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ON INHERITED PROPERTIES FOR SET-VALUED MAPS AND EXISTENCE THEOREMS FOR GENERALIZED VECTOR EQUILIBRIUM PROBLEMS

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Dedicated to Professor Wataru Takahashi on his sixtieth birthday

Abstract. The authors have showed some properties of scalarizing functions for set-valued maps. Fan's inequality for set-valued maps are proved by using those properties. In this paper, we present new inherited properties which are obtained generalized convexity and cone-semicontinuity assumptions for set-valued maps. By applying those new properties to Fan's inequality for set-valued maps, we prove existence theorems for generalized vector equilibrium problems.

1. Introduction

This paper is concerned with a generalization of an existence theorem for the generalized vector equilibrium problem in [1], in which Ansari and Yao proved an existence result by using Fan-Browder type fixed point theorem. It is relative to a vector-valued Fan's inequality for set-valued maps in [4, 5].

In this paper, we consider the following two kinds of generalized vector equilibrium problems:

$$\text{find } \bar{x} \in K \text{ such that } F(\bar{x}, y) \not\subset -\operatorname{int} C(\bar{x}) \quad \text{for every } y \in K \tag{1.1}$$

and

$$\text{find } \bar{x} \in K \text{ such that } F(\bar{x}, y) \cap (-\operatorname{int} C(\bar{x})) = \emptyset \quad \text{for every } y \in K \tag{1.2}$$

where E and Y are two topological vector spaces, K is a nonempty convex subset of E, $F : K \times K \to 2^Y$ is a multifunction, $C : K \to 2^Y$ is a multifunction such that for each $x \in K$, $C(x)$ is a closed convex cone with $\operatorname{int} C(x) \neq \emptyset$. We show existence theorems of these problems by using Fan's inequality. Our proofs of Theorems 3.1 and 3.2 are quite different from that in [1] and in the proofs we use a result of Georgiev and Tanaka [4, Theorem 2.3] which follows from a two-function result of Simons [11, Theorem 1.2].

By applying the two-function result for special scalarizing functions possessing quasiconvexity and semicontinuity, we establish the proofs of the main theorems. For such a reason, it is necessary for those scalarizing functions to have such convexity and semicontinuity. It is, therefore, important and useful to study what kind of scalarizing functions can inherit properties of such kind of convexity and semicontinuity from multifunctions.

To shows some results on the inherited properties, we consider certain generalizations and modifications of convexity and semicontinuity for multifunctions in a

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topological vector space with respect to a cone preorder in the target space, which have motivated by [6, 7] and studied in [4] for generalizing the classical Fan's inequality. Convexity and semicontinuity for multifunctions are inherited by the following scalarizing functions;

$$\inf\{h_C(x,y;k) \mid y \in F(x)\} \tag{1.3}$$

and

$$\sup\{h_C(x,y;k) \mid y \in F(x)\} \tag{1.4}$$

where $h_C(x,y;k) = \inf\{t \mid y \in tk - C(x)\}$, $F : E \to 2^Y$ is a multifunction, $C(x)$ a closed convex cone with $\operatorname{int} C(x) \neq \emptyset$, x and y are vectors in two topological vector spaces E and Y, respectively, and $k \in \operatorname{int} C(x)$. Note that $h_C(x,\cdot;k)$ is positively homogeneous and subadditive for every fixed $x \in E$ and $k \in \operatorname{int} C(x)$, and that $h_C(x,y;k) \leq 0$ for $y \in -C(x)$, remark that $-h_C(x,-y;k) = \sup\{t \mid y \in tk + C(x)\}$. This function $h_C(x,y;k)$ has been treated in some papers. Essentially, $h_C(x,y;k)$ is equivalent to the smallest strictly monotonic function defined by Luc [8]. For each $y \in Y$, $h_C(x,y;k) \cdot k$ corresponds the minimum vector of upper bounds of y with respect to the cone $C(x)$ restricted to the direction k. Similarly, $-h_C(x,-y;k) \cdot k$ corresponds the maximum vector of lower bounds of y with respect to the cone $C(x)$ restricted to the direction k.

