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HEMIVARIATIONAL INEQUALITIES MODELING VISCOUS INCOMPRESSIBLE FLUIDS

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Abstract. In this paper we study a class of inequality problems for Navier-Stokes equations related to the model of motion of a viscous incompressible fluid in a bounded domain. It is assumed that on the boundary of the domain the tangential components of the velocity vector are prescribed and there is a Clarke subdifferential relation between the pressure and the normal components of the velocity. We prove the existence and uniqueness of weak solutions to the model by using the theories of pseudomonotone mappings and differential inclusions.

1. Introduction

This paper is an extended version of a talk presented by the author at The Third International Conference on Nonlinear Analysis and Convex Analysis which held in Tokyo, August 25-29, 2003. The goal is to report on the results of our recent studies of a class of inequality problems for Navier-Stokes equations related to the model of motion of a viscous incompressible fluid. We deal with the problem of stationary flow of inhomogeneous viscous fluid in a regular bounded domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$. The Navier-Stokes equations are the following

$$-\nu \sum_{j=1}^{d} \frac{\partial^2 u_i}{\partial x_j^2} + \sum_{j=1}^{d} u_j \frac{\partial u_i}{\partial x_j} + \frac{\partial p}{\partial x_i} = f_i \quad \text{for } i = 1, \ldots, d \quad \text{in } \Omega, \tag{1}$$

$$\sum_{j=1}^{d} \frac{\partial u_j}{\partial x_j} = 0 \quad \text{in } \Omega. \tag{2}$$

This system describes the flow of a viscous incompressible fluid which occupies the domain Ω, $u = \{u_i\}$, $i = 1, \ldots, d$ denotes the velocity of the fluid, p is the pressure, $f = \{f_i\}$ is the volume density of external forces and ν is a positive constant representing the coefficient of kinematic viscosity. Using the standard Lamb formulation we rewrite (1)-(2) in an equivalent form (see also (6)-(7) in Section 3):

$$-\nu \operatorname{rot} \operatorname{rot} u + \operatorname{rot} u \times u + \nabla h = f, \quad \operatorname{div} u = 0 \quad \text{in } \Omega, \tag{3}$$

where a function $h = p + \frac{1}{2}|u|^2$ denotes the dynamic pressure. We consider this problem under the following boundary conditions

$$h \in \partial j(x, u_N) \quad \text{and} \quad u_\tau = 0 \quad \text{on } \partial\Omega. \tag{4}$$

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Here u_N and u_τ denote the normal and the tangential component of u on the boundary, $u_N = u \cdot n$, $u_\tau = u - u_N n$, n being the unit outward normal on $\partial\Omega$ and ∂j is the Clarke subdifferential of a locally Lipschitz function $j(x, \cdot)$.

We emphasize that the subdifferential boundary condition in particular cases reduces to the classical boundary conditions (see [3, 14, 13]). If the function $j(x, \cdot)$ is assumed to be convex the problem has been studied in papers by Chebotarev [3, 4]. Next, still in a convex setting, Chebotarev [5] considered the boundary conditions (4) for the Boussinesq equations and Konovalova [9] studied the evolution counterpart of (3)-(4). In all these papers the considered problems were formulated as variational inequalities involving maximal monotone operators (recall that the subdifferential of a convex function is a maximal monotone map, cf. e.g. [16, 7]). In the present paper, due to the absence of convexity of the superpotential j, the formulation of (3)-(4) is not longer a variational inequality and it leads to the expression called hemivariational inequality. The latter have been introduced and studied by P.D. Panagiotopoulos in the early eighties as variational formulations for several classes of mechanical problems with nonsmooth and nonconvex energy superpotentials. Since that time the notion of hemivariational inequality proved to be a useful and powerful tool for formulation and solving several problems coming from mechanics and engineering. In mechanics the hemivariational inequalities express the principles of virtual work or power, see e.g. unilateral contact problems in nonlinear elasticity and viscoelasticity, problems describing frictional and adhesive effects, problem of delamination of plates, loading and unloading problems in engineering structures in Panagiotopoulos [14], Naniewicz and Panagiotopoulos [13] and Migorski [11].

