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DUALITY IN DC-CONSTRAINED PROGRAMMING VIA DUALITY IN REVERSE CONVEX PROGRAMMING

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Abstract. This paper establishes a duality theory associated with primal problem

$$\inf\{g_1(x) - g_2(x) : \quad h_1(x) - h_2(x) \notin Y_+ \}$$

where $g_1, g_2 : X \longrightarrow I\!R \cup \{-\infty, +\infty\}$ are two convex functions on the Hausdorff locally convex real vector space X and $h_1, h_2 : X \longrightarrow Y \cup \{+\infty\}$ are two convex vector valued mappings taking their values in a real partially ordered topological vector space Y. The partial order is induced by a convex cone $Y_+ \subset Y$. The approach for dealing with this duality is based on the use of an important formula of nonconvex duality, due to B. Lemaire [2], associated with a reverse convex programming problem.

1. Introduction

A wide class of constrained DC-programming problems has recently received particular attention. It enables to study, in a general and unified way, a large number of problems arising in economics, optimization and operations research. That is minimizing a difference of two extended real-valued convex functions subject to a DC-constraint i.e it concerns the primal problem

$$(\mathcal{P}) \qquad \inf\{g_1(x) - g_2(x) : \quad h_1(x) - h_2(x) < 0 \}$$

where g_1, g_2, h_1 and h_2 are extended real-valued convex functions on the Hausdorff locally convex real vector space X.

In a recent work [3], B. Lemaire and M. Volle presented a duality formula associated with problem $(\mathcal{P})$ by introducing a suitable dual problem defined from the Legendre-Fenchel conjugates of the data functions g_1, g_2, h_1 and h_2. The approach adopted by B. Lemaire and M. Volle for proving their formula is based on the use of convex analysis theory and essentially the " inf sup" Theorem of J. J. Moreau [5]. Let us note that this large class contains an important subclass of programming problems namely reverse convex optimization problems by taking $g_2 \equiv 0$ and $h_1 \equiv 0$ and B. Lemaire, in his recent paper [2], has established a duality theory associated with this subclass.

In the present work, we address a main question, that is : how to obtain the duality theory associated with constrained DC-programming problem $(\mathcal{P})$ via duality in reverse convex programming? Indeed, the answer of this question gives us a new approach totaly different from the technique used by B. Lemaire and M. Volle for obtaining the corresponding dual problem linked to $(\mathcal{P})$ and allows to transform the problem $(\mathcal{P})$ into a reverse convex programming problem on some appropriate

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space. Our approach, also enables us to establish a similar duality result associated with a broad class of constrained DC-programming problems including problem $(\mathcal{P})$. That is minimizing the difference of two convex functions subject to a vector DC-constraint i.e.

$$(\mathcal{Q}) \qquad \alpha := \inf\{g_1(x) - g_2(x) : \quad h_1(x) - h_2(x) \notin Y_+ \}$$

where $g_1, g_2 : X \longrightarrow I\!R \cup \{-\infty, +\infty\}$ are two extended real-valued convex functions and $h_1, h_2 : X \longrightarrow Y \cup \{ +\infty\}$ are two convex vector valued mappings taking values in a real partially ordered topological vector space Y equipped with a preorder induced by a convex cone Y_+. Obviously, in the case when $Y = I\!R$ and $Y_+ = I\!R_+$ we found the problem $(\mathcal{P})$ as a particular case.
The approach that we will adopt for stating our main result is based on an equivalent transformation of the problem $(\mathcal{Q})$ into a minimization problem given by

$$\alpha = \inf_{(x,y,x^*)\in X\times Y\times X^*} \{G(x,y) + g_2^*(x^*) : \quad H(x,y) > 0\}$$

in which appears the following reverse convex programming problem

$$\theta := \inf_{(x,y)\in X\times Y} \{G(x,y) : \quad H(x,y) > 0\}$$

where G and H are two auxiliary convex functions defined on the product space $X \times Y$ expressed both by means of the data functions g_1, g_2, h_1 and h_2. This allow to derive our desired result by applying directly Lemaire's duality formula [2]. As consequence, we recapture particularly in the case when $Y = I\!R$ and $Y_+ = I\!R_+$, the duality result due to B. Lemaire and M. Volle [3].