2. Inherited Properties for Set-Valued Maps

Further let E and Y be topological vector spaces and F and $C : E \to 2^Y$ two multifunctions. Denote $B(x) = \operatorname{co}((\operatorname{int} C(x)) \cap (2S \setminus \overline{S}))$ (which plays a role of base for $\operatorname{int} C(x)$ without uniqueness), where S is a neighborhood of 0 in Y. We observe the following four types of scalarizing functions:

$$\psi_C^F(x;k) := \sup_{y \in F(x)} h_C(x,y;k), \qquad \varphi_C^F(x;k) := \inf_{y \in F(x)} h_C(x,y;k);$$

$$-\varphi_C^{-F}(x;k) = \sup_{y \in F(x)} -h_C(x,-y;k), \qquad -\psi_C^{-F}(x;k) = \inf_{y \in F(x)} -h_C(x,-y;k).$$

The first and fourth functions have symmetric properties and then results for the fourth function $-\psi_C^{-F}(x;k)$ can be easily proved by those for the first function $\psi_C^F(x;k)$. Similarly, the results for the third function $-\varphi_C^{-F}(x;k)$ can be deduced by those for the second function $\varphi_C^F(x;k)$. By using these four functions we measure each image of multifunction F with respect to its 4-tuple of scalars, which can be regarded as standpoints for the evaluation of the image. To avoid confusion for properties of convexity, we consider the constant case of $C(x) = C$ (a convex cone) and $B(x) = B$ (a convex set), and $h_C(x,y;k) = h_C(y;k) := \inf\{t \mid y \in tk - C\}$.

To begin with, we recall some kinds of convexity for multifunctions.

Definition 2.1. A multifunction $F : E \to 2^Y$ is called C-*quasiconvex*, if the set $\{x \in E \mid F(x) \cap (a - C) \neq \emptyset\}$ is convex (or empty) for every $a \in Y$. If $-F$ is C-quasiconvex, then F is said to be C-*quasiconcave*, which is equivalent to a $(-C)$-quasiconvex mapping.

Remark 2.1. The above definition is exactly that of *Ferro type* (-1)-*quasiconvex* mapping in [7, Definition 3.5].

Definition 2.2. A multifunction $F : E \to 2^Y$ is called (in the sense of [7, Definition 3.7])

(a) *type-*(iii) *C-naturally quasiconvex* if for every two points $x_1, x_2 \in E$ and every $\lambda \in (0,1)$, there exists $\mu \in [0,1]$ such that
$$\mu F(x_1) + (1-\mu)F(x_2) \subset F(\lambda x_1 + (1-\lambda)x_2) + C;$$

(b) *type-*(v) *C-naturally quasiconvex*, if for every two points $x_1, x_2 \in E$ and every $\lambda \in (0,1)$, there exists $\mu \in [0,1]$ such that
$$F(\lambda x_1 + (1-\lambda)x_2) \subset \mu F(x_1) + (1-\mu)F(x_2) - C.$$

If $-F$ is type-(iii) [resp., type-(v)] C-naturally quasiconvex, then F is said to be *type-*(iii) [resp., *type-*(v)] *C-naturally quasiconcave*, which is equivalent to a type-(iii) [resp., type-(v)] $(-C)$-naturally quasiconvex mapping.

However, there is no relationship between those for types (iii) and (v) in general.

Proposition 2.1 (See [7, Theorem 3.1])**.** *For a multifunction $F : E \to 2^Y$, type-*(iii) *C-naturally quasiconvexity implies C-quasiconvexity.*

Proposition 2.2. *For each $x \in E$ and a multifunction $F : E \to 2^Y$,*

(i) *$\psi_C^F(x;k)$ is convex with respect to variable $k \in \operatorname{int} C$;*

(ii) *$\varphi_C^F(x;k)$ is convex with respect to variable $k \in \operatorname{int} C$ if $F(x)$ is a convex set.*

