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COINCIDENCE THEORY FOR $\mathcal{U}_c^\kappa$ MAPS AND INEQUALITIES

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Abstract. Applying a new fixed point theorem for $\mathcal{U}_c^\kappa$ maps in extension type spaces we obtain new coincidence theorems and minimax inequalities.

1. Introduction

This paper establishes new minimax and quasi–variational inequalities for a general class of maps, namely the $\mathcal{U}_c^\kappa$ maps of Park. Along the way new coincidence results and analytic alternatives are also presented. Our results in particular improve those in [2, 3, 4, 6, 7]; for example Theorem 2.13 is a generalization of von Neumann's minimax theorem. The theory relies on a new fixed point theorem [1] in extension type spaces.

For the remainder of this section we present some definitions and known results which will be needed throughout this paper. Of particular importance will be the class $\mathcal{U}_c^\kappa$. Suppose X and Y are Hausdorff topological spaces. Given a class $\mathcal{X}$ of maps, $\mathcal{X}(X,Y)$ denotes the set of maps $F : X \to 2^Y$ (nonempty subsets of Y) belonging to $\mathcal{X}$, and $\mathcal{X}_c$ the set of finite compositions of maps in $\mathcal{X}$. A class $\mathcal{U}$ of maps is defined by the following properties:

(i). $\mathcal{U}$ contains the class $\mathcal{C}$ of single valued continuous functions;
(ii). each $F \in \mathcal{U}_c$ is upper semicontinuous and compact valued; and
(iii). for any polytope P, $F \in \mathcal{U}_c(P,P)$ has a fixed point, where the intermediate spaces of composites are suitably chosen for each $\mathcal{U}$.

Definition 1.1. $F \in \mathcal{U}_c^\kappa(X,Y)$ if for any compact subset K of X, there is a $G \in \mathcal{U}_c(K,Y)$ with $G(x) \subseteq F(x)$ for each $x \in K$.

Recall that $\mathcal{U}_c^\kappa$ is closed under compositions. We also discuss special examples of $\mathcal{U}_c^\kappa$ maps. Let X and Y be subsets of Hausdorff topological vector spaces E_1 and E_2 respectively. We will consider maps $F : X \to K(Y)$; here $K(Y)$ denotes the family of nonempty compact subsets of Y. We say $F : X \to K(Y)$ is <u>Kakutani</u> if F is upper semicontinuous with convex values. A nonempty topological space is said to be acyclic if all its reduced Čech homology groups over the rationales are trivial. Now $F : X \to K(Y)$ is <u>acyclic</u> if F is upper semicontinuous with acyclic values. $F : X \to K(Y)$ is said to be an <u>O'Neill</u> map if F is continuous and if the values of F consist of one or m acyclic components (here m is fixed).

Given two open neighborhoods U and V of the origins in E_1 and E_2 respectively, a (U,V)–approximate continuous selection of $F : X \to K(Y)$ is a continuous function $s : X \to Y$ satisfying

$$s(x) \in (F\left[(x+U)\cap X\right] + V) \cap Y \quad \text{for every } x \in X.$$

We say $F : X \to K(Y)$ is <u>approximable</u> if it is upper semicontinuous and if its restriction $F|_K$ to any compact subset K of X admits a (U,V)–approximate continuous selection for every open neighborhood U and V of the origins in E_1 and E_2 respectively.

For our next definition let X and Y be metric spaces. A continuous single valued map $p : Y \to X$ is called a Vietoris map if the following two conditions are satisfied:

(i). for each $x \in X$, the set $p^{-1}(x)$ is acyclic

(ii). p is a proper map i.e. for every compact $A \subseteq X$ we have that $p^{-1}(A)$ is compact.

Definition 1.2. A multifunction $\phi : X \to K(Y)$ is admissible (strongly) in the sense of Gorniewicz (and we write $\phi \in Ad(X, Y)$), if $\phi : X \to K(Y)$ is upper semicontinuous, and if there exists a metric space Z and two continuous maps $p : Z \to X$ and $q : Z \to Y$ such that

(i). p is a Vietoris map

and

(ii). $\phi(x) = q(p^{-1}(x))$ for any $x \in X$.

