



STRONG CONVERGENCE OF MANN'S TYPE SEQUENCES FOR ONE-PARAMETER NONEXPANSIVE SEMIGROUPS IN GENERAL BANACH SPACES

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ABSTRACT. Let C be a compact convex subset of a Banach space E (which may not be strictly convex) and let $\{T(t) : t \geq 0\}$ be a one-parameter nonexpansive semigroup on C . The purpose of this paper is to study the strong convergence of a sequence $\{x_n\}$ in C generated by $x_1 = x \in C$ and

$$x_{n+1} = \frac{\alpha_n}{t_n} \int_0^{t_n} T(s)x_n ds + (1 - \alpha_n)x_n, \quad n = 1, 2, \dots,$$

where $\{\alpha_n\}$ and $\{t_n\}$ are sequences in $[0, 1]$ and $(0, \infty)$, respectively.

1. INTRODUCTION

In 1982, Miyadera and Kobayasi [5] proved the following nonlinear ergodic theorem for a one-parameter nonexpansive semigroup $\{T(t) : t \geq 0\}$ defined on a bounded closed convex subset C of a uniformly convex Banach space E with a Fréchet differentiable norm. For each $x \in C$,

$$(1) \quad \frac{1}{t} \int_0^t T(s)x ds$$

converges weakly to a common fixed point of $\{T(t) : t \geq 0\}$; for a more general result, see [3]. Recently, in the case when E is strictly convex and C is compact and convex, Atsushiba and Takahashi [1] proved that (1) converges strongly to a common fixed point of $\{T(t) : t \geq 0\}$. However, we do not know whether this theorem would hold without strict convexity; see [7].

In this paper, we define an iteration of Mann's type for a one-parameter nonexpansive semigroup and then study the strong convergence of the sequence in a Banach space without strict convexity. One of main results is as follows: Define a sequence $\{x_n\}$ by $x_1 \in C$ and

$$x_{n+1} = \frac{\alpha_n}{t_n} \int_0^{t_n} T(s)x_n ds + (1 - \alpha_n)x_n, \quad n = 1, 2, \dots,$$

where $\{\alpha_n\} \subset [0, 1]$ and $\{t_n\} \subset (0, \infty)$ satisfy the following conditions:

$$0 < \liminf_{n \rightarrow \infty} \alpha_n \leq \limsup_{n \rightarrow \infty} \alpha_n < 1, \quad \lim_{n \rightarrow \infty} t_n = \infty, \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{t_{n+1}}{t_n} = 1.$$

Then $\{x_n\}$ converges strongly to a common fixed point z_0 of $\{T(t) : t \geq 0\}$.

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2. LEMMAS

Throughout this paper, we denote by \mathbb{N} the set of positive integers. In this section, we give two lemmas, which play important roles in the proofs of theorems in Section 3.

Lemma 1 ([6]). *Let $\{z_n\}$ and $\{w_n\}$ be bounded sequences in a Banach space E and let $\{\alpha_n\}$ be a sequence in $(0, 1)$ with $0 < \liminf_n \alpha_n \leq \limsup_n \alpha_n < 1$. Suppose that $z_{n+1} = \alpha_n w_n + (1 - \alpha_n) z_n$ for all $n \in \mathbb{N}$ and*

$$\limsup_{n \rightarrow \infty} (\|w_n - w_{n+k}\| - \|z_n - z_{n+k}\|) \leq 0$$

for all $k \in \mathbb{N}$. Then $\liminf_n \|w_n - z_n\| = 0$.

Lemma 2. *Let A and B be measurable subsets of $[0, \infty)$ and let $\{t_n\}$ be a sequence in $(0, \infty)$ with $\lim_n t_n = \infty$. Suppose that*

$$\lim_{n \rightarrow \infty} \frac{\mu([0, t_n] \cap A)}{t_n} = 1 \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{\mu([0, t_n] \cap B)}{t_n} = 1,$$

where μ is the Lebesgue measure. Then

$$\lim_{n \rightarrow \infty} \frac{\mu([0, t_n] \cap A \cap B)}{t_n} = 1$$

and $[t, \infty) \cap A \cap B \neq \emptyset$ for all $t > 0$.