Proof. We first note that h_C is convex with respect to two variables $y \in Y$ and $k \in \operatorname{int} C$. To begin with, we shall prove the assertion (i). Let $x \in E$ be given. For every $k_1, k_2 \in \operatorname{int} C$ and $\lambda \in [0,1]$,
$$\begin{aligned}
\psi_C^F(x; \lambda k_1 + (1-\lambda)k_2) &:= \sup_{y \in F(x)} h_C(y; \lambda k_1 + (1-\lambda)k_2) \\
&\le \sup_{y \in F(x)} (\lambda\, h_C(y;k_1) + (1-\lambda)\, h_C(y;k_2)) \\
&\le \lambda \sup_{y \in F(x)} h_C(y;k_1) + (1-\lambda) \sup_{y \in F(x)} h_C(y;k_2) \\
&= \lambda\, \psi_C^F(x;k_1) + (1-\lambda)\, \psi_C^F(x;k_2),
\end{aligned}$$
which shows that $\psi_C^F(x;k)$ is convex with respect to variable $k \in \operatorname{int} C$.

Next, we shall prove the assertion (ii). Let $x \in E$ be given. Assume that $F(x)$ is a convex set. By the definition of φ_C^F, for every $\varepsilon > 0$ and $k_1, k_2 \in \operatorname{int} C$ there exist $y_i \in F(x)$ such that for each $i = 1, 2$,
$$h_C(y_i; k_i) < \varphi_C^F(x; k_i) + \varepsilon.$$
For every $\lambda \in [0,1]$, since $F(x)$ is a convex set, $\lambda y_1 + (1-\lambda)y_2 \in F(x)$. Then
$$\begin{aligned}
\varphi_C^F(x; \lambda k_1 + (1-\lambda)k_2) &:= \inf_{y \in F(x)} h_C(y; \lambda k_1 + (1-\lambda)k_2) \\
&\le h_C(\lambda y_1 + (1-\lambda)y_2; \lambda k_1 + (1-\lambda)k_2) \\
&\le \lambda\, h_C(y_1;k_1) + (1-\lambda)\, h_C(y_2;k_2) \\
&< \lambda(\varphi_C^F(x;k_1) + \varepsilon) + (1-\lambda)(\varphi_C^F(x;k_2) + \varepsilon) \\
&= \lambda\, \varphi_C^F(x;k_1) + (1-\lambda)\, \varphi_C^F(x;k_2) + \varepsilon.
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrarily small, we obtain

$$\varphi_C^F(x; \lambda k_1 + (1-\lambda)k_2) \leq \lambda\, \varphi_C^F(x; k_1) + (1-\lambda)\, \varphi_C^F(x; k_2),$$

which shows that $\varphi_C^F(x; k)$ is convex with respect to variable $k \in \operatorname{int} C$. □

Now, we show some inherited properties of convexity for multifunctions.

Lemma 2.1 (See [9, Theorem 2.6]). *If the multifunction $F : E \to 2^Y$ is type-*(v) *C-naturally quasiconvex, then the function $\psi_C^F(x; k) = \sup\{h_C(y; k) \mid y \in F(x)\}$ is quasiconvex with respect to variable x where $k \in \operatorname{int} C$.*

Lemma 2.2. *If $F : E \to 2^Y$ is type-*(v) *C-naturally quasiconvex, then*

$$\psi^F(x) := \inf_{k \in B} \psi_C^F(x; k) = \inf_{k \in B} \sup_{y \in F(x)} h_C(y; k)$$

is quasiconvex.

Proof. By the definition of ψ^F, for every $\varepsilon > 0$ and $x_1, x_2 \in E$ there exist $k_i \in B$ such that for each $i = 1, 2$,

$$\psi_C^F(x_1; k_i) < \psi^F(x_1) + \varepsilon$$

and

$$\psi_C^F(x_2; k_i) < \psi^F(x_2) + \varepsilon.$$

For every $\lambda \in [0, 1]$, since B is convex, $\lambda k_1 + (1-\lambda)k_2 \in B$. Then

