(1-ts)^{\frac{1}{m+1}}\|y_n-q\|+\varepsilon_n, \quad n\geq 0.$
(iii) $\lim_{n\to\infty}y_n=q$ *if and only if* $\lim_{n\to\infty}\varepsilon_n=0$.

Proof. As in the proof of Theorem 3.1, we conclude that $F(T)=\{q\}$ for some $q\in K$ and that

$$\|x_{n+1}-q\|^{m+1}\leq\Bigg[\{1-(1-r)k\alpha_n\}^{m+1}+\frac{\lambda}{2^m-1}(L\alpha_n)^{m+1} + \left(\frac{\lambda}{2^m-1}L^{m+1}\right)^2(\alpha_n\beta_n)^{m+1}\Bigg]\|x_n-q\|^{m+1} \tag{3.16}$$

for any $n \geq 0$. It follows from (3.12), (3.13) and (3.16) that

$$\begin{aligned}\|x_{n+1}-q\|^{m+1} &\leq \Big\{1-(1-r)k\alpha_n+\frac{\lambda}{2^m-1}(L\alpha_n)^{m+1} \\ &\quad +\left(\frac{\lambda}{2^m-1}L^{m+1}\right)^2(\alpha_n\beta_n)^{m+1}\Big\}\|x_n-q\|^{m+1} \\ &= \Big[1-\alpha_n\Big\{(1-r)k-\frac{\lambda}{2^m-1}L^{m+1}\alpha_n^m \\ &\quad -\left(\frac{\lambda}{2^m-1}L^{m+1}\right)^2\alpha_n^m\beta_n^{m+1}\Big\}\Big]\|x_n-q\|^{m+1} \\ &\leq (1-t\alpha_n)\|x_n-q\|^{m+1} \\ &\leq (1-ts)\|x_n-q\|^{m+1},\end{aligned}$$

which implies that

$$\|x_{n+1}-q\| \leq (1-ts)^{\frac{1}{m+1}}\|x_n-q\| \leq (1-ts)^{\frac{n+1}{m+1}}\|x_0-q\|, \quad n \geq 0.$$

In view of (3.14), (3.15), Lemma 2.1 and Lemma 2.2, we have

$$\begin{aligned}\|w_n-q\|^{m+1} &\leq (1-\beta_n)^{m+1}\|y_n-q\|^{m+1}+\frac{\lambda}{2^m-1}\beta_n^{m+1}\|Ty_n-q\|^{m+1} \\ &\quad +(m+1)(1-\beta_n)^m\beta_n\|y_n-q\|^{m-1}\langle Ty_n-q, j(y_n-q)\rangle \\ &\leq (1-\beta_n)^{m+1}\|y_n-q\|^{m+1}+\frac{\lambda}{2^m-1}(L\beta_n)^{m+1}\|y_n-q\|^{m+1} \\ &\quad +(m+1)(1-\beta_n)^m\beta_n(1-k)\|y_n-q\|^{m+1} \\ &\leq \left\{1+\frac{\lambda}{2^m-1}(L\beta_n)^{m+1}\right\}\|y_n-q\|^{m+1}\end{aligned} \tag{3.17}$$

for all $n \geq 0$. Using (3.12)$\sim$(3.14), (3.17), Lemma 2.1 and Lemma 2.2, we obtain that

(3.18)

$$\begin{aligned}&\|p_n-q\|^{m+1} \\ &\leq (1-\alpha_n)^{m+1}\|y_n-q\|^{m+1}+\frac{\lambda}{2^m-1}\alpha_n^{m+1}\|Tw_n-q\|^{m+1} \\ &\quad +(m+1)(1-\alpha_n)^m\alpha_n\|y_n-q\|^{m-1}\langle Tw_n-q, j(y_n-q)\rangle \\ &\leq (1-\alpha_n)^{m+1}\|y_n-q\|^{m+1}+\frac{\lambda}{2^m-1}(\alpha_nL)^{m+1}\|w_n-q\|^{m+1} \\ &\quad +(m+1)(1-\alpha_n)^m\alpha_n\|y_n-q\|^{m-1} \\ &\quad \times\left\{L\|w_n-y_n\|\|y_n-q\|+(1-k)\|y_n-q\|^2\right\} \\ &\leq \left\{(1-\alpha_n)^{m+1}+\frac{\lambda}{2^m-1}(\alpha_nL)^{m+1}+\left(\frac{\lambda}{2^m-1}L^{m+1}\right)^2(\alpha_n\beta_n)^{m+1}\right\} \\ &\quad \times\|y_n-q\|^{m+1}+(m+1)(1-\alpha_n)^m\alpha_n\|y_n-q\|^{m-1} \\ &\quad \times\{L\beta_n\|y_n-Ty_n\|\|y_n-q\|+(1-k)\|y_n-q\|^2\}\end{aligned}$$

$$\le \left\{(1-\alpha_n)^{m+1} + \frac{\lambda}{2^m-1}(\alpha_n L)^{m+1} + \left(\frac{\lambda}{2^m-1}L^{m+1}\right)^2 (\alpha_n\beta_n)^{m+1}\right\}$$
$$\times \|y_n - q\|^{m+1} + (m+1)(1-\alpha_n)^m \alpha_n\{L(1+L)\beta_n + 1 - k\}\|y_n - q\|^{m+1}$$
$$\le \left[\{1-(1-r)k\alpha_n\}^{m+1} + \frac{\lambda}{2^m-1}(\alpha_n L)^{m+1} + \left(\frac{\lambda}{2^m-1}L^{m+1}\right)^2 (\alpha_n\beta_n)^{m+1}\right]$$
$$\times \|y_n - q\|^{m+1}$$
$$\le \left[1 - \alpha_n\left\{(1-r)k - \frac{\lambda}{2^m-1}\alpha_n^m L^{m+1} - \left(\frac{\lambda}{2^m-1}L^{m+1}\right)^2 \alpha_n^m \beta_n^{m+1}\right\}\right]$$
$$\times \|y_n - q\|^{m+1}$$
$$\le (1 - t\alpha_n)\|y_n - q\|^{m+1}$$
$$\le (1 - ts)\|y_n - q\|^{m+1}$$

for all $n \ge 0$. Combining (3.15) and (3.18), we infer that

$$\|y_{n+1} - q\| \le \|p_n - q\| + \|y_{n+1} - p_n\| \le (1-ts)^{\frac{1}{m+1}}\|y_n - q\| + \varepsilon_n \tag{3.19}$$

for all $n \ge 0$.

Suppose that $\lim_{n\to\infty} y_n = q$. Then (3.15) and (3.18) yield that

$$\varepsilon_n \le \|y_{n+1} - q\| + \|p_n - q\| \le \|y_{n+1} - q\| + (1-ts)^{\frac{1}{m+1}}\|y_n - q\| \to 0,$$

as $n \to \infty$. That is, $\lim_{n\to\infty} \varepsilon_n = 0$.

Conversely, suppose that $\lim_{n\to\infty} \varepsilon_n = 0$. Set $a_n = \|y_n - q\|$, $b_n = 1-(1-ts)^{\frac{1}{m+1}}$ and $c_n = [1-(1-ts)^{\frac{1}{m+1}}]^{-1}\varepsilon_n$ for each $n \ge 0$. It follows from Lemma 2.3 that $\lim_{n\to\infty} a_n = 0$. Hence $\lim_{n\to\infty} y_n = q$. □

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