The goal of the paper is to show the results on the existence and uniqueness of weak solutions to a hemivariational inequality corresponding to the problem (3)-(4). The existence will be proved by employing a surjectivity result for a pseudomonotone and coercive operator. The uniqueness is obtained under the relaxed monotonicity condition imposed on the subdifferential of the associated integral superpotential.

The paper is organized as follows. In Section 2 we recall some notation and present some auxiliary material. In Section 3 we present the formulation of the boundary value problem for the stationary Navier-Stokes equation with a subdifferential boundary condition as a hemivariational inequality. The results on the existence and uniqueness of the weak solution to the hemivariational inequality are delivered in Section 4.

2. Notation and preliminaries

In this section we introduce the notation and recall some definitions needed in the sequel.

Let E be a Banach space. In what follows we denote by $\langle \cdot, \cdot \rangle$ the duality map between E and its dual E^*. Following Clarke [6] we recall

Definition 1. Let $h\colon E \to \mathbb{R}$ be a locally Lipschitz function. The generalized directional derivative of h at $x \in E$ in the direction $v \in E$, denoted by $h^0(x; v)$, is defined by

$$h^0(x; v) = \limsup_{y \to x,\ t \downarrow 0} \frac{h(y + tv) - h(y)}{t}.$$

The generalized gradient of h at x, denoted by $\partial h(x)$, is a subset of a dual space E^* given by

$$\partial h(x) = \{\zeta \in E^* : h^0(x; v) \geq \langle \zeta, v\rangle \quad \text{for all } v \in E\}.$$

The locally Lipschitz function h is called regular (in the sense of Clarke) at $x \in E$ if for all $v \in E$ the one-sided directional derivative $h'(x; v)$ exists and satisfies $h^0(x; v) = h'(x; v)$ for all $v \in E$.

We state the chain rules theorem and for its proof we refer to Theorem 2.3.10 in Clarke [6].

Proposition 1. *Let X and Y be Banach spaces, $L \in \mathcal{L}(Y, X)$ and let $f\colon X \to \mathbb{R} \cup \{+\infty\}$ be a locally Lipschitz function. Then*

$$(i)\ (f \circ L)^0(x; z) \leq f^0(Lx; Lz) \quad \text{and} \quad (ii)\ \partial(f \circ L)(x) \subseteq L^*\partial f(Lx)$$

for x, $z \in Y$, where $L^* \in \mathcal{L}(X^*, Y^*)$ denotes the adjoint operator to L. In in addition either f or $-f$ is regular, then in both (i) and (ii) the equalities hold.

Next, given a reflexive Banach space V we recall

Definition 2. An operator $T\colon V \to V^*$ is said to be pseudomonotone if

(i) it is bounded (i.e. it maps bounded subsets of V into bounded subsets of V^*);
(ii) if $u_n \to u$ weakly in V and $\limsup\langle Tu_n, u_n - u\rangle \leq 0$, then $Tu_n \to Tu$ weakly in V^* and $\lim\langle Tu_n, u_n - u\rangle = 0$.

Definition 3. A multivalued operator $T\colon V \to 2^{V^*}$ is said to be pseudomonotone if the following conditions hold:

(i) the set Tv is nonempty, bounded, closed and convex for all $v \in V$;
(ii) T is usc from each finite dimensional subspace of V into V^* endowed with the weak topology;
(iii) if $v_n \in V$, $v_n \to v$ weakly in V and $v_n^* \in Tv_n$ is such that $\limsup\langle v_n^*, v_n - v\rangle \leq 0$, then to each $y \in V$, there exists $v^*(y) \in Tv$ such that $\langle v^*(y), v - y\rangle \leq \liminf\langle v_n^*, v_n - y\rangle$.

Definition 4. An operator $T\colon V \to 2^{V^*}$ is said to be generalized pseudomonotone if for every sequences $v_n \to v$ weakly in V, $v_n^* \to v^*$ weakly in V^*, $v_n^* \in Tv_n$ and $\limsup\langle v_n^*, v_n - v\rangle \leq 0$, we have $v^* \in Tv$ and $\langle v_n^*, v_n\rangle \to \langle v^*, v\rangle$.

Finally, we state a well known result, cf. Browder and Hess [1] and Zeidler [16].