Proof. From the assumption, we have

$$\begin{aligned} & \liminf_{n \rightarrow \infty} \frac{\mu([0, t_n] \cap A \cap B)}{t_n} \\ &= \liminf_{n \rightarrow \infty} \frac{\mu([0, t_n] \cap A) + \mu([0, t_n] \cap B) - \mu([0, t_n] \cap (A \cup B))}{t_n} \\ &\geq \lim_{n \rightarrow \infty} \frac{\mu([0, t_n] \cap A) + \mu([0, t_n] \cap B) - t_n}{t_n} \\ &= 1. \end{aligned}$$

It is obvious that

$$\limsup_{n \rightarrow \infty} \frac{\mu([0, t_n] \cap A \cap B)}{t_n} \leq 1.$$

Fix $t > 0$. Then there exists $n \in \mathbb{N}$ such that $t_n \geq 2t$ and

$$\frac{\mu([0, t_n] \cap A \cap B)}{t_n} > \frac{1}{2}.$$

From

$$\mu([0, t_n] \cap A \cap B) > \frac{1}{2} t_n \geq t = \mu([0, t]),$$

we have

$$\begin{aligned} 0 &< \mu([0, t_n] \cap A \cap B \setminus [0, t]) \\ &= \mu([t, t_n] \cap A \cap B) \\ &\leq \mu([t, \infty) \cap A \cap B). \end{aligned}$$

This completes the proof. \square

3. MAIN RESULTS

In this section, we prove our main results. Let C be a subset of a Banach space E . A family $\{T(t) : t \geq 0\}$ of mappings of C into itself is called a one-parameter nonexpansive semigroup on C if the following hold:

- (1) For each $x \in C$, the mapping $T(\cdot)x$ is continuous;
- (2) $T(0)x = x$ for all $x \in C$;
- (3) $T(s + t) = T(s) \circ T(t)$ for all $s, t \geq 0$;
- (4) for each $t \geq 0$, $T(t)$ is nonexpansive.

Now, we put

$$M(t, x) = \frac{1}{t} \int_0^t T(s)x \, ds$$

for $t > 0$ and $x \in C$. Note that a mapping $M(t, \cdot)$ on C is nonexpansive for each $t > 0$ because

$$\begin{aligned} \|M(t, x) - M(t, y)\| &= \frac{1}{t} \left\| \int_0^t (T(s)x - T(s)y) \, ds \right\| \\ &\leq \frac{1}{t} \int_0^t \|T(s)x - T(s)y\| \, ds \\ &\leq \frac{1}{t} \int_0^t \|x - y\| \, ds \\ &= \|x - y\| \end{aligned}$$

for all $x, y \in C$.

Theorem 1. *Let C be a compact convex subset of a Banach space E and let $\{T(t) : t \geq 0\}$ be a one-parameter nonexpansive semigroup on C . If $z \in C$ satisfies*

$$\liminf_{t \rightarrow \infty} \left\| \frac{1}{t} \int_0^t T(s)z \, ds - z \right\| = 0,$$

then $z \in \bigcap_{t \geq 0} F(T(t))$.

Before proving Theorem 1, we prove one lemma. For $z \in C$, we put

$$\ell = \limsup_{t \rightarrow \infty} \|T(t)z - z\|$$

and assume $\ell > 0$. Further, put

$$A = \bigcap_{t > 0} C(t),$$

where $C(t)$ is the closure of $\{T(s)z : s \geq t\}$. For $u \in C$, $p \in [0, \infty)$, $q \in (0, \infty]$ with $p < q$, and $\varepsilon \in (0, \ell)$, we also put

$$B(u, p, q, \varepsilon) = \{t \in [p, q) : \|T(t)z - u\| \geq \ell - \varepsilon\}.$$

Lemma 3. *Let U be a finite subset of A . Suppose*

$$B(z, t, \infty, \varepsilon) \cap \left(\bigcap_{u \in U} B(u, t, \infty, \varepsilon) \right) \neq \emptyset$$

for all $t \in (0, \infty)$ and $\varepsilon \in (0, \ell)$. Then there exists $v \in A$ such that $\|v - z\| = \ell$ and $\|v - u\| \geq \ell$ for all $u \in U$.

Proof. For $u \in \{z\} \cup U$ and $\varepsilon \in (0, \ell)$, define

$$B(u, \varepsilon) = \{x \in C : \|x - u\| \geq \ell - \varepsilon\}.$$

Then, by the assumption, the family of closed subsets of C consisting of

$$\{C(t) : t > 0\} \quad \text{and} \quad \{B(u, \varepsilon) : u \in \{z\} \cup U, \varepsilon \in (0, \ell)\}$$

has the finite intersection property. So, there exists a point $v \in C$ such that $v \in A$ and $\|v - u\| \geq \ell$ for all $u \in \{z\} \cup U$. We can also obtain a sequence $\{t_n\}$ such that $\lim_n t_n = \infty$ and $\lim_n T(t_n)z = v$. Then we have