It should be noted that ϕ upper semicontinuous is redundant in Definition 1.2. Notice the Kakutani maps, the acyclic maps, the O'Neill maps, the approximable maps and the maps admissible in the sense of Gorniewicz are examples of $\mathcal{U}_c^\kappa$ maps.

For a subset K of a topological space, we denote by $Cov_X(K)$ the directed set of all coverings of K by open sets of X (usually we write $Cov(K) = Cov_X(K)$). Given two maps $F, G : X \to 2^Y$ and $\alpha \in Cov(Y)$, F and G are said to be α–close, if for any $x \in X$ there exists $U_x \in \alpha$, $y \in F(x) \cap U_x$ and $w \in G(x) \cap U_x$.

By a space we mean a Hausdorff topological space. A space Y is an extension space for Q (written $Y \in ES(Q)$) if for any pair (X, K) in Q with $K \subseteq X$ closed, any continuous function $f_0 : K \to Y$ extends to a continuous function $f : X \to Y$.

A space Y is an approximate extension space for Q (and we write $Y \in AES(Q)$) if for any $\alpha \in Cov(Y)$ and any pair (X, K) in Q with $K \subseteq X$ closed, and any continuous function $f_0 : K \to Y$, there exists a continuous function $f : X \to Y$ such that $f|_K$ is α–close to f_0.

Definition 1.3. Let V be a subset of a Hausdorff topological vector space E. Then we say V is Schauder admissible if for every compact subset K of V and every covering $\alpha \in Cov_V(K)$, there exists a continuous function (called the Schauder projection) $\pi_\alpha : K \to V$ such that

(i). π_α and $i : K \to V$ are α–close;

(ii). $\pi_\alpha(K)$ is contained in a subset $C \subseteq V$ with $C \in AES$(compact).

Remark 1.1. In Definition 1.3 we may replace E a Hausdorff topological vector space with E a uniform space.

Examples. Normed spaces, absolute retracts (AR's) and complete metric topological vector spaces admissible in the sense of Klee are examples of ES(compact) spaces. Convex subsets of locally convex topological vector spaces are AES(compact). Spaces admissible in the sense of Klee and AES(compact) spaces are Schauder admissible.

The following fixed point results were established in [1].

Theorem 1.1. *Let V be a Schauder admissible subset of a Hausdorff topological vector space E and $F \in \mathcal{U}_c^\kappa(V,V)$ a compact upper semicontinuous map with closed valued. Then F has a fixed point.*

Theorem 1.2. *Let $V \in ES(compact)$ and $F \in \mathcal{U}_c^\kappa(V,V)$ a compact map. Then F has a fixed point.*

Remark 1.2. Notice in Theorem 1.1 we could replace Hausdorff topological vector space with uniform space. In Theorem 1.2 we only assume that V is a Hausdorff topological space.

Next let Z and W be subsets of Hausdorff topological vector spaces Y_1 and Y_2 and F a multifunction. We say $F \in PK(Z,W)$ if W is convex, and there exists a map $S : Z \to W$ with

$$Z = \cup \left\{ int\, S^{-1}(w) : \ w \in W \right\}, \ \ co\,(S(x)) \subseteq F(x) \ \text{ for } \ x \in Z;$$

here $S^{-1}(w) = \{z : \ w \in S(z)\}$. We recall the following selection theorem [5].

Theorem 1.3. *If Z is paracompact, W is convex, and $F \in PK(Z,W)$. Then there exists a continuous (single valued) function $f : Z \to W$ with $f(x) \in F(x)$ for each $x \in Z$.*

2. Coincidence and minimax inequalities

We begin this section by presenting some coincidence theorems. These will then be used to establish some analytic alternatives which in turn will be used to derive new minimax inequalities. Four coincidence theorems will be presented and the proof in each case relies on Theorem 1.1 (and sometimes Theorem 1.3).

