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A NEW CHARACTERIZATION OF UNIFORMLY CONVEX BANACH SPACES

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Abstract. We will prove a new characterization of uniformly convex Banach spaces and also we will give another proof of Khamsi's characterization of uniformly convex Banach spaces.

1. Introduction

In 1936, Clarkson[5] introduced the concept of uniform convexity in Banach spaces and proved that $L^p (1 < p < \infty)$ is uniformly convex. Since then, uniformly convex Banach spaces have played an important role in Banach space theory, fixed point theory, approximation theory, ergodic theory, probability theory, differential equation theory and so on. Therefore, it is very important to characterize uniformly convex Banach spaces in such fields. Especially, some characterizations of uniformly convex Banach spaces bring us fruitful results in ergodic theory and approximation theory for nonexpansive mappings and accretive operators(cf. [7, 8]).

Also Bruck[3, 4] obtained the following theorem.

Theorem 1.1 ([4]). *Let E be a uniformly convex Banach space, $d > 0$ and $\varepsilon > 0$. Then there exists $\delta > 0$ such that $\overline{co}F_\delta(T) \subset F_\varepsilon(T)$ for any nonempty closed convex subset C of E with $diam(C) = d$ and any nonexpansive mapping T of C into itself.*

Theorem 1.2 ([3]). *Let E be a uniformly convex Banach space and let $d > 0$. Then there exists $\gamma_0 \in \Gamma$ such that any nonexpansive mapping T of C into itself is type (γ) with γ_0 for any nonempty closed convex subset C of E with $diam(C) = d$.*

Using these results, Bruck[3, 4] proved nonlinear ergodic theorems for nonexpansive mappings in uniformly convex Banach spaces, which extended Baillon's nonlinear ergodic theorem[2] in Hilbert spaces to that in Banach spaces. On the other hand, Khamsi[6] proved that Theorem 1.2 characterizes the uniform convexity of Banach spaces.

In this paper, we first prove that Theorem 1.1 also characterizes the uniform convexity of Banach spaces(Theorem 3.5). Using this result, we give another proof of the above result of Khamsi[6].

2. Preliminaries

Throughout this paper, we denote by $\mathbf{N}$ the set of positive integers and by $\mathbf{R}$ the set of real numbers. Let E be a Banach space and let C be a nonempty subset of E. We denote by $diam(C)$ the diameter of C. A mapping T of C into itself is said to be *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for every $x, y \in C$. Let T be a mapping of C into itself and let $\varepsilon > 0$. Then we denote by $F(T)$ the set of fixed points of

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T and by $F_{\varepsilon}(T)$ the set of ε-approximate fixed points of T, i.e. $\|x - Tx\| \le \varepsilon$ for every $x \in F_{\varepsilon}(T)$. Then, for every ε with $0 \le \varepsilon \le 2$, the modulus of convexity $\delta(\varepsilon)$, of E is defined by

$$\delta_E(\varepsilon) = \inf\left\{1 - \frac{\|x+y\|}{2} \mid \|x\| \le 1, \|y\| \le 1, \|x-y\| \ge \varepsilon\right\}.$$

A Banach space E is said to be uniformly convex if $\delta_E(\varepsilon) > 0$ for every $\varepsilon > 0$. A Banach space E is said to be strictly convex if $\left\|\frac{x+y}{2}\right\| < 1$ for every $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \ne y$. Uniformly convex Banach spaces are strictly convex. We know that if E is a strictly convex Banach space and

$$\|x\| = \|y\| = \|(1-\lambda)x + \lambda y\| \text{ for } x, y \in E \text{ and } \lambda \in (0,1),$$

then $x = y$.

Khamsi[6] proved the following theorem.

Theorem 2.1 ([6]). *Let E be a Banach space. The following are equivalent.*

1. *E is strictly convex;*
2. *for any nonempty bounded closed and convex subset C of E and any nonexpansive mapping T of C into itself, $F(T)$ is closed convex.*

3. Main Results

We need the following lemmas.