$$\|v - z\| = \lim_{n \rightarrow \infty} \|T(t_n)z - z\| \leq \ell$$

and hence $\|v - z\| = \ell$. □

Proof of Theorem 1. Assume $\ell > 0$. By the definition of ℓ , there exists a sequence $\{t_n\}$ such that $\lim_n t_n = \infty$ and $\lim_n \|T(t_n)z - z\| = \ell$. Since C is compact, there exists a subsequence $\{t_{n_i}\}$ of $\{t_n\}$ such that $\{T(t_{n_i})z\}$ converges strongly to $u_1 \in C$. Then, we get $u_1 \in A$ and $\|u_1 - z\| = \ell$. We also have by the assumption that there exists an increasing sequence $\{t_n\}$ in $(0, \infty)$ such that $\lim_n t_n = \infty$ and $\{M(t_n, z)\}$ converges strongly to z . We first show

$$(2) \quad \lim_{n \rightarrow \infty} \frac{\mu(B(u, 0, t_n, \varepsilon))}{t_n} = 1$$

for $u \in A$ with $\|u - z\| = \ell$ and $\varepsilon \in (0, \ell)$. Fix $\varepsilon \in (0, \ell)$. For an arbitrary $\delta > 0$, from $\limsup_t \|T(t)z - z\| = \ell$, we obtain $s_0 \in [0, \infty)$ such that $\|T(t)z - z\| \leq \ell + \delta$ for all $t \in [s_0, \infty)$. Further, from $u \in A$, we can choose $s_1 \in [s_0, \infty)$ such that $\|T(s_1)z - u\| \leq \delta$. Then for $t > 2s_1$, we have

$$\begin{aligned} \|T(t)z - u\| &\leq \|T(t)z - T(s_1)z\| + \|T(s_1)z - u\| \\ &\leq \|T(t - s_1)z - z\| + \|T(s_1)z - u\| \\ &\leq \ell + 2\delta. \end{aligned}$$

Take t_n with $t_n > 2s_1$ and put $D = 2 \cdot \sup\{\|y\| : y \in C\}$. Then from

$$\begin{aligned} \ell &= \|z - u\| \\ &\leq \|z - M(t_n, z)\| + \|M(t_n, z) - u\| \\ &= \|z - M(t_n, z)\| + \left\| \frac{1}{t_n} \int_0^{t_n} (T(t)z - u) dt \right\| \\ &\leq \|z - M(t_n, z)\| + \frac{1}{t_n} \int_0^{t_n} \|T(t)z - u\| dt \\ &\leq \|z - M(t_n, z)\| + \frac{2s_1}{t_n} D + \frac{1}{t_n} \int_{2s_1}^{t_n} \|T(t)z - u\| dt \end{aligned}$$

and

$$\begin{aligned}
& \frac{1}{t_n} \int_{2s_1}^{t_n} \|T(t)z - u\| dt \\
&= \frac{1}{t_n} \int_{B(u, 2s_1, t_n, \varepsilon)} \|T(t)z - u\| dt + \frac{1}{t_n} \int_{[2s_1, t_n] \setminus B(u, 2s_1, t_n, \varepsilon)} \|T(t)z - u\| dt \\
&\leq \frac{1}{t_n} \mu(B(u, 2s_1, t_n, \varepsilon)) (\ell + 2\delta) + \frac{1}{t_n} \mu([2s_1, t_n] \setminus B(u, 2s_1, t_n, \varepsilon)) (\ell - \varepsilon) \\
&\leq \frac{1}{t_n} \mu(B(u, 0, t_n, \varepsilon)) (\ell + 2\delta) + \frac{1}{t_n} \mu([0, t_n] \setminus B(u, 0, t_n, \varepsilon)) (\ell - \varepsilon) \\
&= \frac{1}{t_n} \mu(B(u, 0, t_n, \varepsilon)) (\ell + 2\delta) + \frac{1}{t_n} \left(t_n - \mu(B(u, 0, t_n, \varepsilon)) \right) (\ell - \varepsilon) \\
&= \ell - \varepsilon + \frac{1}{t_n} \mu(B(u, 0, t_n, \varepsilon)) (\varepsilon + 2\delta),
\end{aligned}$$

we have

$$\ell \leq \|z - M(t_n, z)\| + \frac{2s_1}{t_n} D + \ell - \varepsilon + \frac{\mu(B(u, 0, t_n, \varepsilon))}{t_n} (\varepsilon + 2\delta).$$