Theorem 2.1. *Let Ω be a paracompact Schauder admissible subset of a Hausdorff topological vector space and Y a convex subset of a Hausdorff topological vector space. Suppose $F \in \mathcal{U}_c^\kappa(Y,\Omega)$ is a upper semicontinuous map with closed values and $G \in PK(\Omega, Y)$ is a compact map. Then G and F^{-1} have a coincidence. That is there exists $(x_0, y_0) \in \Omega \times Y$ with $y_0 \in G(x_0) \cap F^{-1}(x_0)$ (i.e. there exists $(x_0,y_0) \in \Omega \times Y$ with $y_0 \in G(x_0)$ and $x_0 \in F(y_0)$).*

Proof. From Theorem 1.3 there exists a continuous selection $g : \Omega \to Y$ of G. Now since $\mathcal{U}_c^\kappa$ is closed under compositions we notice $J = F \circ g \in \mathcal{U}_c^\kappa(\Omega,\Omega)$ is a compact upper semicontinuous map with closed values. Now Theorem 1.1 guarantees that there exists $x_0 \in \Omega$ with $x_0 \in F\,g(x_0)$. Let $y_0 = g(x_0)$. Then $x_0 \in F(y_0)$ and $y_0 \in G(x_0)$. □

Theorem 2.2. *Let Ω be a Schauder admissible subset of a Hausdorff topological vector space and Y a subset of a Hausdorff topological space. Suppose $F \in \mathcal{U}_c^\kappa(Y,\Omega)$ and $G \in \mathcal{U}_c^\kappa(\Omega, Y)$ are upper semicontinuous compact maps with closed values. Then G and F^{-1} have a coincidence.*

Proof. Notice $J = F \circ G \in \mathcal{U}_c^\kappa(\Omega,\Omega)$ is a upper semicontinuous compact map with closed values so Theorem 1.1 guarantees that there exists $x_0 \in \Omega$ with $x_0 \in F\,G(x_0)$. That is $x_0 \in F\,y_0$ for some $y_0 \in G\,x_0$. □

Theorem 2.3. *Let Ω be a convex Schauder admissible subset of a Hausdorff topological vector space and Y a paracompact subset of a Hausdorff topological space. Suppose $F \in \mathcal{U}_c^\kappa(\Omega, Y)$ is a upper semicontinuous compact map with closed values and $G \in PK(Y, \Omega)$ is a compact maps. Then F and G^{-1} have a coincidence i.e. there exists $(x_0, y_0) \in \Omega \times Y$ with $y_0 \in F(x_0)$ and $x_0 \in G(y_0)$.*

Proof. From Theorem 1.3 there exists a continuous selection $g : Y \to \Omega$ of G. Notice $J = g \circ F \in \mathcal{U}_c^\kappa(\Omega, \Omega)$ is a compact map. Theorem 1.1 guarantees that there exists $x_0 \in \Omega$ with $x_0 \in g\,F\,(x_0)$. Thus there exists $y_0 \in F\,x_0$ with $x_0 = g(y_0)$. □

Remark 2.1. In Theorems 2.1–2.3 we could replace Ω is a subset of a Hausdorff topological vector space with Ω is a subset of a uniform space. If Ω is ES(compact) then we need only assume Ω is a Hausdorff topological space in Theorems 2.1–2.3 and any mention of upper semicontinuous with closed values can be deleted in the statement of Theorems 2.2–2.3.

Example. Suppose $\Omega \in ES$(compact) (for example suppose $\Omega \in AR$), Y a Hausdorff topological space and assume $F \in Ad(Y, \Omega)$, $G \in Ad(\Omega, Y)$ are compact maps. Then G and F^{-1} have a coincidence.

Theorem 2.4. *Let Ω be a paracompact convex Schauder admissible subset of a Hausdorff topological vector space and Y a paracompact convex subset of a Hausdorff topological vector space. Suppose $F \in PK(Y, \Omega)$ and $G \in PK(\Omega, Y)$ are compact maps. Then G and F^{-1} have a coincidence.*

Proof. From Theorem 1.2 there exists a continuous selection $g : \Omega \to Y$ of G and a continuous selection $f : Y \to \Omega$ of F. Notice $J = f \circ g \in \mathcal{U}_c^\kappa(\Omega, \Omega)$ is a continuous compact map. Theorem 1.1 guarantees that there exists $x_0 \in \Omega$ with $x_0 = f\,g\,(x_0)$. □

Next we present new analytic alternatives and minimax inequalities. Our results in particular improve [3, 4, 6, 7]. We first establish four analytic alternatives.