2. Notations and definitions

Throughout this paper X and Y denote two topological real vector spaces with respectives topological duals X^* and Y^*. For the sake of simplicity we use the same symbol $\langle \ , \ \rangle$ to denote the bilinear pairing between X and X^* (resp. Y and Y^*). Naturally, we obtain a duality between $X^* \times Y^*$ and $X \times Y$ given by

$$\langle (x^*, y^*), (x, y)\rangle := \langle x^*, x\rangle + \langle y^*, y\rangle$$

for every $(x, y) \in X \times Y$ and $(x^*, y^*) \in X^* \times Y^*$. By Y_+ we denote a convex cone in Y which makes Y a partially ordered topological vector space given by

$$y \leq_Y z \iff z - y \in Y_+.$$

By Y_+^* we denote the dual positive cone

$$Y_+^* := \{y^* \in Y^* : \langle y^*, y\rangle \geq 0, \quad \forall y \in Y_+\}.$$

We adjoint to Y a greatest abstract element denoted by $+\infty$ i.e. for any $y \in Y$

$$y \leq_Y +\infty.$$

In what follows, we use the following extensions of the addition and the product in $I\!R \cup \{-\infty, +\infty\}$:

$$(+\infty) + (-\infty) = (-\infty) + (+\infty) = +\infty, \quad 0 \times (-\infty) = 0, \quad 0 \times (+\infty) = +\infty.$$

A mapping $h : X \longrightarrow Y \cup \{ +\infty\}$ is said to be Y_+-convex if

$$h(\lambda x_1 + (1 - \lambda)x_2) \leq_Y \lambda h(x_1) + (1 - \lambda)h(x_2)$$

for each $x_1, x_2 \in X$ and each $\lambda \in [0,1]$. By

$$dom\ h := \{x \in X:\ h(x) \in Y\},$$

and

$$Epi\ h := \{(x,y) \in X \times Y:\ h(x) \leq_Y y\}$$

we denote respectively its effective domain and its epigrah. To each extended real-valued function $f : X \longrightarrow I\!R \cup \{-\infty, +\infty\}$ corresponds its Legendre-Fenchel transform f^* defined on the topological dual X^* by

$$f^*(x^*) := \sup_{x \in X}\{\langle x^*, x\rangle - f(x)\}$$

for any $x^* \in X^*$. If C is a nonempty subset of X, then its indicator function $\delta_C : X \longrightarrow I\!R \cup \{+\infty\}$ is defined for every $x \in X$ by

$$\delta_C(x) := \begin{cases} 0 & \text{if } x \in C \\ +\infty, & \text{otherwise} \end{cases}$$

Given a function $g : Y \longrightarrow I\!R \cup \{-\infty,\ +\infty\}$ and a mapping $h : X \longrightarrow Y \cup \{\ +\infty\}$, by $g \circ h$ we set the composition function as

$$(g \circ h)(x) := \begin{cases} g(h(x)) & \text{if } x \in h^{-1}(Y) \\ \sup_{y \in Y} g(y), & \text{otherwise.} \end{cases} \tag{2.1}$$

In the sequel we denote by $\Gamma(X)$ the set of lower semicontinuous proper convex real functions plus the constants $+\infty$ and $-\infty$.

3. Primal problem and preliminary result

Let $g_1, g_2 : X \longrightarrow I\!R \cup \{-\infty, +\infty\}$ be two extended real-valued convex functions and $h_1, h_2 : X \longrightarrow Y \cup \{\ +\infty\}$ be two Y_+-convex mappings. We are interested in finding the infimal value

$$(\mathcal{P}) \quad \alpha := \inf\{g_1(x) - g_2(x):\ h_1(x) - h_2(x) \notin Y_+\}$$

In the case when $Y = I\!R,\quad Y_+ = I\!R_+,\quad g_2 \equiv 0$ and $h_1 \equiv 0$, we obtain a reverse convex programming problem and B. Lemaire in his recent paper [2] established for this scalar reverse convex minimization problem an important formula of duality given by

Theorem 3.1. *Let X be a Hausdorff locally convex vector space and $g, h : X \longrightarrow I\!R \cup \{-\infty, +\infty\}$ two convex functions with $h \in \Gamma(X)$. Then*

$$\inf_{h(x)>0} g(x) = \inf_{x^* \in X^*} \max_{\lambda \geq 0}\{\lambda h^*(x^*) - g^*(\lambda x^*):\ h^*(x^*) - \delta^*_{domg}(x^*) < 0\}$$