Lemma 3.1. *Let E be a Banach space and let β, β_0 be fixed numbers such that $0 \le \beta < 1/2$ and $\beta_0 > \beta$. Suppose that there exist $x, y \in E$ such that $\|x\| = \|y\| = 1, \|x+y\|/2 = 1-\beta$ and $\|x-y\|/2 > \beta_0$. Then for every $a, c \in \mathbf{R}$,*

$$|c| \le \frac{\beta_0 + \beta}{(1-\beta)(\beta_0 - \beta)} \left\| \frac{a(x-y)}{2} + \frac{c(x+y)}{2} \right\|.$$

Proof. To simplify the computation, we will denote $\dfrac{x+y}{2}$ by $\vec{j}$ and $\dfrac{x-y}{2}$ by $\vec{i}$. By the Hahn-Banach theorem there exists $f \in E^*$ such that $\|f\| = 1$ and $(*)$ $f(\vec{j}) = 1 - \beta$. Then we have

$$\begin{aligned} f(x) + f(y) &= 2f(\vec{j}) \\ &= 2 - 2\beta \\ &> 1. \end{aligned}$$

Since $|f(x)| \le 1$ and $|f(y)| \le 1$, $f(x)$ and $f(y)$ are positive. Therefore, there exist non-negative numbers s, s' such that $s + s' = 2\beta$, $f(x) = 1 - s$ and $f(y) = 1 - s'$. Then, We have $|f(\vec{i})| = |s - s'|/2 \le \beta$ and $|a| < \|a\vec{i}\|/\beta_0$ for every $a \in \mathbf{R}$. Since

$$|c| = \frac{|f(c\vec{j})|}{1-\beta} \le \frac{|f(a\vec{i} + c\vec{j})|}{1-\beta} + \frac{|f(a\vec{i})|}{1-\beta}$$

and

$$|f(a\vec{i})| < |a|\beta \le \frac{\beta\|a\vec{i}\|}{\beta_0} \le \frac{\beta\|a\vec{i} + c\vec{j}\| + \beta\|c\vec{j}\|}{\beta_0} \le \frac{\beta\|a\vec{i} + c\vec{j}\|}{\beta_0} + \frac{\beta(1-\beta)|c|}{\beta_0},$$

then we have

$$\begin{aligned}|c| &\le \frac{\|a\vec{\imath}+c\vec{\jmath}\|}{1-\beta}+\frac{|f(a\vec{\imath})|}{1-\beta}\\ &\le \frac{\|a\vec{\imath}+c\vec{\jmath}\|}{1-\beta}+\frac{\beta\,\|a\vec{\imath}+c\vec{\jmath}\|}{\beta_0\,(1-\beta)}+\frac{\beta|c|}{\beta_0}\\ &\le \frac{(\beta_0+\beta)\,\|a\vec{\imath}+c\vec{\jmath}\|}{\beta_0\,(1-\beta)}+\frac{\beta|c|}{\beta_0}.\end{aligned}$$

Hence we get

$$|c| \le \frac{\beta_0+\beta}{(1-\beta)(\beta_0-\beta)}\left\|\frac{a\,(x-y)}{2}+\frac{c\,(x+y)}{2}\right\|.$$

This completes the proof. ☐

Lemma 3.2. *Let E be a Banach space and let $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. Then for every $|c_1|, |c_2| \in [0,1]$, it holds*

$$\left\|\left(|c_1|\left(\frac{x-y}{2}\right)+c_1\left(\frac{x+y}{2}\right)\right)-\left(|c_2|\left(\frac{x-y}{2}\right)+c_2\left(\frac{x+y}{2}\right)\right)\right\| \le |c_1-c_2|.$$

Proof. (i) Case of $c_1, c_2 > 0$. We have

$$\begin{aligned}&\left\|\left(|c_1|\left(\frac{x-y}{2}\right)+c_1\left(\frac{x+y}{2}\right)\right)-\left(|c_2|\left(\frac{x-y}{2}\right)+c_2\left(\frac{x+y}{2}\right)\right)\right\|\\ &\qquad=\left\|(|c_1|-|c_2|)\left(\frac{x-y}{2}\right)+(c_1-c_2)\left(\frac{x+y}{2}\right)\right\|\\ &\qquad=\left\|(c_1-c_2)\left(\frac{x-y}{2}\right)+(c_1-c_2)\left(\frac{x+y}{2}\right)\right\|\\ &\qquad=\|(c_1-c_2)x\|\\ &\qquad=|c_1-c_2|.\end{aligned}$$