Proposition 2. *If $T\colon V \to 2^{V^*}$ is a generalized pseudomonotone operator which is bounded and has nonempty, closed and convex values, then T is pseudomonotone.*

3. Classical model and variational formulation

In this section we present the classical stationary Navier-Stokes equations which will be considered with a subdifferential boundary condition. We give a variational formulation of this problem and applying a surjectivity result we establish the existence of weak solutions. Then we provide conditions under which the solution to our problem is unique.

Let Ω be a bounded simply connected domain in $\mathbb{R}^d$, $d = 2, 3$ with connected boundary Γ of class C^2. We consider the following system of stationary Navier-Stokes equations

$$-\nu\Delta u + (u \cdot \nabla)u + \nabla p = f, \quad \nabla \cdot u = 0 \quad \text{in } \Omega. \tag{5}$$

This system describes the steady state flow of incompressible viscous fluid occupying the volume Ω subjected to given volume forces f. Here $u = \{u_i(x)\}_{i=1}^d$ is the velocity field, p the pressure, $\nu > 0$ the kinematic viscosity of the fluid ($\nu = 1/Re$, where Re is the Reynolds number), $f = \{f_i(x)\}_{i=1}^d$ the density of external forces. The convective term $(u\cdot\nabla)u$ in (5) is a symbolic notation for the vector $\{\sum_{j=1}^{d} u_j \dfrac{\partial u_i}{\partial x_j}\}_{i=1}^d$. The divergence free condition in (5) is the equation for law of mass conservation and it states that the motion is incompressible. By using the following two identities (see Chapter I of Girault and Raviart [8])

$$(u \cdot \nabla)u = \operatorname{rot} u \times u + \frac{1}{2}\nabla(u \cdot u), \quad -\Delta u = \operatorname{rot}\operatorname{rot} u - \nabla \operatorname{div} u$$

and the incompressibility condition $\operatorname{div} u = 0$ in Ω, from (5) we have

$$\nu \operatorname{rot}\operatorname{rot} u + \operatorname{rot} u \times u + \nabla h = f \quad \text{in } \Omega, \tag{6}$$

$$\operatorname{div} u = 0 \quad \text{in } \Omega, \tag{7}$$

where the total head of the fluid, sometimes referred to as "total pressure" or "Bernoulli pressure", is given by $h = p + \frac{1}{2}|u|^2$.

We suppose that on the boundary Γ the tangential components of the velocity vector are known and without loss of generality we put them equal to zero (the so-called nonslip boundary condition):

$$u_\tau = u - u_N n = 0 \quad \text{on } \Gamma, \tag{8}$$

where $n = \{n_i\}_{i=1}^d$ is the unit outward normal on Γ and $u_N = u\cdot n = \sum u_i n_i$ denotes the normal component of the vector u. Moreover, we assume on the boundary the following subdifferential boundary condition

$$h(x) \in \partial j(x, u_N(x)) \quad \text{for } x \in \Gamma. \tag{9}$$

Here $j \colon \Gamma \times \mathbb{R} \to \mathbb{R}$ is called a superpotential and denotes the function which is locally Lipschitz in the second variable and ∂j is the subdifferential of $j(x, \cdot)$ in the sense of Clarke. The dependence of j on the first argument means that the subdifferential boundary condition can be of different character on different parts of Γ (cf. [9, 12]).

In order to give the weak formulation of the problem (6)-(9) we introduce the following notation

$$\mathcal{W} = \{w \in C^\infty(\Omega; \mathbb{R}^d) : \operatorname{div} w = 0 \text{ in } \Omega,\ w_\tau = 0 \text{ on } \Gamma\}.$$

We denote by V and H the closure of $\mathcal{W}$ in the norms of $H^1(\Omega; \mathbb{R}^d)$ and $L^2(\Omega; \mathbb{R}^d)$, respectively. Multiplying the equation of motion (6) by $v \in V$ and applying the

Green formula, we obtain

$$\nu \int_\Omega \operatorname{rot} u \cdot \operatorname{rot} v dx + \int_\Omega (\operatorname{rot} u \times u) \cdot v dx + \int_\Gamma h v_N d\sigma(x) = \int_\Omega f \cdot v dx. \tag{10}$$