4. The main result

In order to state a similar duality formula related to problem $(\mathcal{Q})$, we start with the following lemma

Lemma 4.1. *If we set for any* $y \in Y$

$$E_y := \{x \in X : \ \delta_{-Y_+}(h_2(x) - y) > 0 \ \text{and} \ \ h_1(x) - y \in -Y_+ \ \}$$

and we suppose that dom $h_1 = X$ *then we have*

$$\{x \in X : \ h_1(x) - h_2(x) \notin Y_+ \ \} = \bigsqcup_{y \in Y} E_y.$$

Proof. Let $x \in X$ such that $h_1(x) - h_2(x) \notin Y_+$. By putting $y = h_1(x)$ we obtain $x \in E_y$. Conversely, let $x \in \bigsqcup_{y \in Y} E_y$, there exists some $y \in Y$ satisfying $h_2(x) - y \notin -Y_+$ and $h_1(x) - y \in -Y_+$. If we suppose $h_1(x) - h_2(x) \in Y_+$ we get

$$h_2(x) - y = h_2(x) - h_1(x) + h_1(x) - y \in -Y_+ - Y_+ = -Y_+$$

which contradicts the fact that $h_2(x) - y \notin -Y_+$. □

By introducing the following minimization problem

$$\gamma := \inf_{(x,y) \in X \times Y} \{g_1(x) + \delta_{-Y_+}(h_1(x) - y) - g_2(x) : \ \delta_{-Y_+}(h_2(x) - y) > 0 \ \}$$

which will play a crucial role for stating our main result, we get

Proposition 4.1. *If we suppose that dom* $h_1 = X$ *then we have* $\alpha = \gamma$.

Proof. According to Lemma 4.1, it follows that

$$\begin{aligned} \alpha &= \inf\{g_1(x) - g_2(x) : \ x \in \bigsqcup_{y \in Y} E_y \ \} \\ &= \inf_{(x,y) \in X \times Y} \{g_1(x) - g_2(x) : \ x \in E_y \ \} \\ &= \inf_{(x,y) \in X \times Y} \{g_1(x) - g_2(x) + \delta_{-Y_+}(h_1(x) - y) : \ \delta_{-Y_+}(h_2(x) - y) > 0 \ \} = \gamma. \end{aligned}$$ □

Let us consider the following auxiliary functions $G, \ H : X \times Y \longrightarrow I\!R \cup \{-\infty, +\infty\}$ given by

$$\begin{aligned} H(x, y) &:= \delta_{-Y_+}(h_2(x) - y) \\ G(x, y) &:= g_1(x) + \delta_{-Y_+}(h_1(x) - y) - \langle x^*, x \rangle \end{aligned}$$

where $x^* \in X^*$ is arbitrary.

In the proof of the main theorem we will need the following lemma

Lemma 4.2. *For any* $(p^*, y^*) \in X^* \times Y^*$, *we have*

1) $G^*(p^*, y^*) = (g_1 - y^* \circ h_1)^*(p^* + x^*) + \delta_{Y_+^*}(-y^*)$.
2) $H^*(p^*, y^*) = (-y^* \circ h_2)^*(p^*) + \delta_{Y_+^*}(-y^*)$.
3) $\delta^*_{domG}(p^*, y^*) = (\delta_{domg_1} - y^* \circ h_1)^*(p^*) + \delta_{Y_+^*}(-y^*)$.
4) $Epi \ H = Epi \ h_2 \times [0, +\infty[$.

Proof. For any $(p^*, y^*) \in X^* \times Y^*$, we have

1)

$$G^*(p^*, y^*) = \sup_{(x,y)\in X\times Y} \{\langle p^*, x\rangle + \langle y^*, y\rangle - g_1(x) - \delta_{-Y_+}(h_1(x) - y) + \langle x^*, x\rangle\}.$$

If there exists some $x \in X$ such that $g_1(x) = -\infty$, then according to the previous conventions, we have obviously

$$G^*(p^*, y^*) = (g_1 - y^* \circ h_1)^*(p^* + x^*) + \delta_{Y_+^*}(-y^*) = +\infty.$$

