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REMARKS ON $(0, k)$-EPI MAPS WITH NONZERO DEGREE

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Abstract. We prove that a map of the form $id - f$ with a countably ℓ-condensing operator f is $(0, k)$-epi for $k \leq 1 - \ell$ if and only if it has nonzero degree on a bounded open and connected set in a Banach space. It is based on a result of M. Väth [5] on 0-epi maps.

1. Introduction

The study of 0-epi maps was initiated by M. Furi, M. Martelli, and A. Vignoli [4]. 0-epi maps have analogous properties to that of degree, such as existence, normalization, and homotopy invariance. The theory of 0-epi maps can be applied to solvability of boundary value problems and a spectral theory for nonlinear operators; see [3,4,6,8].

It is well-known that a map of the form $id - f$ is 0-epi if and only if it has nonzero degree whenever $f : \overline{\Omega} \to X$ is compact on a Jordan domain Ω in an infinite-dimensional Banach space X. In [12] it is shown that this still holds for countably 1/2-condensing maps f. More generally, M. Väth [5] proved that a map $id - f$ with a strictly countably condensing operator f is 0-epi if and only if it has nonzero degree on a component of its domain of definition.

The class of 0-epi maps is not stable under noncompact perturbations. To reduce the gap, the concept of $(0, k)$-epi maps was introduced by E.U. Tarafdar and H.B. Thompson [9]. The aim of this paper is to establish a connection between $(0, k)$-epi maps and degree theory. To do this, we introduce a variant of the notion of $(0, k)$-epi maps and a degree of countably condensing maps due to M. Väth [5,10,11]. We show that a map of the form $id - f$ with a countably ℓ-condensing operator f is $(0, k)$-epi for $k \leq 1 - \ell$ if and only if it has nonzero degree on a bounded open and connected set in a Banach space. Analyzing a simple example leads us to facilitate the proof of the sufficiency. For the necessity we follow the basic line of the proof in [5].

Given a nonempty subset Ω of a metric space X, the closure and the boundary of Ω in X are denoted by $\overline{\Omega}$ and $\partial\Omega$, respectively. For a metric space Y, a continuous map $f : \overline{\Omega} \to Y$ is said to be *compact* if its range $f(\overline{\Omega})$ is contained in a compact subset of Y.

Let X and Y be normed spaces and Ω a bounded open subset of X. A continuous map $F : \overline{\Omega} \to Y$ is said to be 0-*epi* on Ω if

(1) $F(x) \neq 0$ for all $x \in \partial\Omega$; and

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(2) for any compact map $h : \overline{\Omega} \to Y$ with $h(x) = 0$ for all $x \in \partial\Omega$, the equation $F(x) = h(x)$ has a solution in Ω.

Let (X, d) be a metric space. Given a bounded set $A \subset X$, the *Kuratowski measure of noncompactness* of A, $\alpha(A)$, is defined as the infimum of all $\varepsilon > 0$ such that A can be covered by a finite number of sets of diameter less than ε; see [1,2].

Let X and Y be metric spaces and Ω a nonempty subset of X. Given a real number $k \geq 0$, a continuous map $f : \overline{\Omega} \to Y$ is said to be a *k-set contraction* if $\alpha(f(A)) \leq k\alpha(A)$ for each bounded subset A of Ω. Given $k > 0$, a continuous map f is called *k-condensing* (with respect to α) if $\alpha(f(A)) < k\alpha(A)$ for each bounded subset A of Ω with $\alpha(A) > 0$. More generally, a continuous map f is called *countably k-condensing* (with respect to α) if $\alpha(f(A)) < k\alpha(A)$ for each countable bounded subset A of Ω with $\alpha(A) > 0$. In case when $k = 1$, f is usually called *condensing* and *countably condensing*, respectively.

Note that every compact map is k-condensing for any $k > 0$. Any k-condensing map $f : \overline{\Omega} \to Y$ is a k-set contraction when X is a complete metric space. But the converse is not true in general. An example of a 1-set contraction which is not 1-condensing can be found in [2, Example II.7].

In this regard, we introduce a variant of the notion of $(0, k)$-epi map due to E.U. Tarafdar and H.B. Thompson [9], where the condition h is "k-set contractive" in [9] is replaced by "k-condensing".