Let us introduce the following operators $A\colon V \to V^*$ and $B[\cdot]\colon V \to V^*$ by

$$\langle Au, v\rangle = \nu \int_\Omega \operatorname{rot} u \cdot \operatorname{rot} v dx,$$

$$\langle B(u,v), w\rangle = \int_\Omega (\operatorname{rot} u \times v) \cdot w dx, \quad B[v] = B(v,v)$$

for $u, v, w \in V$ and let $\langle F, v\rangle = \int_\Omega f \cdot v dx$ for $v \in V$. On the other hand, from (9), by using the definition of the Clarke subdifferential we have

$$\int_\Gamma h v_N d\sigma(x) \le \int_\Gamma j^0(x, u_N(x); v_N(x)) d\sigma(x),$$

where $j^0(x, \xi; \eta)$ denotes the directional derivative of $j(x, \cdot)$ at the point $\xi \in \mathbb{R}$ in the direction $\eta \in \mathbb{R}$. Hence and from (10) we arrive to the following weak formulation of the problem: find $u \in V$ such that

$$\langle Au + B[u], v\rangle + \int_\Gamma j^0(x, u_N(x); v_N(x)) d\sigma(x) \ge \langle F, v\rangle \quad \text{for every } v \in V. \tag{11}$$

The relation (11) is called the hemivariational inequality. We have shown that the hemivariational inequality (11) is derived from (6)-(9). The following remark states that in some sense the converse statement also holds.

Remark 1. If $u \in V$ is a solution to the hemivariational inequality (11) and u is sufficiently smooth, then there exists a distribution h such that the conditions (6)-(9) hold. Indeed, since $u \in V$ from the definition of V we have $\operatorname{div} u = 0$ in Ω and $u_\tau = 0$ on Γ. Let us now take $v = \pm w$, $w \in V \cap C_0^\infty(\Omega; \mathbb{R}^d)$ in (11). Since w is arbitrary and $j^0(x, u_N; 0) = 0$, we obtain $\langle Au + B[u], w\rangle = \langle F, w\rangle$. From Proposition 1.1 in Chapter I of Temam [15] it follows that $Au + B[u] + \nabla h = F$ which implies (6). Next let $v \in V$. After multiplying the last equation by v and integrating by parts over Ω we have

$$\langle Au + B[u], v\rangle + \int_\Gamma h v_N d\sigma(x) = \langle F, w\rangle.$$

Comparing this equality with (11) entails $\int_\Gamma \left[j^0(x, u_N(x); v_N(x)) - h v_N \right] d\sigma(x) \ge 0$ for every $v \in V$. Hence we deduce $j^0(x, u_N(x); v_N(x)) \ge h v_N$ on Γ. This shows that the subdifferential condition (9) holds.

4. Main results

The aim of this section is to provide the main result of the paper on the existence of solutions to the hemivariational inequality (11). This result will be a consequence of a theorem in which we establish the existence of solutions to an abstract inclusion associated to (11).

First we recall that since Ω is supposed to be simply connected domain, by a result of Bykhovski and Smirnov [2], the bilinear form

$$((u,v))_V = \int_\Omega \operatorname{rot} u \cdot \operatorname{rot} v dx$$

generates a norm in V, $\|u\|_V = ((u,u))_V^{1/2}$, which is equivalent to the $H^1(\Omega;\mathbb{R}^d)$-norm. Therefore it is easy to deduce that the operator A satisfies the following condition:

$\underline{H(A)}$: $A\colon V \to V^*$ is a linear, bounded, symmetric operator such that

$$\langle Av, v\rangle \geq \alpha \|v\|_V^2 \quad \text{for } v \in V \text{ with } \alpha > 0.$$

Since the above mentioned norms are equivalent, it is clear that $\alpha = \nu k$ with some $k > 0$. Then we introduce the trilinear form $b\colon [H^1(\Omega;\mathbb{R}^d)]^3 \to \mathbb{R}$ defined by

$$b(u,v,w) = \langle B(u,v), w\rangle \quad \text{for } u,v,w \in H^1(\Omega;\mathbb{R}^d).$$

Similarly as in Lemmas 1.1, 1.3 and 1.5 in Chapter II of Temam [15], we can show that the form b is continuous, $b(u,v,w) = -b(u,w,v)$, $b(u,v,v) = 0$ for $u,v,w \in H^1(\Omega;\mathbb{R}^d)$ and that if $u_n \to u$ weakly in V, then

$$b(u_n,u_n,v) \to b(u,u,v) \quad \text{for all } v \in V.$$

This means that the bilinear operator $B\colon V \times V \to V^*$ satisfies:

$\underline{H(B)}$: $B[v] = B(v,v)$, $B\colon V \times V \to V^*$ is a bilinear continuous operator such that $\langle B(u,v), v\rangle = 0$ for $u,v \in V$ and the map $B[\cdot]\colon V \to V^*$ is weakly continuous.