If we assume now that g_1 does not take the value $-\infty$, then we get

$$G^*(p^*, y^*) = \sup_{\substack{(x,y)\in X\times Y \\ h_1(x)-y\in -Y_+}} \{\langle p^*, x\rangle + \langle y^*, y\rangle - g_1(x) + \langle x^*, x\rangle\}.$$

By setting $z := h_1(x) - y$, we obtain

$$\begin{aligned} G^*(p^*, y^*) &= \sup_{\substack{z\in -Y_+ \\ x\in h_1^{-1}(Y)}} \{\langle p^* + x^*, x\rangle - g_1(x) + (y^* \circ h_1)(x) - \langle y^*, z\rangle\} \\ &= \sup_{x\in h_1^{-1}(Y)} \{\langle p^* + x^*, x\rangle - g_1(x) + (y^* \circ h_1)(x)\} + \delta^*_{-Y_+}(-y^*) \\ &= \sup_{x\in h_1^{-1}(Y)} \{\langle p^* + x^*, x\rangle - (g_1 - y^* \circ h_1)(x)\} + \delta_{Y_+^*}(-y^*). \end{aligned}$$

Now, we claim that

$$G^*(p^*, y^*) = (g_1 - y^* \circ h_1)^*(p^* + x^*) + \delta_{Y_+^*}(-y^*). \tag{4.1}$$

Indeed, if $-y^* \notin Y_+^*$ then $\delta_{Y_+^*}(-y^*) = +\infty$ and hence the equality (4.1) holds.
If $y^* \equiv 0$ then by virtue of convention (2.1) we have $y^* \circ h_1 \equiv 0$ and thus we get

$$\begin{aligned} G^*(p^*, 0) &= \sup_{x\in X}\{\langle p^* + x^*, x\rangle - g_1(x)\} \\ &= g_1^*(p^* + x^*). \end{aligned}$$

If $-y^* \in Y_+^*\backslash\{0\}$ then there exists some $y_0 \in Y_+$ such that $\langle -y^*, y_0\rangle > 0$ and since

$$\lambda\langle -y^*, y_0\rangle \le \sup_{y\in Y}\langle -y^*, y\rangle, \quad \forall\lambda > 0$$

it follows from convention (2.1), by letting $\lambda \longrightarrow +\infty$, that

$$(-y^* \circ h_1)(x) = +\infty, \qquad \text{if } h_1(x) = +\infty$$

which yields

$$(g_1 - y^* \circ h_1)(x) = +\infty, \qquad \text{if } h_1(x) = +\infty$$

and therefore we get the required result (4.1).

2) By taking in the above result 1) $g_1 \equiv 0$, $x^* \equiv 0$ and replacing h_1 by h_2 we obtain

$$H^*(p^*, y^*) = (-y^* \circ h_2)^*(p^*) + \delta_{Y_+^*}(-y^*).$$

3) At first, it is easy to see that

$$\mathit{dom}\ G = (\mathit{dom}\ g_1 \times Y) \cap \mathit{Epi}\ h_1$$

and therefore for any $(p^*, y^*) \in X^* \times Y^*$ we have

$$\begin{aligned}\delta^*_{domG}(p^*, y^*) &= \sup_{(x,y)\in X\times Y} \{\langle p^*, x\rangle + \langle y^*, y\rangle - \delta_{(domg_1\times Y)\cap Epih_1}(x, y)\ \} \\ &= \sup_{(x,y)\in X\times Y} \{\langle p^*, x\rangle + \langle y^*, y\rangle - \delta_{domg_1\times Y}(x, y) - \delta_{Epih_1}(x, y)\ \} \\ &= \sup_{(x,y)\in X\times Y} \{\langle p^*, x\rangle + \langle y^*, y\rangle - \delta_{domg_1}(x) - \delta_{-Y_+}(h_1(x) - y)\ \}.\end{aligned}$$

Now, we are in a position to apply 1) by replacing g_1 by δ_{domg_1} and taking $x^* \equiv 0$ and hence we obtain the result

$$\delta^*_{domG}(p^*, y^*) = (\delta_{domg_1} - y^* \circ h_1)^*(p^*) + \delta_{Y_+^*}(-y^*).$$

4) It is obvious. □

Remark 4.1. If we authorise the mapping h_2 taking at some points the value $-\infty$, then according to definitions of $Epi\ h_2$ and the mapping H we obtain

$$Epi\ H = (Epi\ h_2 \backslash (h_2^{-1}(-\infty) \times Y)) \times [0, +\infty[$$

which is not closed in general although $Epi\ h_2$ is closed.