Let X and Y be Banach spaces and Ω a bounded open subset of X. A continuous map $F : \overline{\Omega} \to Y$ is said to be *$(0, k)$-epi* on Ω if

(1) $F(x) \neq 0$ for all $x \in \partial\Omega$; and
(2) for any k-condensing map $h : \overline{\Omega} \to Y$ with $h(x) = 0$ for all $x \in \partial\Omega$, the equation $F(x) = h(x)$ has a solution in Ω.

We give the following simple example which clarifies the definition.

Example 1.1. Let Ω be the open unit ball in a Banach space X and $0 < c < 1$. If a map $f : \overline{\Omega} \to X$ is defined by

$$f(x) := cx \quad \text{for } x \in \overline{\Omega}$$

then $F = id - f$ is $(0, k)$-epi on Ω for any $k \leq 1 - c$.

Proof. Let $\varphi : \overline{\Omega} \to X$ be any k-condensing map such that $\varphi(x) = 0$ for all $x \in \partial\Omega$. Consider a homotopy $h : [0, 1] \times \overline{\Omega} \to X$ defined by

$$h(t, x) := f(x) + t\varphi(x) \quad \text{for } (t, x) \in [0, 1] \times \overline{\Omega}.$$

Then h is a condensing map and $x \neq h(t, x)$ for all $(t, x) \in [0, 1] \times \partial\Omega$. In fact, for any subset A of Ω with $\alpha(A) > 0$, we have

$$\begin{aligned}\alpha(h([0, 1] \times A)) &\leq \alpha(cA + \mathrm{co}\,(\varphi(A) \cup \{0\})) \\ &\leq c\alpha(A) + \alpha(\varphi(A)) < (c + k)\alpha(A) \leq \alpha(A),\end{aligned}$$

where co B denotes the convex hull of a set B. Since the condensing map $f : \overline{\Omega} \to X$ is odd and has no fixed points on $\partial\Omega$, Borsuk's theorem implies that the degree $\mathrm{Deg}\,(id - f, \Omega)$ is odd; see e.g. [1, Theorem 3.2.7]. In particular, $\mathrm{Deg}\,(id - f, \Omega)$ is different from zero. The homotopy invariance of the degree (see Lemma 3.1 below) implies that

$$\mathrm{Deg}\,(id - f - \varphi, \Omega) = \mathrm{Deg}\,(id - f, \Omega) \neq 0.$$

Hence there exists a point x_0 in Ω such that $x_0 - f(x_0) - \varphi(x_0) = 0$. Thus the equation $x - f(x) = \varphi(x)$ is solvable. Therefore, F is $(0,k)$-epi on Ω. This completes the proof. □

2. $(0,k)$-Epi Maps

In this section we begin with the following extension of k-condensing maps. For the case of k-set contractions, see [9, Theorem 1.3].

Lemma 2.1. *Let X and Y be metrizable topological vector spaces and Ω a bounded open subset of X such that X is complete. Let $h : \overline{\Omega} \to Y$ be a k-condensing map with $k > 0$ such that $h(x) = 0$ for all $x \in \partial\Omega$. If a map $\tilde{h} : X \to Y$ is defined by*

$$\tilde{h}(x) := \begin{cases} h(x) & \text{for } x \in \overline{\Omega} \\ 0 & \text{for } x \notin \Omega \end{cases}$$

then $\tilde{h}$ is k-condensing.

Proof. Let $A \subset X$ be any bounded set such that $\alpha(A) > 0$. If $A \cap \Omega = \emptyset$, then $\tilde{h}(A) = \{0\}$ and so $\alpha(\tilde{h}(A)) = 0 < k\alpha(A)$. Suppose $A \cap \Omega \neq \emptyset$. There are two cases to consider. If $\alpha(A \cap \Omega) \neq 0$, then we have

$$\begin{aligned} \alpha(\tilde{h}(A)) &\leq \alpha(h(A \cap \Omega) \cup \{0\}) = \alpha(h(A \cap \Omega)) \\ &< k\alpha(A \cap \Omega) \leq k\alpha(A). \end{aligned}$$

Now let $\alpha(A \cap \Omega) = 0$. Since X is complete and h is continuous, the set $A \cap \Omega$ is relatively compact and so is $h(A \cap \Omega)$. From $\alpha(h(A \cap \Omega)) = 0$ it follows that

$$\alpha(\tilde{h}(A)) \leq \alpha(h(A \cap \Omega)) = 0 < k\alpha(A).$$

Therefore, in all possible cases, $\tilde{h}$ is k-condensing. This completes the proof. □

Now we consider some of basic properties of $(0,k)$-epi maps in our sense; see [9].