Concerning the superpotential let us consider the functional $J\colon L^2(\Gamma;\mathbb{R}^d) \to \mathbb{R}$ defined by

$$J(v) = \int_\Gamma j(x, v_N(x))d\sigma(x) \quad \text{for } v \in L^2(\Gamma;\mathbb{R}^d) \tag{12}$$

and for its integrand j let us assume the following hypothesis:

$\underline{H(j)}$: $j\colon \Gamma \times \mathbb{R} \to \mathbb{R}$ is a function such that

(i) $j(\cdot,\xi)$ is measurable on Γ for each $\xi \in \mathbb{R}$ and $j(\cdot,0) \in L^1(\Gamma)$;
(ii) $j(x,\cdot)$ is locally Lipschitz on $\mathbb{R}$ for each $x \in \Gamma$;
(iii) $|\eta| \leq c_1(1 + |\xi|^\rho)$ for all $\eta \in \partial j(x,\xi)$, $(x,\xi) \in \Gamma \times \mathbb{R}$ with $c_1 > 0$ and $0 \leq \rho < 1$.

To continue the formulation of the problem in the form of an operator inclusion, we need to introduce an operator of the subdifferential type. To this end we define the space Z to be the closure of $\mathcal{W}$ in the norm of $H^\delta(\Omega;\mathbb{R}^d)$ with $\delta \in (\frac{1}{2}, 1)$. We have

$$V \subset Z \subset H \simeq H^* \subset Z^* \subset V^*$$

with all embeddings being dense and compact. Denoting by $i\colon V \to Z$ the embedding injection and by $\gamma\colon Z \to L^2(\Gamma;\mathbb{R}^d)$ and $\gamma_0\colon H^1(\Omega;\mathbb{R}^d) \to H^{1/2}(\Gamma;\mathbb{R}^d) \subset L^2(\Gamma;\mathbb{R}^d)$ the trace operators, for all $v \in V$ we get $\gamma_0 v = \gamma(iv)$. For simplicity we omit the notation of the embedding i and we write $\gamma_0 v = \gamma v$ for $v \in V$.

We consider the following operator inclusion:

$$\text{find } u \in V \text{ such that } Au + B[u] + \gamma^*(\partial J(\gamma u)) \ni F. \tag{13}$$

We say that an element $u \in V$ is a solution to (13) if and only if there exists $\eta \in Z^*$ such that $Au + B[u] + \eta = F$ and $\eta \in \gamma^* (\partial J(\gamma u))$.

Some properties of the nonconvex integral functional (12) are collected below (for the proof see Migorski and Ochal [12]).

Lemma 1. *Assume that the function $j\colon \Gamma \times \mathbb{R} \to \mathbb{R}$ satisfies $H(j)$. Then the functional J defined by (12) has the following properties:*

$\underline{H(J)}$: $J\colon L^2(\Gamma;\mathbb{R}^d) \to \mathbb{R}$ *is a functional such that*

(i) *J is well defined and Lipschitz on bounded subsets of $L^2(\Gamma;\mathbb{R}^d)$;*

(ii) $\|\zeta\|_{L^2(\Gamma;\mathbb{R}^d)} \le \widetilde{c}\left(1 + \|v\|^{\rho}_{L^2(\Gamma;\mathbb{R}^d)}\right)$ *for all $\zeta \in \partial J(v)$, $v \in L^2(\Gamma;\mathbb{R}^d)$ with $\widetilde{c} > 0$;*

(iii) *for all u, $v \in L^2(\Gamma;\mathbb{R}^d)$, we have*

$$J^0(u;v) \le \int_\Gamma j^0(x, u_N(x); v_N(x))d\sigma(x), \tag{14}$$

where $J^0(u;v)$ denotes the directional derivative of J at a point $u \in L^2(\Gamma;\mathbb{R}^d)$ in the direction $v \in L^2(\Gamma;\mathbb{R}^d)$.