Now, we are in position to state and prove the main result of this paper.

Theorem 4.1. *Let X and Y be two Hausdorff locally convex vector spaces, $g_1, g_2 : X \longrightarrow I\!R \cup \{-\infty, +\infty\}$ are two convex functions and $h_1 : X \longrightarrow Y$, $h_2 : X \longrightarrow Y \cup \{+\infty\}$ are two Y_+-convex mappings. We assume that $g_2 \in \Gamma(X)$ and h_2 has a closed epigraph. Then we have*

$$\alpha = \inf_{\substack{(x^*,p^*)\in X^*\times X^* \\ y^*\in Y_+^*\backslash\{0\}}} \max_{\lambda\geq 0}\{g_2^*(x^*) + \lambda(y^* \circ h_2)^*(p^*) - (g_1 + \lambda y^* \circ h_1)^*(x^* + \lambda p^*) : \\ (y^* \circ h_2)^*(p^*) - (\delta_{domg_1} + y^* \circ h_1)^*(p^*) < 0\}.$$

Proof. Since $g_2 \in \Gamma(X)$, we have for any $x \in X$

$$g_2(x) = g_2^{**}(x) = \sup_{x^*\in X^*} \{\langle x^*, x\rangle - g_2^*(x^*)\ \}$$

and hence it follows, according to Proposition 4.1, that

$$\alpha = \inf_{(x,y,x^*)\in X\times Y\times X^*} \{G(x, y) + g_2^*(x^*) :\ H(x, y) > 0\ \}. \tag{4.2}$$

Let us observe that in the above minimization problem appears the following reverse convex programming problem

$$\theta := \inf_{(x,y)\in X\times Y} \{G(x, y) :\ H(x, y) > 0\ \}.$$

defined on the product space $X \times Y$ and as $Epi\ H$ is closed (using Lemma 4.2 and the fact that $Epi\ h_2$ is closed), it follows from Lemaire's duality Theorem 3.1 that

$$\theta = \inf_{(p^*,y^*)\in X^*\times Y^*} \max_{\lambda\geq 0}\{\lambda H^*(p^*, y^*) - G^*(\lambda p^*, \lambda y^*) :\ H^*(p^*, y^*) - \delta^*_{domG}(p^*, y^*) < 0\}.$$

If we combine the above equality with Lemma 4.2 and by noticing that $\lambda\delta_{Y_+^*} = \delta_{Y_+^*}$ and $-y^* \in Y_+^*$ then we obtain

$$\alpha = \inf_{\substack{(x^*,p^*)\in X^*\times X^* \\ y^*\in Y_+^*}} \max_{\lambda\geq 0}\{g_2^*(x^*) + \lambda(y^*\circ h_2)^*(p^*) - (g_1+\lambda y^*\circ h_1)^*(x^*+\lambda p^*) :$$
$$(y^*\circ h_2)^*(p^*) - (y^*\circ h_1 + \delta_{domg_1})^*(p^*) < 0\}.$$

Now, it remains to show that the infimum over $y^* \in Y_+^*$ is, in fact, taken over $Y_+^*\backslash\{0\}$ i.e. the following strict inequality

$$(y^*\circ h_2)^*(p^*) - (\delta_{domg_1} + y^*\circ h_1)^*(p^*) < 0, \tag{4.3}$$

does not hold for $y^* = 0$. Suppose the contrary and by using the convention (2.1), the strict inequality (4.3) becomes

$$\delta_{\{0\}}(p^*) - \delta^*_{domg_1}(p^*) < 0$$

i.e

$$\delta^*_{domg_1}(0) = \sup_{x\in X}\{-\delta_{domg_1}(x)\} > 0.$$

This contradicts the fact that $\delta^*_{domg_1}(0) \leq 0$ since $\delta_{domg_1}(x) \geq 0$ for any $x \in X$. The proof is complete. □

Remark 4.2. 1) The right-hand side in the formula given by Theorem 4.1 can be expressed by means of each datum Legendre-Fenchel conjugate by developing the expressions $(g_1 + \lambda y^*\circ h_1)^*$ and $(\delta_{domg_1} + y^*\circ h_1)^*$ as infimal convolution under additional constraint qualification (see [1]).