Lemma 2.2. *For Banach spaces X, Y and a bounded open subset Ω of X, the class of $(0,k)$-epi maps has the following properties:*

(a) *(Existence) If $F : \overline{\Omega} \to Y$ is $(0,k)$-epi, then the equation $F(x) = 0$ has a solution in Ω.*
(b) *(Normalization) Let $0 < k \leq 1$. Then the inclusion map $i : \overline{\Omega} \to X$ is $(0,k)$-epi if and only if $0 \in \Omega$.*
(c) *(Restriction) If $F : \overline{\Omega} \to Y$ is a $(0,k)$-epi map on Ω such that $F^{-1}(0)$ is contained in an open set $\Omega_1 \subset \Omega$, then the restriction of F to $\overline{\Omega}_1$, $F|_{\overline{\Omega}_1} : \overline{\Omega}_1 \to Y$, is $(0,k)$-epi on Ω_1.*
(d) *(Homotopy) Suppose that $F : \overline{\Omega} \to Y$ is a $(0,k)$-epi map and $H : [0,1] \times \overline{\Omega} \to Y$ is a p-set contraction with $0 \leq p < k < 1$ such that $H(0,x) = 0$ for each $x \in \overline{\Omega}$. If $F(x) + H(t,x) \neq 0$ for all $(t,x) \in [0,1] \times \partial\Omega$, then the map $F(\cdot) + H(1,\cdot) : \overline{\Omega} \to Y$ is $(0,k-p)$-epi.*

Proof. Statement (a) follows directly from the definition of $(0,k)$-epi map with $h \equiv 0$.

(b) Let $0 < k \le 1$. If i is $(0,k)$-epi, the existence property (a) implies that $0 \in \Omega$. Conversely, suppose that $0 \in \Omega$. Let $h : \overline{\Omega} \to X$ be any k-condensing map such that $h(x) = 0$ for all $x \in \partial\Omega$. Consider a map $\hat{h} : X \to X$ defined by

$$\hat{h}(x) := \begin{cases} h(x) & \text{for } x \in \overline{\Omega} \\ 0 & \text{for } x \notin \Omega. \end{cases}$$

By Lemma 2.1, $\hat{h}$ is k-condensing with $k \in (0,1]$ and hence condensing. Since $0 \in \Omega$, the equation $i(x) = h(x)$ has a solution in Ω if and only if $\hat{h}$ has a fixed point. Since $\hat{h}$ is a condensing map defined on the Banach space X, it is known that $\hat{h}$ has a fixed point; see [1, Theorem 1.5.11] or [7, Satz 4.2.6]. Hence, the inclusion map i is $(0,k)$-epi.

(c) Let $h : \overline{\Omega}_1 \to Y$ be any k-condensing map that vanishes on $\partial\Omega_1$. If a map $\tilde{h} : X \to Y$ is defined by

$$\tilde{h}(x) := \begin{cases} h(x) & \text{for } x \in \overline{\Omega}_1 \\ 0 & \text{for } x \notin \Omega_1 \end{cases}$$

then $\tilde{h}$ is k-condensing by Lemma 2.1. Hence the restriction h_1 of $\tilde{h}$ to $\overline{\Omega}$ is also k-condensing and h_1 vanishes on $\partial\Omega$. Since F is $(0,k)$-epi on Ω, the equation $F(x) = h_1(x)$ has a solution x_0 in Ω. The inclusion $F^{-1}(0) \subset \Omega_1$ implies that $x_0 \in \Omega_1$ and hence $F(x_0) = h(x_0)$. Therefore, the map $F|_{\overline{\Omega}_1}$ is $(0,k)$-epi on Ω_1.