$$\begin{aligned}
\psi^F(\lambda x_1 + (1-\lambda)x_2) &:= \inf_{k \in B} \psi_C^F(\lambda x_1 + (1-\lambda)x_2; k) \\
&\leq \psi_C^F(\lambda x_1 + (1-\lambda)x_2; \lambda k_1 + (1-\lambda)k_2) \\
&\leq \lambda\, \psi_C^F(\lambda x_1 + (1-\lambda)x_2; k_1) \\
&\quad + (1-\lambda)\, \psi_C^F(\lambda x_1 + (1-\lambda)x_2; k_2) \quad \text{(by (i) of Prop. 2.2)} \\
&\leq \max\{\psi_C^F(\lambda x_1 + (1-\lambda)x_2; k_1), \psi_C^F(\lambda x_1 + (1-\lambda)x_2; k_2)\} \\
&\leq \max\{\max\{\psi_C^F(x_1; k_1), \psi_C^F(x_2; k_1)\}, \\
&\qquad\quad \max\{\psi_C^F(x_1; k_2), \psi_C^F(x_2; k_2)\}\} \quad \text{(by Lemma 2.1)} \\
&< \max\{\psi^F(x_1), \psi^F(x_2)\} + \varepsilon.
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrarily small, we obtain

$$\psi^F(\lambda x_1 + (1-\lambda)x_2) \leq \max\{\psi^F(x_1), \psi^F(x_2)\},$$

which shows that ψ^F is quasiconvex. □

Lemma 2.3 (See [9, Theorem 2.5]). *If the multifunction $F : E \to 2^Y$ is C-quasiconvex, then the function $\varphi_C^F(x; k) = \inf\{h_C(y; k) \mid y \in F(x)\}$ is quasiconvex with respect to variable x where $k \in \operatorname{int} C$.*

Lemma 2.4. *If $F : E \to 2^Y$ is convex-valued and C-quasiconvex, then*

$$\varphi^F(x) := \inf_{k \in B} \varphi_C^F(x; k) = \inf_{k \in B} \inf_{y \in F(x)} h_C(y; k)$$

is quasiconvex.

Proof. Since F is convex-valued, by (ii) of Proposition 2.2, $\varphi_C^F(x;k)$ is convex with respect to variable $k \in B$. By Lemma 2.3, $\varphi_C^F(x;k)$ is quasiconvex with respect to variable x where $k \in B$. From these properties, we can verify that φ^F is quasiconvex in the same way as the proof of Lemma 2.2. □

Remark 2.2. When we replace F by $-F$ in Lemmas 2.2, 2.4, it leads to the quasiconcavity of scalarizing functions $-\psi^{-F}$ and $-\varphi^{-F}$. By Proposition 2.1 and Lemma 2.4, if $F : E \to 2^Y$ is convex-valued and type-(iii) C-naturally quasiconvex, then φ^F is quasiconvex.

Next we show some inherited properties from some kinds of semicontinuity. We introduce two types of cone-semicontinuity for multifunctions, which are regarded as extensions of the ordinary lower semicontinuity for real-valued functions; see [6].

Definition 2.3. Let $\hat{x} \in E$. A multifunction F is called $C(\hat{x})$-*upper semicontinuous* at x_0, if for every $y \in C(\hat{x}) \cup (-C(\hat{x}))$ satisfying with $F(x_0) \subset y + \operatorname{int} C(\hat{x})$, there exists an open $U \ni x_0$ such that $F(x) \subset y + \operatorname{int} C(\hat{x})$ for every $x \in U$.

Definition 2.4. Let $\hat{x} \in E$. A multifunction F is called $C(\hat{x})$-*lower semicontinuous* at x_0, if for every open V such that $F(x_0) \cap V \neq \emptyset$, there exists an open $U \ni x_0$ such that $F(x) \cap (V + \operatorname{int} C(\hat{x})) \neq \emptyset$ for every $x \in U$.

Remark 2.3. In the two definitions above, the notions for single-valued functions are equivalent to the ordinary notion of lower semicontinuity of real-valued ones, whenever $Y = \mathbf{R}$ and $C(x) = [0, \infty)$. Usual upper semicontinuous multifunction is also (cone-) upper semicontinuous. When the cone $C(\hat{x})$ consists only of the zero of the space, the notion in Definition 2.4 coincides with that of lower semicontinuous multifunction. Moreover, it is equivalent to the cone-lower semicontinuity defined in [6], based on the fact that $V + \operatorname{int} C(\hat{x}) = V + C(\hat{x})$; see [13, Theorem 2.2].