(ii) Case of $c_1, c_2 < 0$. We have

$$\begin{aligned}&\left\|\left(|c_1|\left(\frac{x-y}{2}\right)+c_1\left(\frac{x+y}{2}\right)\right)-\left(|c_2|\left(\frac{x-y}{2}\right)+c_2\left(\frac{x+y}{2}\right)\right)\right\|\\ &\qquad=\left\|(|c_1|-|c_2|)\left(\frac{x-y}{2}\right)+(c_1-c_2)\left(\frac{x+y}{2}\right)\right\|\\ &\qquad=\left\|(c_2-c_1)\left(\frac{x-y}{2}\right)+(c_1-c_2)\left(\frac{x+y}{2}\right)\right\|\\ &\qquad=\|(c_1-c_2)y\|\\ &\qquad=|c_1-c_2|.\end{aligned}$$

(iii) Case of $c_1 > 0, c_2 < 0$. We have

$$\left\|\left(|c_1|\left(\frac{x-y}{2}\right)+c_1\left(\frac{x+y}{2}\right)\right)-\left(|c_2|\left(\frac{x-y}{2}\right)+c_2\left(\frac{x+y}{2}\right)\right)\right\|$$

$$
\begin{aligned}
&= \left\| (|c_1| - |c_2|) \left(\frac{x-y}{2} \right) + (c_1 - c_2) \left(\frac{x+y}{2} \right) \right\| \\
&= \left\| (c_1 + c_2) \left(\frac{x-y}{2} \right) + (c_1 - c_2) \left(\frac{x+y}{2} \right) \right\| \\
&= \|c_1 x - c_2 y\| \\
&\leq |c_1| + |c_2| \\
&= c_1 - c_2 \\
&= |c_1 - c_2|.
\end{aligned}
$$

(iv) Case of $c_1 < 0, c_2 > 0$. We have

$$
\begin{aligned}
&\left\| \left(|c_1| \left(\frac{x-y}{2} \right) + c_1 \left(\frac{x+y}{2} \right) \right) - \left(|c_2| \left(\frac{x-y}{2} \right) + c_2 \left(\frac{x+y}{2} \right) \right) \right\| \\
&= \left\| (|c_1| - |c_2|) \left(\frac{x-y}{2} \right) + (c_1 - c_2) \left(\frac{x+y}{2} \right) \right\| \\
&= \left\| (-c_1 - c_2) \left(\frac{x-y}{2} \right) + (c_1 - c_2) \left(\frac{x+y}{2} \right) \right\| \\
&= \|c_1 y - c_2 x\| \\
&\leq |c_1| + |c_2| \\
&= -c_1 + c_2 \\
&= |c_1 - c_2|.
\end{aligned}
$$

This completes the proof. □

Using Lemma 3.1 and Lemma 3.2, we first obtain the following new characterization of uniformly convex Banach spaces.

Theorem 3.3. *Let E be a Banach space. The following are equivalent.*

(1) *E is uniformly convex;*
(2) *for each $\varepsilon > 0$, then there exists $\delta > 0$ such that $\lambda F_\delta(T) + (1-\lambda)F_\delta(T) \subset F_\varepsilon(T)$ for any $\lambda \in [0,1]$, nonempty closed convex subset C of E with diam$(C) = 1$ and any nonexpansive mapping T of C into itself.*

Proof. Bruck[4] proved that (1) implies (2). So, we shall prove that (2) implies (1). We assume that E is not uniformly convex. Then, there exist a real number $\beta_0 > 0$, $\{\beta_n\} \subset (0,1)$ and $\{x_n\}, \{y_n\} \subset E$ such that $\|x_n\| = \|y_n\| = 1$ and

$$
\left\| \frac{x_n - y_n}{2} \right\| > \beta_0, \left\| \frac{x_n + y_n}{2} \right\| = 1 - \beta_n \text{ and } \beta_n \leq \frac{\beta_0}{(2+\beta_0)n}.
$$

for all $n \in \mathbf{N}$. We define sequences $\{C_n\}$ and $\{D_n\}$ of closed convex subsets of E by

$$
C_n = \left\{ a \left(\frac{x_n - y_n}{2} \right) + c \left(\frac{x_n + y_n}{2} \right) : a \in [0,1], |c| \in [0,1] \right\}
$$