Theorem 2.5. *Let Ω be a paracompact Schauder admissible subset of a Hausdorff topological vector space and Y a convex subset of a Hausdorff topological vector space. Let $f, g : \Omega \times Y \to \mathbf{R}$ be such that*

$$g(x, y) \le f(x, y) \quad \textit{for all} \quad (x, y) \in \Omega \times Y. \tag{2.1}$$

Fix $\alpha \in \mathbf{R}$ and let

$$G(x) = \{y \in Y : \; f(x, y) > \alpha\}$$

and

$$F(y) = \{x \in \Omega : \; g(x, y) \le \alpha\}.$$

Suppose $F \in \mathcal{U}_c^\kappa(Y, \Omega)$ is upper semicontinuous with compact values. Also assume if $G(x) \ne \emptyset$ for every $x \in \Omega$ then $G \in PK(\Omega, Y)$. If F and G are compact maps then either

(A1). there exists $z_0 \in \Omega$ with $f(z_0, y) \le \alpha$ for all $y \in Y$

or

(A2). there exists $(x_0, y_0) \in \Omega \times Y$ with $g(x_0, y_0) \le \alpha < f(x_0, y_0)$

occur.

Proof. Either $G(x) \neq \emptyset$ for every $x \in \Omega$ or not. If $G(x) \neq \emptyset$ for every $x \in \Omega$ then $G \in PK(\Omega, Y)$ so Theorem 2.1 implies that there exists $(x_0, y_0) \in \Omega \times Y$ with $x_0 \in F(y_0)$ and $y_0 \in G(x_0)$ i.e. $(A2)$ occurs. If $G(x) \neq \emptyset$ for every $x \in \Omega$ does not hold, then there exists $z_0 \in \Omega$ with $G(z_0) = \emptyset$. That is $f(z_0, y) \leq \alpha$ for every $y \in Y$ so $(A1)$ occurs. □

Remark 2.2. If $g \equiv f$ in Theorem 2.5 then $(A2)$ cannot occur.

Remark 2.3. If we replace $F \in \mathcal{U}_c^\kappa(Y, \Omega)$ in Theorem 2.5 by

$$\text{if } F(y) \neq \emptyset \text{ for every } y \in Y \text{ then } F \in \mathcal{U}_c^\kappa(Y, \Omega),$$

then the conclusion in Theorem 2.5 is that either $(A1)$, $(A2)$ or

$(A3)$. there exists $w_0 \in Y$ with $g(x, w_0) > \alpha$ for all $x \in \Omega$

occur. We note in this case if there exists $w_0 \in Y$ with $F(w_0) = \emptyset$ then $G(x) \neq \emptyset$ for every $x \in \Omega$, since if there exists $z_0 \in \Omega$ with $G(z_0) = \emptyset$ then in particular $g(z_0, w_0) > \alpha$ and $f(z_0, w_0) \leq \alpha$, so $f(z_0, w_0) \leq \alpha < g(z_0, w_0)$ which contradicts (2.1).

Essentially the same reasoning as in Theorem 2.5 except now we use Theorem 2.3 gives the following result.

Theorem 2.6. *Let Ω be a convex Schauder admissible subset of a Hausdorff topological vector space and Y a paracompact subset of a Hausdorff topological space. Let f, $g : \Omega \times Y \to \mathbf{R}$ be such that* (2.1) *occurs. Fix $\alpha \in \mathbf{R}$ and let*

$$G(y) = \{x \in \Omega : \ f(x, y) > \alpha\}$$

and

$$F(x) = \{y \in Y : \ g(x, y) \leq \alpha\}.$$

Suppose $F \in \mathcal{U}_c^\kappa(\Omega, Y)$ is upper semicontinuous with compact values. Also assume if $G(y) \neq \emptyset$ for every $y \in Y$ then $G \in PK(Y, \Omega)$. If F and G are compact maps then either

$(A1)$. *there exists $w_0 \in Y$ with $f(x, w_0) \leq \alpha$ for all $x \in \Omega$*

or

$(A2)$. *there exists $(x_0, y_0) \in \Omega \times Y$ with $g(x_0, y_0) \leq \alpha < f(x_0, y_0)$*

occur.