Moreover, if additionally either j or $-j$ is regular in the sense of Clarke, then J or $-J$ is regular, respectively and the inequality (14) becomes equality.

Remark 2. If the functional J is of the form (12) and $H(j)$ holds, then every solution to (13) is also a solution to the inequality (11). Moreover, if j or $-j$ is regular, then the converse is also true. Indeed, if $u \in V$ solves (13), then for every $v \in V$, we have $\langle Au + B[u], v\rangle + \langle \eta, v\rangle_{Z^*\times Z} = \langle F, v\rangle$ with $\eta = \gamma^*\zeta$ and $\zeta \in \partial J(\gamma u)$. From the definition of the subdifferential we obtain $\langle \zeta, z\rangle_{L^2(\Gamma;\mathbb{R}^d)} \le J^0(\gamma u; z)$ for all $z \in Z$ and therefore by using $H(J)(iii)$ we get

$$\langle \eta, v\rangle_{Z^*\times Z} = \langle \gamma^*\zeta, v\rangle_{Z^*\times Z} = \langle \zeta, \gamma v\rangle_{L^2(\Gamma;\mathbb{R}^d)} \le$$
$$\le J^0(\gamma u;\gamma v) \le \int_\Gamma j^0(x, u_N(x); v_N(x))d\sigma(x) \quad \text{for every } v \in V.$$

Hence u is also a solution to (11). Now we will show that under regularity of j or $-j$ every solution to (11) solves also (13). From Lemma 1 we have

$$\langle F - Au - B[u], v\rangle \le \int_\Gamma j^0(x, u_N(x); v_N(x))d\sigma(x) = J^0(\gamma u;\gamma v).$$

By the chain rule (see Proposition 1) we get $\partial(J\circ\gamma)(v) = \gamma^* \circ \partial J(\gamma v)$ so

$$F - Au - B[u] \in \partial(J\circ\gamma)(v) = \gamma^*(\partial J(\gamma v))$$

which implies (13).

The main result of this paper reads now as follows.

Theorem 1. *Under hypothesis $H(j)$ and $f \in V^*$, the hemivariational inequality (11) admits a solution.*

Proof. It follows from Remark 2 that in order to establish the existence of solutions to (11), it is enough to obtain the existence result for the inclusion (13). The existence result for the latter is based on a surjectivity theorem (cf. Theorem 1.3.70

of Denkowski et al. [7]) for a multivalued coercive and pseudomonotone operator between a reflexive Banach space and its dual. To this end we define the operator $\mathcal{F}\colon V \to 2^{V^*}$ by

$$\mathcal{F}v = Av + B[v] + \gamma^* \left(\partial J(\gamma v)\right) \quad \text{for } v \in V.$$

We will show that $\mathcal{F}$ is coercive and pseudomonotone. We begin with the properties of the operator $T\colon Z \to 2^{Z^*}$ given by

$$Tz = \gamma^* \left(\partial J(\gamma z)\right) \quad \text{for } z \in Z. \tag{15}$$

Claim: The operator T satisfies the following

(i) The values of T are nonempty, convex and weakly compact subsets of Z^*;
(ii) The graph of T is closed in $Z \times (w\text{–}Z^*)$ topology;
(iii) $\|Tz\|_{Z^*} \le \overline{c}\left(1 + \|z\|_Z^\rho\right)$ for all $z \in Z$ with $\overline{c} > 0$,

where $(w\text{–}Z^*)$ denotes the space Z^* equipped with the weak topology.

Indeed, the nonemptiness and convexity of values of T follow immediately from the analogous properties of the Clarke subdifferential. It can be also easily proved that the values of T are weakly compact.