(d) Let $g : \overline{\Omega} \to Y$ be any $(k-p)$-condensing map such that $g(x) = 0$ for all $x \in \partial\Omega$. Consider the set

$$S := \{x \in \overline{\Omega} : F(x) + H(t,x) = g(x) \quad \text{for some } t \in [0,1]\}.$$

Then the set S is closed in $\overline{\Omega}$ because the maps F, H and g are continuous and $[0,1]$ is compact. Note that g is obviously a k-condensing map. Since F is $(0,k)$-epi on Ω, it follows that $F(x_0) = g(x_0)$ for some $x_0 \in \Omega$ and hence $F(x_0) + H(0,x_0) = g(x_0)$ and therefore S is not empty. Since $\overline{\Omega}$ is normal and $S \cap \partial\Omega = \emptyset$, there exists a continuous function $\varphi : \overline{\Omega} \to [0,1]$ such that $\varphi(x) = 1$ for every $x \in S$ and $\varphi(x) = 0$ for every $x \in \partial\Omega$. Now consider a map $h : \overline{\Omega} \to Y$ defined by

$$h(x) := g(x) - H(\varphi(x), x) \qquad \text{for } x \in \overline{\Omega}.$$

Then h is k-condensing and $h(x) = 0$ for all $x \in \partial\Omega$. In fact, for any subset A of Ω with $\alpha(A) > 0$, some properties of the Kuratowski measure α of noncompactness imply that

$$\begin{aligned} \alpha(h(A)) &\le \alpha(g(A)) + \alpha(H(\{(\varphi(x), x) : x \in A\})) \\ &< (k-p)\alpha(A) + p\,\alpha(\{(\varphi(x), x) : x \in A\}) \\ &= (k-p)\alpha(A) + p\,\alpha(A) = k\alpha(A). \end{aligned}$$

Here recall that if A is a bounded subset of a metric space (X,d) and C is a subset of $[0,1] \times A$, then $\alpha(\{x \in A : (t,x) \in C\}) = \alpha(C)$, where the metric ρ on $\mathbb{R} \times X$ is given by $\rho((t,x),(t',x')) = \max\{|t-t'|, d(x,x')\}$; see [9, Lemma 1.1].

Since F is $(0,k)$-epi on Ω, the equation $F(x) = h(x)$ has a solution x_0 in Ω; that is, $F(x_0) = g(x_0) - H(\varphi(x_0), x_0)$. From $x_0 \in S$ it follows that $F(x_0) + H(1, x_0) = g(x_0)$. Therefore, the map $F(\cdot) + H(1,\cdot)$ is $(0, k-p)$-epi on Ω. This completes the proof. □

Remark. It is remarkable that the normalization property holds even for $k = 1$, in contrast to [9].

Motivated by Example 1.1, we show which map is $(0,k)$-epi if it has nonzero degree. Later we will see in Theorem 3.5 below that the converse is true in this situation.

Theorem 2.3. *Let Ω be a bounded open subset of a Banach space X and $0 < \ell < 1$. Let $f : \overline{\Omega} \to X$ be a countably ℓ-condensing map with respect to α that is fixed point free on $\partial\Omega$. If $Deg(id - f, \Omega) \neq 0$, then $F = id - f$ is $(0,k)$-epi on Ω for any $k \leq 1 - \ell$.*

Proof. Fix a real number k with $k \leq 1 - \ell$. Let $\varphi : \overline{\Omega} \to X$ be any k-condensing map such that $\varphi(x) = 0$ for all $x \in \partial\Omega$. Consider a homotopy $h : [0,1] \times \overline{\Omega} \to X$ defined by

$$h(t,x) := f(x) + t\varphi(x) \quad \text{for } (t,x) \in [0,1] \times \overline{\Omega}.$$

Then h is countably condensing with respect to α and $x \neq h(t,x)$ for all $(t,x) \in [0,1] \times \partial\Omega$. In fact, for each countable subset C of Ω with $\alpha(C) > 0$, we have

$$\begin{aligned}\alpha(h([0,1] \times C)) &\leq \alpha(f(C)) + \alpha(\mathrm{co}\,(\varphi(C) \cup \{0\})) \\ &< \ell\alpha(C) + k\alpha(C) \leq \alpha(C).\end{aligned}$$

The homotopy invariance of the degree (see Lemma 3.1 below) implies that

$$\mathrm{Deg}\,(id - f - \varphi, \Omega) = \mathrm{Deg}\,(id - f, \Omega).$$

If $\mathrm{Deg}\,(id - f, \Omega) \neq 0$, the fixed point property of the degree implies that the equation $x - f(x) = \varphi(x)$ has a solution in Ω. We conclude that $id - f$ is $(0,k)$-epi on Ω. This completes the proof. □

Corollary 2.4. *Let Ω be a bounded open subset of a Banach space X and $0 < 2k \leq 1$. Let $f : \overline{\Omega} \to X$ be a countably k-condensing map with respect to α that is fixed point free on $\partial\Omega$. If $Deg(id - f, \Omega) \neq 0$, then $F = id - f$ is $(0,k)$-epi on Ω; in particular, F is 0-epi on Ω.*

3. Nonzero Degree

To develop the theory of countably condensing maps in a more general setting, we need measures of noncompactness; see [5,12].