Proposition 2.3 (See [10, Proposition 2])**.** *Assume that there exists a compact subset $D \subset Y$ satisfying* (i) $A \subset \operatorname{cone} D$ *where* $\operatorname{cone} D := \{\lambda x | \lambda \geq 0, x \in D\}$ *and* (ii) $D \subset \operatorname{int} C(x_0)$ *for some $x_0 \in E$. If $W(\cdot) := Y \setminus \{\operatorname{int} C(\cdot)\}$ has a closed graph, then there exists an open set $U \ni x_0$ such that $A \subset C(x)$ for every $x \in U$. In particular C is lower semicontinuous.*

Lemma 2.5. *Suppose that $W : E \to 2^Y$ defined as $W(x) = Y \setminus \operatorname{int} C(x)$ has a closed graph. If F is $(-C(x))$-upper semicontinuous at x for each $x \in E$ and there exists a compact-valued multifunction $D : E \to 2^Y$ such that for each $x_0 \in E$,* (i) $D(x_0) \subset \operatorname{int} C(x_0)$ *and* (ii) *for every $t \in \mathbf{R}$, $k \in B(x_0)$ and $x \in E$ satisfying with $tk - F(x) \subset \operatorname{int} C(x_0)$, $tk - F(x) \subset \operatorname{cone} D(x_0)$. Then*

$$\psi^F(x) := \inf_{k \in B(x)} \sup_{y \in F(x)} h_C(x, y; k)$$

is upper semicontinuous. If the mapping C is constant-valued, then ψ^F is upper semicontinuous without assumptions on D above.

Proof. Let $\varepsilon > 0$ and $x_0 \in E$ be given. In the case $\psi^F(x_0) = \infty$, it is clear that ψ^F is upper semicontinuous at x_0. So we assume $\psi^F(x_0) \neq \infty$. By the definition of ψ^F, there exists $k_0 \in B(x_0)$ such that

$$\sup_{y \in F(x_0)} h_C(x_0, y; k_0) < \psi^F(x_0) + \varepsilon.$$

Since $\sup_{y\in F(x_0)} h_C(x_0, y; k_0) = \inf\{t \mid F(x_0) \subset tk_0 - C(x_0)\}$, we can take

$$\inf\{t \mid F(x_0) \subset tk_0 - C(x_0)\} < t_0 < \psi^F(x_0) + \varepsilon.$$

Therefore

$$F(x_0) \subset t_0k_0 - \operatorname{int} C(x_0).$$

Since F is $(-C(x_0))$-upper semicontinuous at x_0, there exists an open $U_1 \ni x_0$ such that

$$F(x) \subset t_0k_0 - \operatorname{int} C(x_0) \quad \text{for every } x \in U_1.$$

Therefore

$$t_0k_0 - F(U_1) \subset \operatorname{int} C(x_0).$$

Hence, from the assumption, there exists a compact $D(x_0)$ such that

$$D(x_0) \subset \operatorname{int} C(x_0)$$

and

$$t_0k_0 - F(U_1) \subset \operatorname{cone} D(x_0).$$

By Proposition 2.3, there exists an open $U_2 \ni x_0$ such that

$$t_0k_0 - F(U_1) \subset C(x) \quad \text{for every } x \in U_2.$$

Hence, for $t_0 < t' < \psi^F(x_0) + \varepsilon$, there exists an open $U_3 := U_1 \cap U_2$ ($U_3 \ni x_0$) such that

$$F(x) \subset t'k_0 - \operatorname{int} C(x) \quad \text{and} \quad k_0 \in B(x) \quad \text{for every } x \in U_3.$$

Then

$$\begin{aligned}
\psi^F(x) &= \inf_{k\in B(x)} \sup\{h_C(x, y; k) \mid y \in F(x)\} \\
&\le \sup\{h_C(x, y; k_0) \mid y \in F(x)\} \\
&\le \sup\{h_C(x, y; k_0) \mid y \in t'k_0 - C(x)\} \\
&= \sup\{h_C(x, t'k_0 - c; k_0) \mid c \in C(x)\} \\
&= h_C(x, t'k_0; k_0) + \sup\{h_C(x, -c; k_0) \mid c \in C(x)\} \\
&\le t' \\
&\le \psi^F(x_0) + \varepsilon.
\end{aligned}$$