Let $\{z_n\} \subset Z$, $\{z_n^*\} \subset Z^*$ be such that $z_n^* \in Tz_n$, $z_n \to z$ in Z and $z_n^* \to z^*$ weakly in Z^*. We will show that $z^* \in Tz$. By assumption we have $z_n^* = \gamma^* w_n$ and $w_n \in \partial J(\gamma z_n)$. Using the fact that $\partial J\colon L^2(\Gamma;\mathbb{R}^d) \to 2^{L^2(\Gamma;\mathbb{R}^d)}$ is a bounded map (cf. $H(J)(ii)$), we may assume that $w_n \to w_0$ weakly in $L^2(\Gamma;\mathbb{R}^d)$. Hence $z_n^* = \gamma^* w_n \to \gamma^* w_0 = z^*$ weakly in Z^*. From the closedness of the graph of ∂J in $L^2(\Gamma;\mathbb{R}^d) \times (w\text{–}L^2(\Gamma;\mathbb{R}^d))$ topology (cf. [6]), passing to the limit in the relation $w_n \in \partial J(\gamma z_n)$ we obtain $w_0 \in \partial J(\gamma z)$. This together with $z^* = \gamma^* w_0$ implies $z^* \in \gamma^*(\partial J(\gamma z)) = Tz$ and proves the closedness of the graph of T in $Z \times (w\text{–}Z^*)$ topology.

By using $H(J)$(ii), for all $z \in Z$ we have

$$\|Tz\|_{Z^*} \le \|\gamma^*\| \|\partial J(\gamma z)\|_{L^2(\Gamma;\mathbb{R}^d)} \le \|\gamma^*\| \widetilde{c} \left(1 + \|\gamma z\|^\rho_{L^2(\Gamma;\mathbb{R}^d)}\right) \le$$
$$\le \widetilde{c} \|\gamma^*\| \left(1 + \|\gamma\|^\rho \|z\|_Z^\rho\right) \le \widehat{c}\left(1 + \|z\|_Z^\rho\right)$$

with a positive constant $\widehat{c} > 0$, where $\|\gamma\| = \|\gamma^*\| = \|\gamma\|_{\mathcal{L}(Z;L^2(\Gamma;\mathbb{R}^d))}$. This shows that (iii) holds and completes the proof of the claim.

Next, for the proof of coerciveness of $\mathcal{F}$, we observe that by $H(A)$ and $H(B)$, we have

$$\langle \mathcal{F}v, v\rangle = \langle Av, v\rangle + \langle B(v,v), v\rangle + \langle \zeta, v\rangle_{Z^* \times Z} \ge \alpha \|v\|_V^2 + \langle \zeta, v\rangle_{Z^* \times Z} \quad \text{for all } v \in V$$

with $\zeta \in Tv$. By the condition (iii) of Claim, we easily get

$$|\langle \zeta, v\rangle_{Z^* \times Z}| \le \|\zeta\|_{Z^*} \|v\|_Z \le \overline{c}\left(1 + \|v\|_Z^\rho\right) \|v\|_Z =$$
$$= \overline{c}\|v\|_Z + \overline{c}\|v\|_Z^{\rho+1} \le \overline{c}\beta\|v\|_V + \overline{c}\beta^{\rho+1}\|v\|_V^{\rho+1},$$

where $\beta > 0$ is such that $\|\cdot\|_Z \le \beta \|\cdot\|_V$. Hence

$$\langle \zeta, v\rangle_{Z^* \times Z} \ge -\overline{c}\beta\|v\|_V - \overline{c}\beta^{\rho+1}\|v\|_V^{\rho+1}.$$

Since $0 \le \rho < 1$ it follows that the map $\mathcal{F}$ is coercive.

We will show now the pseudomonotonicity of $\mathcal{F}$. This will be done by applying Proposition 2. From (i) of Claim, it is clear that $\mathcal{F}$ has nonempty, closed and convex values. Moreover, from the condition (iii) of Claim, $H(A)$ and $H(B)$, we conclude that $\mathcal{F}$ is a bounded map. In order to prove the theorem, it is sufficient to show that $\mathcal{F}$ is a generalized pseudomonotone operator. For this, let us assume $v_n \to v$ weakly in V, $v_n^* \to v^*$ weakly in V^*, $v_n^* \in \mathcal{F}v_n$ and $\limsup\langle v_n^*, v_n - v\rangle \le 0$. We will show that $v^* \in \mathcal{F}v$ and $\langle v_n^*, v_n\rangle \to \langle v^*, v\rangle$. Since $v_n^* \in \mathcal{F}v_n$ we have $v_n^* = Av_n + B[v_n] + \zeta_n$ with $\zeta_n \in Tv_n$. The continuity of the embedding $V \subset Z$ implies that

$$v_n \to v \quad \text{in } Z. \tag{16}$$

Hence and from the boundedness of T (see again (iii) of Claim), we may suppose, by passing to a subsequence if necessary, that

$$\zeta_n \to \zeta \quad \text{weakly in } Z^* \text{ with } \zeta \in Z^*. \tag{17}$$