Let X be a Banach space. A function $\gamma : \{M \subset X : M \text{ is bounded}\} \to [0,\infty)$ is said to be a *measure of noncompactness* on X if it has the following properties:

(1) $\gamma(\overline{\mathrm{co}}\, M) = \gamma(M)$;
(2) $\gamma(N) \leq \gamma(M)$ if $N \subset M$;
(3) $\gamma(\lambda M) = |\lambda|\gamma(M)$;
(4) $\gamma(M + N) \leq \gamma(M) + \gamma(N)$; and
(5) $\gamma(M \cup \{x\}) = \gamma(M)$ for $x \in X$.

In what follows Γ denotes the class of all measures of noncompactness on X in the above sense. Note that the Kuratowski or the Hausdorff measure of noncompactness on a Banach space has the above properties; see [1,2].

Let Ω be a bounded open subset of a Banach space X. We say that a continuous map $h : [0,1] \times \overline{\Omega} \to X$ is *countably condensing* with respect to Γ if for each countable set $C \subset \Omega$ that is not precompact, the set $h([0,1] \times C)$ is bounded and there is some $\gamma \in \Gamma$ with $\gamma(h([0,1] \times C)) \not\geq \gamma(C)$. If there is a constant $k < 1$ such that for each countable set $C \subset \Omega$ that is not precompact, there is some $\gamma \in \Gamma$ with $\gamma(h([0,1] \times C)) \not\geq k\gamma(C)$, we call h *strictly countably condensing* with respect to Γ. Similarly, we call a continuous map $f : \overline{\Omega} \to X$ *countably condensing* (resp. *strictly countably condensing*) with respect to Γ if the constant homotopy $h(t,x) = f(x)$ for all $t \in [0,1]$ has this property.

We consider a degree of countably condensing maps which was introduced in [10,11]; see also [5, Theorem 3.1].

Lemma 3.1. *For every countably condensing map $f : \overline{\Omega} \to X$ with respect to Γ which has no fixed points on $\partial\Omega$, the degree $Deg(id - f, \Omega)$ is an integer with the following properties:*

(a) *(Fixed point property) If $Deg(id - f, \Omega) \neq 0$, then f has a fixed point in Ω.*
(b) *(Normalization) If f is compact, then $Deg(id - f, \Omega)$ is the Leray-Schauder degree.*
(c) *(Excision) If $f : \overline{\Omega} \to X$ is fixed point free outside an open set $\Omega_0 \subset \Omega$, then*
$$Deg(id - f, \Omega) = Deg(id - f, \Omega_0).$$
(d) *(Homotopy invariance) If $h : [0,1] \times \overline{\Omega} \to X$ is a countably condensing homotopy with respect to Γ such that $x \neq h(t,x)$ for all $(t,x) \in [0,1] \times \partial\Omega$, then*
$$Deg(id - h(0,\cdot), \Omega) = Deg(id - h(1,\cdot), \Omega).$$

To show the next theorem we need the following result [5, Lemma 3.2].

Lemma 3.2. *Let Ω be a bounded open and connected subset of a Banach space X. Let $f : \overline{\Omega} \to X$ be a countably condensing map with respect to Γ such that f has no fixed points on $\partial\Omega$. Then there exists an open connected set Ω_0 with $\overline{\Omega}_0 \subset \Omega$ that contains all fixed points of f. Moreover, there is a compact map $f_0 : \overline{\Omega} \to X$ such that the convex homotopy $h : [0,1] \times \overline{\Omega} \to X$ defined by $h(t,x) := (1-t)f(x) + tf_0(x)$ is fixed point free on $\partial\Omega_0$.*

Now we can give a slight modification of [5, Theorem 3.2] for the case of $(0,k)$-epi maps in our sense. We follow the basic line of the proof in [5].