The proof of the second statement (when C is constant-valued) is similar, however in this case there is no need to use Proposition 2.3. □

Lemma 2.6. *Suppose that $W : E \to 2^Y$ defined as $W(x) = Y \setminus \operatorname{int} C(x)$ has a closed graph. If F is $(-C(x))$-lower semicontinuous for each $x \in E$ and there exists a compact-valued multifunction $D : E \to 2^Y$ such that for each $x_0 \in E$,* (i) *$D(x_0) \subset \operatorname{int} C(x_0)$ and* (ii) *for every $t < t^* \in \mathbf{R}$, $k \in B(x_0)$, $x \in E$ and $y \in F(x_0)$ satisfying with $F(x) \cap [y + tk - \operatorname{int} C(x_0)] \neq \emptyset$, $F(x) \cap [y + t^*k - \operatorname{cone} D(x_0)] \neq \emptyset$. Then*

$$\varphi^F(x) := \inf_{k\in B(x)} \inf_{y\in F(x)} h_C(x, y; k)$$

is upper semicontinuous. If the mapping C is constant-valued, then φ^F is upper semicontinuous without assumptions on D above.

Proof. Let $\varepsilon > 0$ and $x_0 \in E$ be given. In the case $\varphi^F(x_0) = -\infty$, we can easy verify that there exists an open $U \in x_0$ such that $\varphi^F(x) = -\infty$ for every $x \in U$. Therefore, in this case it is clear that φ^F is upper semicontinuous at x_0. So we assume $\varphi^F(x_0) \neq -\infty$. By the definition of φ^F, for $\varphi^F(x_0) < t_0 < \varphi^F(x_0) + \varepsilon$, there exists $k_0 \in B(x_0)$ ($k_0 \in \operatorname{int} C(x_0)$) and $z_0 \in F(x_0)$ such that

$$t_0 k_0 - z_0 \in \operatorname{int} C(x_0).$$

By Proposition 2.3, there exists an open $U_1 \ni x_0$ such that

$$t_0 k_0 - z_0 \in \operatorname{int} C(x) \quad \text{and} \quad k_0 \in \operatorname{int} C(x) \quad \text{for every } x \in U_1.$$

Therefore

$$h_C(x, z_0; k_0) \le t_0 \quad \text{for every } x \in U_1. \tag{2.1}$$

Let $0 < \gamma < \frac{\varepsilon}{2}$. By $(-C(x_0))$-lower semicontinuity of F, there exists an open $U_2 \subset U_1$ ($U_2 \ni x_0$) such that

$$F(x) \cap (z_0 + \gamma k_0 - \operatorname{int} C(x_0)) \neq \emptyset \quad \text{for every } x \in U_2.$$

Since $z_0 + \gamma k_0 - \operatorname{int} C(x_0) \subset z_0 + 2\gamma k_0 - \operatorname{int} C(x_0)$, from the assumption, there exists a compact $D(x_0)$ such that

$$D(x_0) \subset \operatorname{int} C(x_0)$$

and

$$G(x) := F(x) \cap (z_0 + 2\gamma k_0 - \operatorname{cone} D(x_0)) \neq \emptyset \quad \text{for every } x \in U_2.$$

Therefore

$$\emptyset \neq G(U_2) \subset z_0 + 2\gamma k_0 - \operatorname{cone} D(x_0).$$

So we have

$$z_0 + 2\gamma k_0 - G(U_2) \subset \operatorname{cone} D(x_0).$$

By Proposition 2.3, there exists an open $U_3 \ni x_0$ such that

$$z_0 + 2\gamma k_0 - G(U_2) \subset \operatorname{int} C(x) \quad \text{for every } x \in U_3.$$

Hence, there exists an open $U_4 := U_2 \cap U_3$ ($U_4 \ni x_0$) such that

$$\emptyset \neq G(U_2) \subset z_0 + 2\gamma k_0 - \operatorname{int} C(x) \quad \text{and} \quad k_0 \in \operatorname{int} C(x) \quad \text{for every } x \in U_4.$$

This implies

$$F(x) \cap (z_0 + 2\gamma k_0 - \operatorname{int} C(x)) \neq \emptyset \quad \text{for every } x \in U_4.$$