Remark 2.4. In Theorems 2.5–2.6 we could replace Ω is a subset of a Hausdorff topological vector space with Ω is a subset of a uniform space. If Ω is ES(compact) then we need only assume Ω is a Hausdorff topological space in Theorems 2.5–2.6 and any mention of upper semicontinuous with closed values can be deleted in the statement of Theorems 2.5–2.6.

Theorem 2.7. *Let Ω be a paracompact convex Schauder admissible subset of a Hausdorff topological vector space and Y a paracompact convex subset of a Hausdorff topological vector space. Let f, $g : \Omega \times Y \to \mathbf{R}$ be such that* (2.1) *occurs. Fix $\alpha \in \mathbf{R}$ and let*

$$G(x) = \{y \in Y : \ f(x, y) < \alpha\}$$

and

$$F(y) = \{x \in \Omega : \ g(x, y) > \alpha\}.$$

If $G(x) \neq \emptyset$ *for every* $x \in \Omega$ *suppose* $G \in PK(\Omega, Y)$*. Also assume if* $F(y) \neq \emptyset$ *for every* $y \in Y$ *then* $F \in PK(Y, \Omega)$*. If* F *and* G *are compact maps then either*

$(A1)$*. there exists* $z_0 \in \Omega$ *with* $f(z_0, y) \geq \alpha$ *for all* $y \in Y$

or

$(A2)$*. there exists* $w_0 \in Y$ *with* $g(x, w_0) \leq \alpha$ *for all* $x \in \Omega$

occur.

Proof. There are three cases to consider,
Case (a). $G(x) \neq \emptyset$ for every $x \in \Omega$ and $F(y) \neq \emptyset$ for every $y \in Y$.

In this case Theorem 2.4 guarantees that there exists $(x_0, y_0) \in \Omega \times Y$ with $f(x_0, y_0) < \alpha$ and $g(x_0, y_0) > \alpha$. This contradicts (2.1).
Case (b). Suppose $G(x) \neq \emptyset$ for every $x \in \Omega$ does not hold.

Then there exists $z_0 \in \Omega$ with $G(z_0) = \emptyset$. That is $f(z_0, y) \geq \alpha$ for all $y \in Y$ i.e. $(A1)$ occurs.
Case (c). Suppose $F(y) \neq \emptyset$ for every $y \in Y$ does not hold.

Then there exists $w_0 \in Y$ with $F(w_0) = \emptyset$ i.e. $(A2)$ occurs. □

It is also easy to see that the analytic alternative Theorem 2.7 has a "dual version".

Theorem 2.8. *Let* Ω *be a paracompact convex Schauder admissible subset of a Hausdorff topological vector space and* Y *a paracompact convex subset of a Hausdorff topological vector space. Let* $f, g : \Omega \times Y \to \mathbf{R}$ *be such that* (2.1) *occurs. Fix* $\alpha \in \mathbf{R}$ *and let*

$$G(x) = \{y \in Y : \ g(x, y) > \alpha\}$$

and

$$F(y) = \{x \in \Omega : \ f(x, y) < \alpha\}.$$

If $G(x) \neq \emptyset$ *for every* $x \in \Omega$ *suppose* $G \in PK(\Omega, Y)$*. Also assume if* $F(y) \neq \emptyset$ *for every* $y \in Y$ *then* $F \in PK(Y, \Omega)$*. If* F *and* G *are compact maps then either*

$(A1)$*. there exists* $z_0 \in \Omega$ *with* $g(z_0, y) \leq \alpha$ *for all* $y \in Y$

or

$(A2)$*. there exists* $w_0 \in Y$ *with* $f(x, w_0) \geq \alpha$ *for all* $x \in \Omega$

occur.

Remark 2.5. It is also easy to construct an analytic alternative modelled off Theorem 2.2. We leave the details to the reader.