Theorem 3.3. *Let Ω be a bounded open and connected subset of a Banach space X. Let $f : \overline{\Omega} \to X$ be a countably condensing map with respect to Γ such that $F = id - f$ is 0-epi on Ω. If one of the following conditions is satisfied:*

(a) *F is $(0,k)$-epi on Ω for some $k \in (0,1)$ and f is a p-set contraction for some real number $p > 0$,*
(b) *f is strictly countably condensing with respect to Γ,*

then $Deg(F, \Omega) \neq 0$.

Proof. Suppose that $f : \overline{\Omega} \to X$ is a countably condensing map with respect to Γ and F is 0-epi on Ω. We remark that if f is compact and F is 0-epi on Ω, then $\mathrm{Deg}\,(F,\Omega) \neq 0$; see the proof of [5, Theorem 3.2]. Since f has no fixed points on $\partial\Omega$, by Lemma 3.2, there exists an open connected set Ω_0 with $\overline{\Omega}_0 \subset \Omega$ that contains all fixed points of f and there is a compact map $f_0 : \overline{\Omega} \to X$ such that the convex homotopy $h : [0,1] \times \overline{\Omega} \to X$ given by $h(t,x) = (1-t)f(x) + tf_0(x)$ has no fixed points on $\partial\Omega_0$.

We will claim that $G := id - f_0$ is 0-epi on Ω_0 and $\mathrm{Deg}\,(F,\Omega) = \mathrm{Deg}\,(G,\Omega_0)$. Since f_0 is compact and $\Omega_0(\subset \Omega)$ is an open connected set, the above remark says that $\mathrm{Deg}\,(G,\Omega_0) \neq 0$ which implies that $\mathrm{Deg}\,(F,\Omega) \neq 0$, as desired.

Claim 1: $\mathrm{Deg}\,(F,\Omega) = \mathrm{Deg}\,(G,\Omega_0)$.

Since f is fixed point free outside Ω_0, the excision of the degree implies $\mathrm{Deg}\,(F,\Omega) = \mathrm{Deg}\,(F,\Omega_0)$. Then the homotopy h is countably condensing with respect to Γ because f is countably condensing with respect to Γ and f_0 is compact. Since h has no fixed points on $\partial\Omega_0$, the homotopy invariance of Lemma 3.1 implies that $\mathrm{Deg}\,(F,\Omega_0) = \mathrm{Deg}\,(G,\Omega_0)$. Consequently, we obtain $\mathrm{Deg}\,(F,\Omega) = \mathrm{Deg}\,(G,\Omega_0)$.

Claim 2: G is 0-epi on Ω_0.

There are two cases to consider. First we suppose that F is $(0,k)$-epi on Ω for some $k \in (0,1)$ and f is a p-set contraction for some $p > 0$. Since Ω_0 contains all fixed points of f, Lemma 2.2 implies that F is $(0,k)$-epi on Ω_0. Fix a real number $\lambda \in (0,1)$ with $\lambda < k/p$. Consider the homotopy $H_0 : [0,1] \times \overline{\Omega} \to X$ given by

$$H_0(t,x) := t\lambda(f_0(x) - f(x)).$$

Then H_0 is a k_0-set contraction with $k_0 := \lambda p < k$ and $H_0(0,\cdot) = 0$ and $F(x) \neq H_0(t,x)$ for all $(t,x) \in [0,1] \times \partial\Omega_0$ because h has no fixed points on $\partial\Omega_0$. Since $0 \leq k_0 < k < 1$, the homotopy property of $(0,k)$-epi maps stated in Lemma 2.2 implies that $F_\lambda := F - H_0(1,\cdot)$ is $(0, k-k_0)$-epi on Ω_0. Let a map $f_\lambda : \overline{\Omega}_0 \to X$ be defined by

$$f_\lambda(x) := (1-\lambda)f(x) + \lambda f_0(x) \quad \text{for } x \in \overline{\Omega}_0.$$

In particular, $F_\lambda = id - f_\lambda$ is 0-epi on Ω_0. Since f is countably condensing with respect to Γ, we conclude that f_λ is strictly countably condensing with respect to Γ, with constant $1-\lambda \in (0,1)$. Let a map $h_\lambda : [0,1] \times \overline{\Omega}_0 \to X$ be defined by

$$h_\lambda(t,x) := h((1-t)\lambda + t\cdot 1, x) = (1-t)f_\lambda(x) + tf_0(x) \quad \text{for } (t,x) \in [0,1] \times \overline{\Omega}_0.$$