Take $x \in U_4$, $y_x \in F(x) \cap (z_0 + 2\gamma k_0 - \operatorname{int} C(x))$. Therefore $y_x = z_0 + 2\gamma k_0 + c_x$ where $c_x \in -\operatorname{int} C(x)$. Then, we obtain

$$\begin{aligned}
\varphi^F(x_0) + \varepsilon &\ge t_0 \\
&\ge h_C(x, z_0; k_0) \quad \text{(by (2.1))} \\
&= h_C(x, y_x - 2\gamma k_0 - c_x; k_0) \\
&\ge h_C(x, y_x; k_0) - h_C(x, 2\gamma k_0; k_0) - h_C(x, c_x; k_0) \begin{pmatrix} \text{by subadditivity} \\ \text{of } h_C(x, \cdot; k_0) \end{pmatrix} \\
&\ge h_C(x, y_x; k_0) - 2\gamma \\
&\ge \varphi^F(x) - \varepsilon.
\end{aligned}$$

Hence

$$\varphi^F(x_0) + 2\varepsilon \geq \varphi^F(x) \quad \text{for every } x \in U_4.$$

The proof of the second statement (when C is constant-valued) is similar, however in this case there is no need to use Proposition 2.3. □

Remark 2.4. When we replace F by $-F$ in the two lemmas above, it leads to the lower semicontinuity of scalarizing functions $-\psi^{-F}$ and $-\varphi^{-F}$.

3. Existence Results

Firstly, we introduce our main tool, which is presented in [4, Theorem 2.3], for proving the main results in this paper.

Lemma 3.1 (See [4, Theorem 2.3]). *Let X be a nonempty compact convex subset of a topological vector space, $a : X \times X \to \mathbf{R}$ lower semicontinuous in its second variable, $b : X \times X \to \mathbf{R}$ quasiconvex in its second variable, and*

$$x, y \in X \quad and \quad a(x,y) > 0 \Rightarrow b(y,x) < 0.$$

If $\inf_{x\in X} b(x,x) \geq 0$, then there exists $z \in X$ such that $a(x,z) \leq 0$ for every $x \in X$.

Now we present two existence results for generalized vector equilibrium problems.

Theorem 3.1. *Let K be a nonempty convex subset of a topological vector space E, Y a topological vector space. Let $F : K \times K \to 2^Y$ be a multifunction. Assume that*

(i) *$C : K \to 2^Y$ is a multifunction such that for every $x \in K$, $C(x)$ is a closed convex cone in Y with $\operatorname{int} C(x) \neq \emptyset$;*
(ii) *$W : K \to 2^Y$ is a multifunction defined as $W(x) = Y \setminus (-\operatorname{int} C(x))$, and the graph of W is closed in $K \times Y$;*
(iii) *for every $x, y \in K$, $F(\cdot, y)$ is $(-C(x))$-upper semicontinuous at x;*
(iv) *there exists a multifunction $G : K \times K \to 2^Y$ such that*
  (a) *for every $x \in K$, $G(x,x) \not\subset -\operatorname{int} C(x)$,*
  (b) *for every $x, y \in K$, $F(x,y) \subset -\operatorname{int} C(x)$ implies $G(x,y) \subset -\operatorname{int} C(x)$,*
  (c) *$G(x,\cdot)$ is type-(v) $C(x)$-naturally quasiconvex on K for every $x \in K$,*
  (d) *$G(x,y)$ is compact, if $G(x,y) \subset -\operatorname{int} C(x)$;*
(v) *there exists a nonempty compact convex subset P of K such that for every $x \in K \setminus P$, there exists $y \in P$ with $F(x,y) \subset -\operatorname{int} C(x)$;*
(vi) *there exists a compact-valued multifunction $D : K \to 2^Y$ such that for each $x_0 \in K$,*
  (a) *$D(x_0) \subset \operatorname{int} C(x_0)$,*
  (b) *for every $t \in \mathbf{R}$, $k \in B(x_0)$ and $x, y \in K$ satisfying with $tk - F(x,y) \subset \operatorname{int} C(x_0)$, $tk - F(x,y) \subset \operatorname{cone} D(x_0)$.*

Then, the solutions set

$$S = \{x \in K \mid F(x,y) \not\subset -\operatorname{int} C(x), \text{ for every } y \in K\}$$

is a nonempty and compact subset of P.