So the property (ii) of Claim implies $\zeta \in Tv$. Next, by the equality

$$\langle v_n^*, v_n - v\rangle = \langle Av_n, v_n - v\rangle + \langle B[v_n], v_n - v\rangle + \langle \zeta_n, v_n - v\rangle_{Z^*\times Z}$$

by using (17) and (16), we obtain

$$\lim\left(\langle Av_n, v_n - v\rangle + \langle B[v_n], v_n - v\rangle\right) = \lim\langle v_n^*, v_n - v\rangle \le 0. \tag{18}$$

On the other hand by $H(B)$ we have

$$\langle B[v_n], v_n - v\rangle = \langle B[v_n], v_n\rangle - \langle B[v_n], v\rangle = -\langle B[v_n], v\rangle \to -\langle B[v], v\rangle = 0.$$

Hence and from (18) we deduce

$$\limsup\langle Av_n, v_n - v\rangle = \limsup\langle Av_n, v_n - v\rangle + \lim\langle B[v_n], v_n - v\rangle =$$

$$= \limsup\langle Av_n + B[v_n], v_n - v\rangle \le 0.$$

This clearly yields (cf. (ii) of Definition 2)

$$Av_n \to Av \quad \text{weakly in } V^* \tag{19}$$

and $\lim\langle Av_n, v_n - v\rangle = 0$. Exploiting (17) and (19), and passing to the limit in the equality $v_n^* = Av_n + B[v_n] + \zeta_n$ we have $v^* = Av + B[v] + \zeta$ which together with $\zeta \in Tv$ implies that $v^* \in Tv$.

Finally, from (17)–(19), we deduce

$$\lim\langle v_n^*, v_n\rangle = \lim\langle Av_n, v_n\rangle + \lim\langle B[v_n], v_n\rangle + \lim\langle \zeta_n, v_n\rangle_{Z^*\times Z} =$$

$$= \langle Av + B[v], v\rangle + \langle \zeta, v\rangle_{Z^*\times Z} = \langle v^*, v\rangle$$

which completes the proof of the generalized pseudomonotonicity of $\mathcal{F}$. The proof of the theorem is finished. □

We now comment on the uniqueness of solutions to the inclusion (13). We need an additional hypothesis on the functional J.

$\underline{H(J)_1}$: $J\colon L^2(\Gamma;\mathbb{R}^d) \to \mathbb{R}$ satisfies $H(J)$ and the following relaxed monotonicity condition: $(z_1 - z_2, w_1 - w_2)_{L^2(\Gamma;\mathbb{R}^d)} \ge -m\|w_1 - w_2\|^2_{L^2(\Gamma;\mathbb{R}^d)}$ for all $z_i \in \partial J(w_i)$, $w_i \in L^2(\Gamma;\mathbb{R}^d)$, $i = 1, 2$ with $m > 0$.

Proposition 3. *Let the operators A and B satisfy $H(A)$ and $H(B)$, let $H(J)$ hold, $f \in V^*$ and let $u \in V$ be a solution to (13). Then there exists a constant $C > 0$ such that $\|u\|_V \leq C$. Moreover, if $H(J)_1$ holds and $\alpha - m\beta^2\|\gamma\|^2 - c_b C > 0$, where $c_b > 0$ is the continuity constant of the form b associated to the operator B, then the solution to problem (13) is unique.*

We remark that when $J \equiv 0$, the uniqueness of solutions was obtained in Theorem 1.3, p.167 of Temam [15]

Finally, we mention that the existence and uniqueness results can be obtained also in the case when $\rho = 1$ in $H(j)(iii)$. For these results, the examples of functionals which satisfy hypothesis $H(J)_1$ and for the issues concerning the stability of the solution set of the hemivariational inequality with respect to perturbations in the boundary conditions, we refer to Migorski and Ochal [12].

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