Since h has no fixed points on $\partial\Omega_0$, it follows that h_λ has no fixed points on $\partial\Omega_0$. Consider the homotopy $H_\lambda : [0,1] \times \overline{\Omega}_0 \to X$ given by

$$H_\lambda(t,x) := t(f_0(x) - f_\lambda(x)) \quad \text{for } (t,x) \in [0,1] \times \overline{\Omega}_0.$$

Observing the fact that $F_\lambda(x) = H_\lambda(t,x)$ is equivalent to $x = h_\lambda(t,x)$, we have $F_\lambda(x) \neq H_\lambda(t,x)$ for $(t,x) \in [0,1] \times \partial\Omega_0$. To apply Theorem 2.2 of [5], we will check that the following compactness condition is satisfied. To see this, choose a partition $0 = t_0 < t_1 < \cdots < t_n = 1$ such that

$$c_i := t_i - t_{i-1} \leq \frac{1}{1-\lambda} - 1 \qquad \text{for } i = 1, \cdots, n.$$

For each map $\psi : \overline{\Omega}_0 \to X$ such that ψ vanishes on $\partial\Omega_0$ and $\overline{\text{co}}(\psi(\overline{\Omega}_0))$ is compact let

$$F_{\lambda i}(x) = x - (1 - t_{i-1})f_\lambda(x) - t_{i-1}f_0(x),$$

$$H_{\lambda i,\psi}(t,x) = (t - t_{i-1})(f_0(x) - f_\lambda(x)) + \psi(x) \quad \text{for } t \in [t_{i-1}, t_i] \text{ and } x \in \overline{\Omega}_0.$$

Let $C \subset \Omega_0$ be any countable set that the following relation holds:

$$\overline{\text{co}(H_{\lambda i,\psi}([t_{i-1}, t_i] \times C) \cup \{0\}) \cap F_{\lambda i}(\Omega_0)} \subset \overline{F_{\lambda i}(C)}$$
$$\subset \overline{\overline{\text{co}}(H_{\lambda i,\psi}([t_{i-1}, t_i] \times C) \cup \{0\}) \cap F_{\lambda i}(\Omega_0)}.$$

Now it remains to show that $\overline{\text{co}}(H_{\lambda i,\psi}([t_{i-1}, t_i] \times \overline{C}) \cup \{0\})$ is compact. For any $\gamma \in \Gamma$, we obtain

$$\gamma(F_{\lambda i}(C)) \le \gamma(\overline{\text{co}}(H_{\lambda i,\psi}([t_{i-1}, t_i] \times C) \cup \{0\})) = \gamma(H_{\lambda i,\psi}([t_{i-1}, t_i] \times C))$$
$$\le (t_i - t_{i-1})\gamma(f_0(C) - f_\lambda(C)) + \gamma(\psi(C)) \le c_i\gamma(f_\lambda(C)).$$

From $x = F_{\lambda i}(x) + (1 - t_{i-1})f_\lambda(x) + t_{i-1}f_0(x)$ it follows that

$$\gamma(C) \le \gamma(F_{\lambda i}(C)) + (1 - t_{i-1})\gamma(f_\lambda(C))$$

which implies

$$(1 - \lambda)\gamma(C) \le (1 - t_{i-1} + c_i)(1 - \lambda)\gamma(f_\lambda(C))$$
$$\le (1 + c_i)(1 - \lambda)\gamma(f_\lambda(C)) \le \gamma(f_\lambda(C)).$$

Since this estimate holds for any $\gamma \in \Gamma$ and f_λ is strictly countably condensing with respect to Γ, the set C is precompact. Since $H_{\lambda i,\psi}$ maps compact sets into compact sets and X is a Banach space, the set $\overline{\text{co}}(H_{\lambda i,\psi}([t_{i-1}, t_i] \times \overline{C}) \cup \{0\})$ is compact. In view of Theorem 2.2 of [5], since F_λ is 0-epi on Ω_0, the map $F_\lambda - H_\lambda(1, \cdot)$ is 0-epi on Ω_0. From $G = F_\lambda - H_\lambda(1, \cdot)$ it follows that G is 0-epi on Ω_0.