We now indicate how our analytic alternatives (or even our coincidence theorems) generate minimax inequalities. Our first result is modelled off Theorem 2.6.

Theorem 2.9. *Let* Ω *be a convex Schauder admissible subset of a Hausdorff topological vector space and* Y *a paracompact subset of a Hausdorff topological space. Let* $f : \Omega \times Y \to \mathbf{R}$ *and* $\alpha = \sup_{x \in \Omega} \inf_{y \in Y} f(x, y)$*. Also let*

$$G(y) = \{x \in \Omega : \ f(x, y) > \alpha\}$$

and

$$F(x) = \{y \in Y : \ f(x, y) \leq \alpha\}.$$

Suppose $F \in \mathcal{U}_c^\kappa(\Omega, Y)$ *is upper semicontinuous with closed values. Also assume if* $G(y) \neq \emptyset$ *for every* $y \in Y$ *then* $G \in PK(Y, \Omega)$*. If* F *and* G *are compact maps then*

$$\inf_{y \in Y} \sup_{x \in \Omega} f(x, y) = \sup_{x \in \Omega} \inf_{y \in Y} f(x, y).$$

Proof. Let $\beta = \inf_{y \in Y} \sup_{x \in \Omega} f(x, y)$. Clearly $\alpha \le \beta$. Also from Theorem 2.6 (note $(A2)$ cannot occur since $g = f$) that there exists $w_0 \in Y$ with $f(x, w_0) \le \alpha$ for all $x \in \Omega$. As a result $\sup_{x \in \Omega} f(x, w_0) \le \alpha$, so $\beta \le \alpha$. □

Remark 2.6. In Theorem 2.9 we could replace Ω is a subset of a Hausdorff topological vector space with Ω is a subset of a uniform space. If Ω is ES(compact) then we need only assume Ω is a Hausdorff topological space in Theorem 2.9 and any mention of upper semicontinuous with closed values can be deleted in the statement of Theorem 2.9.

It is also possible to generalize Theorem 2.9 using f and g. In addition it is possible to construct an analogue of Theorem 2.9 off Theorem 2.5. Instead of presenting these results we show the technique involved by establishing a minimax inequality modelled off Theorem 2.7.

Theorem 2.10. *Let* Ω *be a paracompact convex Schauder admissible subset of a Hausdorff topological vector space and* Y *a paracompact convex subset of a Hausdorff topological vector space. Let* $f, g : \Omega \times Y \to \mathbf{R}$ *be such that* (2.1) *occurs. For each* $\alpha \in \mathbf{R}$ *let*

$$G_\alpha(x) = \{y \in Y :\ f(x, y) < \alpha\}$$

and

$$F_\alpha(y) = \{x \in \Omega :\ g(x, y) > \alpha\}.$$

For each $\alpha \in \mathbf{R}$*, if* $G_\alpha(x) \neq \emptyset$ *for every* $x \in \Omega$ *assume* $G_\alpha \in PK(\Omega, Y)$ *and if* $F_\alpha(y) \neq \emptyset$ *for every* $y \in Y$ *assume* $F_\alpha \in PK(Y, \Omega)$*. For each* $\alpha \in \mathbf{R}$ *if* F_α *and* G_α *are compact maps then*

$$\beta_0 \equiv \inf_{y \in Y} \sup_{x \in \Omega} g(x, y) \le \sup_{x \in \Omega} \inf_{y \in Y} f(x, y) \equiv \alpha_0.$$

Proof. Let $\alpha_0 < \infty$ and $\beta_0 > -\infty$. Suppose $\beta_0 > \alpha_0$. Then there exists $\alpha \in \mathbf{R}$ with

$$\alpha_0 < \alpha < \beta_0. \tag{2.2}$$

Apply Theorem 2.7. If $(A1)$ occurs then there exists $z_0 \in \Omega$ with $f(z_0, y) \ge \alpha$ for all $y \in Y$, so $\inf_{y \in Y} f(z_0, y) \ge \alpha$. Consequently $\alpha_0 \ge \alpha$, and this contradicts (2.2). If $(A2)$ occurs then there exists $w_0 \in Y$ with $g(x, w_0) \le \alpha$ for all $x \in \Omega$, so $\sup_{x \in \Omega} g(x, w_0) \le \alpha$. Consequently $\beta_0 \le \alpha$, and this contradicts (2.2). In both cases we have a contradiction, so $\beta_0 \le \alpha_0$. □