So, we have

$$\liminf_{n \rightarrow \infty} \frac{\mu(B(u, 0, t_n, \varepsilon))}{t_n} \geq \lim_{n \rightarrow \infty} \frac{-\|z - M(t_n, z)\| - 2s_1 D/t_n + \varepsilon}{\varepsilon + 2\delta} = \frac{\varepsilon}{\varepsilon + 2\delta}.$$

Since $\delta > 0$ is arbitrary, we obtain (2). For each $\varepsilon \in (0, \ell)$, there exists $s_2 \in [0, \infty)$ such that $\|T(s_2)z - u_1\| \leq \varepsilon/2$. So, if $t_n > s_2$ and $t \in B(u_1, s_2, t_n, \varepsilon/2)$, we have

$$\begin{aligned}
\|T(t - s_2)z - z\| &\geq \|T(t)z - T(s_2)z\| \\
&\geq \|T(t)z - u_1\| - \|T(s_2)z - u_1\| \\
&\geq \ell - \varepsilon
\end{aligned}$$

and hence

$$\begin{aligned}
\mu(B(z, 0, t_n, \varepsilon)) &\geq \mu(\{t - s_2 : t \in B(u_1, s_2, t_n, \varepsilon/2)\}) \\
&= \mu(B(u_1, s_2, t_n, \varepsilon/2)) \\
&= \mu(B(u_1, 0, t_n, \varepsilon/2) \setminus [0, s_2]) \\
&\geq \mu(B(u_1, 0, t_n, \varepsilon/2)) - s_2.
\end{aligned}$$

So, we obtain

$$\liminf_{n \rightarrow \infty} \frac{\mu(B(z, 0, t_n, \varepsilon))}{t_n} \geq \lim_{n \rightarrow \infty} \frac{\mu(B(u_1, 0, t_n, \varepsilon/2)) - s_2}{t_n} = 1$$

and hence $\lim_n \mu(B(z, 0, t_n, \varepsilon))/t_n = 1$ for $\varepsilon \in (0, \ell)$. We next find a sequence $\{u_m\}$ in A satisfying $\|u_i - z\| = \ell$ and $\|u_i - u_j\| \geq \ell$ for $i \neq j$. If we find u_1, u_2, \dots, u_m , then we can find u_{m+1} as follows: Since $\lim_n \mu(B(u_i, 0, t_n, \varepsilon))/t_n = 1$ for $i \in \{1, 2, \dots, m\}$ and $\varepsilon \in (0, \ell)$, by Lemma 2, we have

$$\lim_{n \rightarrow \infty} \frac{\mu(B(z, 0, t_n, \varepsilon) \cap (\bigcap_{i=1}^m B(u_i, 0, t_n, \varepsilon)))}{t_n} = 1$$

and

$$B(z, t, \infty, \varepsilon) \cap \left(\bigcap_{i=1}^m B(u_i, t, \infty, \varepsilon) \right) \neq \emptyset$$

for all $t \in [0, \infty)$ and $\varepsilon \in (0, \ell)$. By Lemma 3, we can find $u_{m+1} \in A$ such that $\|u_{m+1} - z\| = \ell$ and $\|u_{m+1} - u_i\| \geq \ell$ for $i \in \{1, 2, \dots, m\}$. Since $\{u_n\} \subset A$ is a sequence in a compact set C , there exists a convergent subsequence of $\{u_n\}$. This is a contradiction. Hence we have $\ell = 0$. So, we have

$$T(t)z = T(t) \left(\lim_{s \rightarrow \infty} T(s)z \right) = \lim_{s \rightarrow \infty} T(t+s)z = z$$

for all $t \in [0, \infty)$. This completes the proof. □

Using Theorem 1, we prove the following.

Theorem 2. *Let C be a compact convex subset of a Banach space E and let $\{T(t) : t \geq 0\}$ be a one-parameter nonexpansive semigroup on C . Let $x_1 \in C$ and define a sequence $\{x_n\}$ in C by*

$$x_{n+1} = \frac{\alpha_n}{t_n} \int_0^{t_n} T(s)x_n ds + (1 - \alpha_n)x_n$$

for $n \in \mathbb{N}$, where $\{\alpha_n\} \subset [0, 1]$ and $\{t_n\} \subset (0, \infty)$ satisfy the following conditions:

$$0 < \liminf_{n \rightarrow \infty} \alpha_n \leq \limsup_{n \rightarrow \infty} \alpha_n < 1, \quad \lim_{n \rightarrow \infty} t_n = \infty, \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{t_{n+1}}{t_n} = 1.$$

Then $\{x_n\}$ converges strongly to a common fixed point z_0 of $\{T(t) : t \geq 0\}$.