Next we suppose that f is strictly countably condensing with respect to Γ. If we replace f_λ, F_λ, h_λ, and H_λ in the first case by f, F, h and H, respectively, where $H(t,x) = t(f_0(x) - f(x))$, a similar argument establishes that $G = F - H(1, \cdot)$ is 0-epi on Ω_0. Consequently, in both cases, Claim 2 is proved. This completes the proof. □

Theorem 3.3 includes [5, Theorem 1.1] as a special case.

Corollary 3.4. *Let Ω be a bounded open and connected subset of a Banach space X and $f : \overline{\Omega} \to X$ a condensing map with respect to α. If $F = id - f$ is $(0,k)$-epi on Ω for some $k \in (0,1)$, then $Deg(F, \Omega) \neq 0$.*

Finally we show that $(0,k)$-epi maps of the form $id - f$ where f is strictly countably condensing are precisely those maps with nonzero degree. For other result on countably 1/2-condensing maps on a Jordan domain, we refer to [12, Theorem 4.3].

Theorem 3.5. *Let Ω be a bounded open and connected subset of a Banach space X and $0 < \ell < 1$. Let $f : \overline{\Omega} \to X$ be a countably ℓ-condensing map with respect to α that has no fixed points on $\partial\Omega$. Then the following statements are equivalent:*

(a) *$F = id - f$ is $(0,k)$-epi on Ω for $k \le 1 - \ell$.*
(b) *F is 0-epi on Ω.*
(c) *$Deg(F, \Omega) \neq 0$.*

Proof. Since f is strictly countably condensing with respect to α, this is an immediate consequence of Theorem 2.3 and Theorem 3.3. □

Remark. It is known that a map of the form $F = id - f$ where f is an ℓ-set contraction with $\ell < 1/2$ is $(0,k)$-epi for any $k < 1 - \ell$ if F is 0-epi; see [5, Corollary 2.1]. Notice that Theorem 3.5 is a sharp version of this fact.

Corollary 3.6. *Let Ω be a bounded open and connected subset of a Banach space X and $f : \overline{\Omega} \to X$ a countably $1/2$-condensing map with respect to α that has no fixed points on $\partial\Omega$. Then $F = id - f$ is $(0,k)$-epi on Ω for $k \leq 1/2$ if and only if $Deg(F,\Omega) \neq 0$.*

References

[1] R.R. Akhmerov, M.I. Kamenskii, A.S. Potapov, A.E. Rodkina, and B.N. Sadovskii, Measures of Noncompactness and Condensing Operators, Birkhäuser, Basel, 1992.

[2] J.M. Ayerbe Toledano, T. Dominguez Benavides, and G. López Acedo, Measures of Noncompactness in Metric Fixed Point Theory, Birkhäuser, Basel, 1997.

[3] W. Feng, *A new spectral theory for nonlinear operators and its applications*, Abstr. Appl. Anal. **2** (1997), 163-183.

[4] M. Furi, M. Martelli, and A. Vignoli, *On the solvability of nonlinear operator equations in normed spaces*, Ann. Mat. Pura Appl. **124** (1980), 321-343.

[5] E. Giorgieri and M. Väth, *A characterization of* 0*-epi maps with a degree*, J. Funct. Anal. **187** (2001), 183-199.

[6] D.H. Hyers, G. Isac, and T.M. Rassias, Topics in Nonlinear Analysis and Applications, World Scientific, Singapore, 1997.

[7] T. Jerofsky, Zur Fixpunkttheorie mengenwertiger Abbildungen, Dissertation A, TU Dresden, 1983.

[8] P. Santucci and M. Väth, *Grasping the phantom– A new approach to nonlinear spectral theory*, Ann. Mat. Pura Appl. **180** (2001), 255-284.

[9] E.U. Tarafdar and H.B. Thompson, *On the solvability of nonlinear noncompact operator equations*, J. Austral. Math. Soc. Ser. A **43** (1987), 103-126.

[10] M. Väth, *Fixed point theorems and fixed point index for countably condensing maps*, Topol. Methods Nonlinear Anal. **13** (1999), 341-363.

[11] M. Väth, *An axiomatic approach to a coincidence index for noncompact function pairs*, Topol. Methods Nonlinear Anal. **16** (2000), 307-338.

[12] M. Väth, *On the connection of degree theory and* 0*-epi maps*, J. Math. Anal. Appl. **257** (2001), 223-237.

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