and

$$
D_n = \frac{C_n}{M}, \ M = diam(C_n)
$$

for all $n \in \mathbf{N}$ and we also define a sequence $\{T_n\}$ of mappings of D_n into itself by, for every $a, |c| \in [0,1]$,

$$T_n\left(\frac{a}{M}\left(\frac{x_n-y_n}{2}\right)+\frac{c}{M}\left(\frac{x_n+y_n}{2}\right)\right) = \frac{|c|}{M}\left(\frac{x_n-y_n}{2}\right)+\frac{c}{M}\left(\frac{x_n+y_n}{2}\right) \text{ for all } n \in \mathbf{N}.$$

By the definiton of $\{D_n\}$, we have $0 \in D_n$ for all $n \in \mathbf{N}$. Further we define a sequence $\{L_n\}$ of mappings of D_n into itself by

$$L_n = \frac{(1-\beta_n)(\beta_0-\beta_n)}{\beta_0+\beta_n}T_n$$

for all $n \in \mathbf{N}$. By Lemma 3.1 and Lemma 3.2, we obtain that L_n is a nonexpansive mapping of D_n into itself for each $n \in \mathbf{N}$. Let $n \in \mathbf{N}$ and $x \in F(T_n)$. Since

$$\begin{aligned}
\|L_n x - x\| &= \left\|\frac{(1-\beta_n)(\beta_0-\beta_n)}{\beta_0+\beta_n}T_n x - x\right\| \\
&= \left\|\left(\frac{(1-\beta_n)(\beta_0-\beta_n)-(\beta_0+\beta_n)}{\beta_0+\beta_n}\right)x\right\| \\
&= \left|\frac{(1-\beta_n)(\beta_0-\beta_n)-(\beta_0+\beta_n)}{\beta_0+\beta_n}\right|\|x\| \\
&\leq \left|\frac{\left(-2\beta_n-\beta_n\beta_0+(\beta_n)^2\right)}{\beta_0+\beta_n}\right| \\
&= \left|\frac{\left(-2-\beta_0+\beta_n\right)\beta_n}{\beta_0+\beta_n}\right| \\
&\leq \left|\frac{\left(2+\beta_0\right)\beta_n}{\beta_0}\right| \\
&\leq \frac{1}{n},
\end{aligned}$$

we have $x \in F_{\frac{1}{n}}(L_n)$, i.e. $F(T_n) \subset F_{\frac{1}{n}}(L_n)$ for each $n \in \mathbf{N}$. Put

$$u = \frac{1}{2}\left(\frac{x_n-y_n}{2}\right)+\frac{1}{2}\left(\frac{x_n+y_n}{2}\right) \text{ and } v = \frac{1}{2}\left(\frac{x_n-y_n}{2}\right)-\frac{1}{2}\left(\frac{x_n+y_n}{2}\right).$$

Then, we have that $u, v \in F(T_n) \subset F_{\frac{1}{n}}(L_n)$ and

$$\frac{1}{2}\left(\frac{x_n-y_n}{2}\right) = \frac{u+v}{2} \in \frac{1}{2}F_{\frac{1}{n}}(L_n)+\frac{1}{2}F_{\frac{1}{n}}(L_n).$$

Since

$$\left\|L_n\left(\frac{1}{2}\left(\frac{x_n-y_n}{2}\right)\right)-\frac{1}{2}\left(\frac{x_n-y_n}{2}\right)\right\| = \left\|\frac{1}{2}\left(\frac{x_n-y_n}{2}\right)\right\| > \frac{\beta_0}{2}$$

for each $n \in \mathbf{N}$, we have

$$\frac{1}{2}\left(\frac{x_n-y_n}{2}\right) \notin F_{\frac{\beta_0}{2}}(L_n)$$

for each $n \in \mathbf{N}$. Since

$$\frac{1}{2}\left(\frac{x_n - y_n}{2}\right) = \frac{u+v}{2} \in \frac{1}{2}F_{\frac{1}{n}}(L_n) + \frac{1}{2}F_{\frac{1}{n}}(L_n)$$

for each $n \in \mathbf{N}$, this contradicts (2). This completes the proof. □

By Theorem 1.1 and Theorem 3.4, we have the following theorem and we obtain the affirmative answer to our question, i.e. the converse of Theorem 3.1 is true.