2) Let us observe that the transformation (4.2) allows us to reduce the problem $(\mathcal{Q})$ to a reverse convex programming problem on the product space $X\times Y\times X^*$. For this, it suffices to set $(x,y,x^*) \to \tilde{G}(x,y,x^*) := G(x,y) + g_2^*(x^*)$ and $(x,y,x^*) \to \tilde{H}(x,y,x^*) := H(x,y)$ which are obviously both convex, and hence according to (4.2) we may rewrite

$$\alpha = \inf\{\tilde{G}(x,y,x^*) : \quad \tilde{H}(x,y,x^*) > 0\}.$$

Let us consider now the case when h_1 is identically equal to zero. In such a case, we get the problem of minimizing a DC-function subject to a vector reverse convex constraint

$$\alpha = \inf\{g_1(x) - g_2(x) : \ h_2(x) \notin -Y_+ \}$$

and hence we deduce from Theorem 4.1

Corollary 4.1. *Let X and Y be two Hausdorff locally convex vector spaces, $g_1, g_2 : X \longrightarrow I\!R \cup \{-\infty,+\infty\}$ are two convex functions and $h_2 : X \longrightarrow Y \cup \{+\infty\}$ is a Y_+-convex mapping. We assume that $g_2 \in \Gamma(X)$ and h_2 with closed epigraph. Then*

$$\alpha = \inf_{\substack{(x^*,p^*)\in X^*\times X^* \\ y^*\in Y_+^*\backslash\{0\}}} \max_{\lambda\geq 0}\{g_2^*(x^*) + \lambda(y^*\circ h_2)^*(p^*) - g_1^*(x^*+\lambda p^*) :$$
$$(y^*\circ h_2)^*(p^*) - \delta^*_{domg_1}(p^*) < 0\}.$$

Recently, M. Laghdir and N. Benkenza studied in [4] the following vector reverse programming problem

$$\alpha = \inf\{g_1(x) : \ h_2(x) \notin -Y_+ \}$$

from the point of view of duality and they stated

Theorem 4.2. [4] *Let X and Y be two Hausdorff locally convex vector spaces, $g_1 : X \longrightarrow I\!R \cup \{-\infty, +\infty\}$ is a convex function and $h_2 : X \longrightarrow Y \cup \{ +\infty\}$ is a Y_+-convex mapping with closed epigraph. If there exists some point $\bar{x} \in X$ such that $h_2(\bar{x}) \in -int\ Y_+$ where "$int\ Y_+$" stands for topological interior of the cone Y_+, then we have*

$$\alpha = \inf_{p^* \in X^*} \max_{\lambda \geq 0} \min_{y^* \in Y_+^* \setminus \{0\}} \{\lambda (y^* \circ h_2)^*(p^*) - g_1^*(\lambda p^*) : \ (y^* \circ h_2)^*(p^*) - \delta^*_{domg_1}(p^*) < 0\}$$

By dropping the constraint qualification "$\exists \bar{x} \in X$ such that $h_2(\bar{x}) \in -int\ Y_+$" in the above Theorem 4.2 we obtain consequently from Corollary 4.1

Corollary 4.2. *Let X and Y be two Hausdorff locally convex vector spaces, $g_1 : X \longrightarrow I\!R \cup \{-\infty, +\infty\}$ is a convex function and $h_2 : X \longrightarrow Y \cup \{+\infty\}$ is a Y_+-convex mapping. We assume that h_2 with closed epigraph. Then*

$$\alpha = \inf_{\substack{p^* \in X^* \\ y^* \in Y_+^* \setminus \{0\}}} \max_{\lambda \geq 0} \{\lambda (y^* \circ h_2)^*(p^*) - g_1^*(\lambda p^*) : \ (y^* \circ h_2)^*(p^*) - \delta^*_{domg_1}(p^*) < 0\}$$

Another consequence of Theorem 4.1, that is the case when we consider h_2 identically equal to zero, then the problem $(\mathcal{Q})$ becomes

$$\alpha = \inf\{g_1(x) - g_2(x) : \ h_1(x) \notin Y_+ \}$$

and hence we get

Corollary 4.3. *Let X and Y be two Hausdorff locally convex vector spaces, $g_1, g_2 : X \longrightarrow I\!R \cup \{-\infty, +\infty\}$ are two convex functions and $h_1 : X \longrightarrow Y$ is an Y_+-convex mapping. If we assume that $g_2 \in \Gamma(X)$, then*

$$\alpha = \inf_{\substack{x^* \in X^* \\ y^* \in Y_+^* \setminus \{0\}}} \max_{\lambda \geq 0} \{g_2^*(x^*) - (g_1 + \lambda y^* \circ h_1)^*(x^*) : \ (\delta_{domg_1} + y^* \circ h_1)^*(0) > 0\}$$