Proof. We know by [2, 8] that $\bigcap_{t \geq 0} F(T(t))$ is nonempty. Fix $w \in \bigcap_{t \geq 0} F(T(t))$. Since

$$\begin{aligned} (3) \quad \|x_{n+1} - w\| &\leq \alpha_n \|M(t_n, x_n) - w\| + (1 - \alpha_n) \|x_n - w\| \\ &\leq \alpha_n \|x_n - w\| + (1 - \alpha_n) \|x_n - w\| \\ &= \|x_n - w\| \end{aligned}$$

for $n \in \mathbb{N}$. So, we have that $\lim_n \|x_n - w\|$ exists. Fix $k, n \in \mathbb{N}$, and put

$$a = \min\{t_n, t_{n+k}\}, \quad b = \max\{t_n, t_{n+k}\}, \quad \text{and} \quad D = \sup_{y \in C} \|y\| < \infty.$$

Then we have

$$\begin{aligned} &\|M(t_n, x_n) - M(t_{n+k}, x_{n+k})\| - \|x_n - x_{n+k}\| \\ &\leq \|M(t_n, x_n) - M(t_n, x_{n+k})\| + \|M(t_n, x_{n+k}) - M(t_{n+k}, x_{n+k})\| - \|x_n - x_{n+k}\| \\ &\leq \|x_n - x_{n+k}\| + \|M(t_n, x_{n+k}) - M(t_{n+k}, x_{n+k})\| - \|x_n - x_{n+k}\| \\ &= \|M(t_n, x_{n+k}) - M(t_{n+k}, x_{n+k})\| \\ &= \left\| \frac{1}{t_n} \int_0^{t_n} T(s)x_{n+k} ds - \frac{1}{t_{n+k}} \int_0^{t_{n+k}} T(s)x_{n+k} ds \right\| \\ &\leq \left(\frac{1}{a} - \frac{1}{b} \right) \left\| \int_0^a T(s)x_{n+k} ds \right\| + \frac{1}{b} \left\| \int_a^b T(s)x_{n+k} ds \right\| \end{aligned}$$

$$\begin{aligned} &\leq \left(\frac{a}{a} - \frac{a}{b} + \frac{b-a}{b} \right) D \\ &= \left(2 - 2 \frac{\min\{t_n, t_{n+k}\}}{\max\{t_n, t_{n+k}\}} \right) D. \end{aligned}$$

By the assumption and induction, we get that

$$\lim_{n \rightarrow \infty} \frac{\min\{t_n, t_{n+k}\}}{\max\{t_n, t_{n+k}\}} = 1$$

for all $k \in \mathbb{N}$. So, we have

$$\limsup_{n \rightarrow \infty} (\|M(t_n, x_n) - M(t_{n+k}, x_{n+k})\| - \|x_n - x_{n+k}\|) \leq 0$$

for all $k \in \mathbb{N}$. By Lemma 1, we obtain $\liminf_n \|M(t_n, x_n) - x_n\| = 0$. It follows from the compactness of C that there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\lim_k \|M(t_{n_k}, x_{n_k}) - x_{n_k}\| = 0$ and $\{x_{n_k}\}$ converges strongly to some point $z_0 \in C$. Since

$$\begin{aligned} &\limsup_{k \rightarrow \infty} \|M(t_{n_k}, z_0) - z_0\| \\ &\leq \limsup_{k \rightarrow \infty} (\|M(t_{n_k}, z_0) - M(t_{n_k}, x_{n_k})\| + \|M(t_{n_k}, x_{n_k}) - x_{n_k}\| + \|x_{n_k} - z_0\|) \\ &\leq \limsup_{k \rightarrow \infty} (2\|x_{n_k} - z_0\| + \|M(t_{n_k}, x_{n_k}) - x_{n_k}\|) = 0, \end{aligned}$$

we obtain

$$\liminf_{t \rightarrow \infty} \|M(t, z_0) - z_0\| = \lim_{k \rightarrow \infty} \|M(t_{n_k}, z_0) - z_0\| = 0.$$

So, by Theorem 1, we have $z_0 \in \bigcap_{t \geq 0} F(T(t))$. Since $\|x_{n+1} - z_0\| \leq \|x_n - z_0\|$ for $n \in \mathbb{N}$, we obtain $\lim_n \|x_n - z_0\| = \lim_k \|x_{n_k} - z_0\| = 0$. This completes the proof. \square

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