Theorem 3.4. *Let E be a Banach space. The following are equivalent.*

(1) *E is uniformly convex;*
(2) *for each $\varepsilon > 0$ and $d > 0$, then there exists $\delta > 0$ such that $\overline{co}F_\delta(T) \subset F_\varepsilon(T)$ for any nonempty closed convex subset C of E with $diam(C) = d$ and nonexpansive mapping T of C into itself.*

Proof. It is trivial that (2) implies (2) in Theorem 3.3. By Theorem 1.1, we have that (1) implies (2). This completes the proof. □

We need the following definitions.

Definition 3.5. A function $\gamma : [0, \infty) \to [0, \infty)$ belongs to class Γ if γ satisfies

(1) $\gamma(0) = 0$,
(2) γ is a continuous convex function,
(3) $s < t$ implies $\gamma(s) < \gamma(t)$.

Definition 3.6. Let C be a convex subset of a Banach space E. A mapping T of C into E is said to be of type(γ) if there exists $\gamma \in \Gamma$ such that for all $x, y \in C$ and all $\lambda \in [0, 1]$,

$$\gamma(\|\lambda T(x) + (1-\lambda)T(y) - T(\lambda x + (1-\lambda)y)\|) \leq \|x - y\| - \|T(x) - T(y)\|.$$

Definition 3.7. Let C be a convex subset of a Banach space E and $\gamma_0 \in \Gamma$. A mapping T of C into E is said to be of type(γ) with γ_0 if for all $x, y \in C$ and all $\lambda \in [0, 1]$,

$$\gamma_0(\|\lambda T(x) + (1-\lambda)T(y) - T(\lambda x + (1-\lambda)y)\|) \leq \|x - y\| - \|T(x) - T(y)\|.$$

Finally, by this theorem and Theorem 3.3, we can obtain the following Khamsi's theorem.

Theorem 3.8 ([6])**.** *Let E be a Banach space. The following are equivalent.*

(1) *E is uniformly convex;*
(2) *there exists $\gamma_0 \in \Gamma$ such that any nonexpansive mapping T of C into itself is of type (γ) with γ_0 for any nonempty closed convex subset C of E with $diam(C) = 1$.*

Proof. By Theorem 1.2, we have that (1) implies (2). By Theorem 3.3, it suffices to show that (2) implies (2) in Theorem 3.3. By our assumption, there exists $\gamma_0 \in \Gamma$ such that any nonexpansive mapping T of C into itself is type (γ) with γ_0 for any nonempty closed convex subset C of E with $diam(C) = 1$. Let $\varepsilon > 0$ and let δ be a positive number such that

$$\delta + \gamma_0^{-1}(2\delta) \leq \varepsilon.$$

Let C be a nonempty closed convex subset of E with $diam(C) = 1$, let T be a nonexpansive mapping of C into itself, let $x, y \in F_\delta(T)$ and let $\lambda \in [0, 1]$. Since

$$\begin{aligned}\|\lambda x + (1-\lambda)y - T(\lambda x + (1-\lambda)y)\| &\le \|\lambda x + (1-\lambda)y - [\lambda Tx + (1-\lambda)Ty])\| \\ &\quad + \|\lambda Tx + (1-\lambda)Ty - T(\lambda x + (1-\lambda)y)\| \\ &\le \lambda \|x - Tx\| + (1-\lambda)\|y - Ty\| \\ &\quad + \gamma_0^{-1}(\|x - y\| - \|Tx - Ty\|) \\ &\le \lambda \|x - Tx\| + (1-\lambda)\|y - Ty\| \\ &\quad + \gamma_0^{-1}(\|x - Tx\| + \|y - Ty\|) \\ &\le \delta + \gamma_0^{-1}(2\delta) \\ &\le \varepsilon,\end{aligned}$$

then we have that $\lambda x + (1-\lambda)y \in F_\varepsilon(T)$. Therefore we obtain $\lambda F_\delta(T) + (1-\lambda)F_\delta(T) \subset F_\varepsilon(T)$ for any $\lambda \in [0, 1]$, nonempty closed convex subset C of E with $diam(C) = 1$ and nonexpansive mapping T of C into itself. This completes the proof. □

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