Proof. In the formula of Theorem 4.1, it suffices to observe that when $h_2 \equiv 0$ we have $(y^* \circ h_2)^*(p^*) = \delta_{\{0\}}(p^*)$ and hence we take necessarily $p^* = 0$. □

In the case when $Y = I\!R$ and $Y_+ = I\!R_+$, we recapture as consequence from Theorem 4.1 a result by B. Lemaire and M. Volle ([3] Theorem 3.1) given by

Corollary 4.4. *Let X be a Hausdorff locally convex vector space, $g_1, g_2 : X \longrightarrow I\!R \cup \{-\infty, +\infty\}$ are convex functions and $h_1 : X \longrightarrow I\!R \cup \{+\infty\}$, $h_2 : X \longrightarrow I\!R \cup \{-\infty, \ +\infty\}$ are two others convex functions. We assume that g_2 and h_2 are in $\Gamma(X)$. Then we have*

$$\inf\ (\mathcal{P}) = \inf_{(x^*, p^*) \in X^* \times X^*} \max_{\lambda \geq 0} \{g_2^*(x^*) + \lambda h_2^*(p^*) - (g_1 + \lambda h_1)^*(x^* + \lambda p^*) : \\ h_2^*(p^*) - (\delta_{domg_1} + h_1)^*(p^*) < 0\}$$

Proof. At first, let us notice that when $Y = I\!R$ and $Y_+ = I\!R_+$ we have obviously $Y_+^* = I\!R_+$ and according to Theorem 4.1, if $dom\ h_1 = X$ and $h_2 : X \longrightarrow I\!R \cup \{+\infty\}$ we obtain

$$\inf\ (\mathcal{P}) = \inf_{\substack{(x^*, p^*) \in X^* \times X^* \\ \beta > 0}} \max_{\lambda \geq 0} \{g_2^*(x^*) + \lambda (\beta h_2)^*(p^*) - (g_1 + \lambda \beta h_1)^*(x^* + \lambda p^*) : \\ (\beta h_2)^*(p^*) - (\delta_{domg_1} + \beta h_1)^*)(p^*) < 0\}.$$

As $(\beta h_2)^*(p^*) = \beta h_2^*(\frac{p^*}{\beta})$ and

$$(\delta_{domg_1} + \beta h_1)^*(p^*) = \beta(\delta_{domg_1} + h_1)^*(\frac{p^*}{\beta})$$

hence by setting $z^* := \frac{p^*}{\beta}$ and $\rho := \lambda\beta$, we recapture the duality result due to B. Lemaire and M. Volle given by

$$\inf\ (\mathcal{P}) = \inf_{(x^*,z^*)\in X^*\times X^*} \max_{\rho\geq 0}\{g_2^*(x^*) + \rho h_2^*(z^*) - (g_1 + \rho h_1)^*(x^* + \rho z^*) : h_2^*(z^*) - (\delta_{domg_1} + h_1)^*)(z^*) < 0\}.$$

If $dom\ h_1 = X$ and $h_2 \equiv -\infty$ (resp. $h_1 \equiv +\infty$ and $h_2 : X \longrightarrow I\!R \cup \{-\infty, +\infty\}$), then we have

$$\{x \in X :\ h_1(x) - h_2(x) < 0\} = \{z^* \in X^* :\ h_2^*(z^*) - (\delta_{domg_1} + h_1)^*(z^*) < 0\} = \emptyset$$

which yields

$$\begin{aligned}\inf\ (\mathcal{P}) &= \inf_{(x^*,z^*)\in X^*\times X^*} \max_{\rho\geq 0}\{g_2^*(x^*) + \rho h_2^*(z^*) - (g_1 + \rho h_1)^*(x^* + \rho z^*) : h_2^*(z^*) - (\delta_{domg_1} + h_1)^*)(z^*) < 0\}\\ &= +\infty.\end{aligned}$$

The proof is complete. □

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