Remark 2.7. If $g = f$ in Theorem 2.10 then

$$\inf_{y \in Y} \sup_{x \in \Omega} f(x, y) = \sup_{x \in \Omega} \inf_{y \in Y} f(x, y).$$

We also have a "dual version" of Theorem 2.10 if we use Theorem 2.8.

Theorem 2.11. *Let Ω be a paracompact convex Schauder admissible subset of a Hausdorff topological vector space and Y a paracompact convex subset of a Hausdorff topological vector space. Let $f,\, g : \Omega \times Y \to \mathbf{R}$ be such that* (2.1) *occurs. For each $\alpha \in \mathbf{R}$ let*

$$G_\alpha(x) = \{y \in Y : \ g(x,y) > \alpha\}$$

and

$$F_\alpha(y) = \{x \in \Omega : \ f(x,y) < \alpha\}.$$

For each $\alpha \in \mathbf{R}$, if $G_\alpha(x) \neq \emptyset$ for every $x \in \Omega$ assume $G_\alpha \in PK(\Omega, Y)$ and if $F_\alpha(y) \neq \emptyset$ for every $y \in Y$ assume $F_\alpha \in PK(Y, \Omega)$. For each $\alpha \in \mathbf{R}$ if F_α and G_α are compact maps then

$$\inf_{x\in\Omega} \sup_{y\in Y} g(x,y) \leq \sup_{y\in Y} \inf_{x\in\Omega} f(x,y).$$

Finally we obtain a minimax theorem modelled off our coincidence Theorem 2.2.

Theorem 2.12. *Let Ω be a Schauder admissible subset of a Hausdorff topological vector space and Y a subset of a Hausdorff topological space. Let $f,\, g : \Omega \times Y \to \mathbf{R}$ be such that* (2.1) *occurs. For each $\alpha \in \mathbf{R}$ let*

$$G_\alpha(x) = \{y \in Y : \ g(x,y) \geq \alpha\}$$

and for each $\beta \in \mathbf{R}$ let

$$F_\beta(y) = \{x \in \Omega : \ f(x,y) \leq \beta\}.$$

For each $\alpha \in \mathbf{R}$, if $G_\alpha(x) \neq \emptyset$ for every $x \in \Omega$ assume $G_\alpha \in \mathcal{U}_c^\kappa(\Omega, Y)$ and for each $\beta \in \mathbf{R}$, if $F_\beta(y) \neq \emptyset$ for every $y \in Y$ assume $F_\beta \in \mathcal{U}_c^\kappa(Y, \Omega)$. For each $\alpha,\, \beta \in \mathbf{R}$ if G_α and F_β are upper semicontinuous compact maps with closed values then

$$\beta_0 \equiv \inf_{x\in\Omega} \sup_{y\in Y} g(x,y) \leq \sup_{y\in Y} \inf_{x\in\Omega} f(x,y) \equiv \alpha_0.$$

Proof. Let $\alpha_0 < \infty$ and $\beta_0 > -\infty$. Suppose $\beta_0 > \alpha_0$. Then there exists $\beta \in \mathbf{R}$ with $\alpha_0 < \beta < \beta_0$. In addition there exists $\epsilon > 0$ with

$$\alpha_0 < \beta < \beta + \epsilon \equiv \alpha < \beta_0. \tag{2.3}$$

There are three cases to consider.

Case (a). $G_\alpha(x) \neq \emptyset$ for every $x \in \Omega$ and $F_\beta(y) \neq \emptyset$ for every $y \in Y$.

Now Theorem 2.2 guarantees that there exists $(x_0, y_0) \in \Omega \times Y$ with $f(x_0,y_0) \leq \beta$ and $g(x_0,y_0) \geq \alpha$. This together with (2.1) gives

$$\alpha \leq g(x_0,y_0) \leq f(x_0,y_0) \leq \beta,$$

which contradicts (2.3).

Case (b). $F_\beta(y) \neq \emptyset$ for every $y \in Y$ does not hold.

Then there exists $w_0 \in Y$ with $f(x, w_0) > \beta$ for all $x \in \Omega$, so

$$\inf_{x\in\Omega} f(x,w_0) \geq \beta.$$

Consequently $\alpha_0 \geq \beta$, which contradicts (2.3).

Case (c). $G_\alpha(x) \neq \emptyset$ for every $x \in \Omega$ does not hold.

Then there exists $z_0 \in \Omega$ with $g(z_0, y) < \alpha$ for all $y \in Y$, so

$$\sup_{y \in Y} g(z_0, y) \leq \alpha.$$

As a result $\beta_0 \leq \alpha$, which contradicts (2.3).

In all cases we have a contradiction, so $\beta_0 \leq \alpha_0$. □

We now present a "dual version" of Theorem 2.12; in particular it is a generalization of von Neumann's minimax theorem.

Theorem 2.13. *Let Ω be a Schauder admissible subset of a Hausdorff topological vector space and Y a subset of a Hausdorff topological space. Let $f,\, g : \Omega \times Y \to \mathbf{R}$ be such that* (2.1) *occurs. For each $\beta \in \mathbf{R}$ let*

$$G_\beta(x) = \{y \in Y : \ f(x, y) \leq \beta\}$$

and for each $\alpha \in \mathbf{R}$ let

$$F_\alpha(y) = \{x \in \Omega : \ g(x, y) \geq \alpha\}.$$

For each $\beta \in \mathbf{R}$, if $G_\beta(x) \neq \emptyset$ for every $x \in \Omega$ assume $G_\beta \in \mathcal{U}_c^\kappa(\Omega, Y)$ and for each $\alpha \in \mathbf{R}$, if $F_\alpha(y) \neq \emptyset$ for every $y \in Y$ assume $F_\alpha \in \mathcal{U}_c^\kappa(Y, \Omega)$. For each $\alpha,\, \beta \in \mathbf{R}$ if G_β and F_α are upper semicontinuous compact maps with closed values then

$$\inf_{y \in Y} \sup_{x \in \Omega} g(x, y) \leq \sup_{x \in \Omega} \inf_{y \in Y} f(x, y).$$

Remark 2.8. In Theorems 2.12–2.13 we could replace Ω is a subset of a Hausdorff topological vector space with Ω is a subset of a uniform space. If Ω is ES(compact) then we need only assume Ω is a Hausdorff topological space in Theorems 2.12–13 and any mention of upper semicontinuous with closed values can be deleted in the statement of Theorem 2.9.

Example. Let Ω be a Schauder admissible subset of a Hausdorff topological vector space and Y a subset of a Hausdorff topological vector space with Ω and Y compact. Suppose $f,\, g : \Omega \times Y \to \mathbf{R}$ are continuous and (2.1) holds. Also assume

(i). for each $\beta \in \mathbf{R}$ and $x \in \Omega$ the set $\{y \in Y : \ f(x, y) \leq \beta\}$ is acyclic

and

(ii). for each $\alpha \in \mathbf{R}$ and $y \in Y$ the set $\{x \in \Omega : \ g(x, y) \geq \alpha\}$ is acyclic.

Then

$$\inf_{y \in Y} \sup_{x \in \Omega} g(x, y) \leq \sup_{x \in \Omega} \inf_{y \in Y} f(x, y).$$

First notice for each $\alpha,\, \beta \in \mathbf{R}$ that the maps F_α and G_β (defined in Theorem 2.13) are compact. Also since f is continuous we see that if $\beta \in \mathbf{R}$ and $G_\beta(x) \neq \emptyset$ for every $x \in \Omega$ then $G_\beta : \Omega \to K(Y)$ is an acyclic map (this is clear since G_β has closed graph). Also if $\alpha \in \mathbf{R}$ and $F_\alpha(y) \neq \emptyset$ for every $y \in Y$ then $F_\alpha : Y \to K(\Omega)$ is an acyclic map. The result now follows from Theorem 2.13.

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