Proof. Put

$$a(x,y) := -\inf_{k\in B(y)} \sup_{z\in F(y,x)} h_C(y,z;k), \quad b(x,y) := \inf_{k\in B(x)} \sup_{z\in G(x,y)} h_C(x,z;k).$$

It is easy to check that

$$a(x,y) > 0 \quad \text{if and only if} \quad F(y,x) \subset -\operatorname{int} C(y)$$

by using condition (vi), and also

$$b(y,x) < 0 \quad \text{if and only if} \quad G(y,x) \subset -\operatorname{int} C(y)$$

by using (d) of the condition (iv), and then $a(x,x) \le 0$ and $b(x,x) \ge 0$.

Denote

$$S_y := \{x \in P \mid F(x,y) \not\subset -\operatorname{int} C(x)\} = \{x \in P \mid a(y,x) \le 0\}\,. \tag{3.1}$$

Since $a(y,\cdot)$ is lower semicontinuous (by Lemma 2.5), the set S_y is closed. Let Y_0 be a finite subset of K. Denote by Z the closed convex hull of $Y_0 \cup P$. Obviously Z is compact and convex. Lemmas 2.2, 2.5 and (b) of the condition (iv) show that the conditions of Lemma 3.1 are satisfied.

Now we apply Lemma 3.1 and obtain a point $z \in Z$ such that $a(y,z) \le 0$ for every $y \in Z$, which means

$$F(z,y) \not\subset -\operatorname{int} C(z) \quad \text{for every } y \in Z. \tag{3.2}$$

The conditions (v) and (3.2) imply that $z \in P$. Relation (3.1) implies that

$$\cap\{S_y \mid y \in Y_0\} \neq \emptyset.$$

So we proved that the family $\{S_y \mid y \in K\}$ has finite intersection property. Since P is compact,

$$\cap\{S_y \mid y \in K\} \neq \emptyset,$$

which means that there exists $x_0 \in K$ such that

$$F(x_0,y) \not\subset -\operatorname{int} C(x_0) \quad \text{for every } y \in K.$$

So we proved that S is nonempty, and since S is a closed subset of P, the proof is completed. □

Remark 3.1. The above theorem is a generalization of the theorem that it is replaced F and G in [4, Theorem 4.1] by $-F$ and $-G$, respectively. The main difference between our result and [4, Theorem 4.1] is (c) of the condition (iv), which is more generalized with respect to convexity.

Theorem 3.2. *Let K be a nonempty convex subset of a topological vector space E, Y a topological vector space. Let $F : K \times K \to 2^Y$ be a multifunction. Assume that*

(i) *$C : K \to 2^Y$ is a multifunction such that for every $x \in K$, $C(x)$ is a closed convex cone in Y with $\operatorname{int} C(x) \neq \emptyset$;*
(ii) *$W : K \to 2^Y$ is a multifunction defined as $W(x) = Y \setminus (-\operatorname{int} C(x))$, and the graph of W is closed in $K \times Y$;*
(iii) *for every $x, y \in K$, $F(\cdot, y)$ is $(-C(x))$-lower semicontinuous at x;*
(iv) *there exists a multifunction $G : K \times K \to 2^Y$ such that*
  (a) *for every $x \in K$, $G(x,x) \cap (-\operatorname{int} C(x)) = \emptyset$,*
  (b) *for every $x, y \in K$, $F(x,y) \cap (-\operatorname{int} C(x)) \neq \emptyset$ implies $G(x,y) \cap (-\operatorname{int} C(x)) \neq \emptyset$,*
  (c) *$G(x,\cdot)$ is $C(x)$-quasiconvex on K for every $x \in K$,*
  (d) *G is convex-valued;*

(v) *there exists a nonempty compact convex subset P of K such that for every $x \in K \setminus P$, there exists $y \in P$ with $F(x,y) \cap (-\operatorname{int} C(x)) \neq \emptyset$;*

(vi) *there exists a compact-valued multifunction $D : K \to 2^Y$ such that for each $x_0 \in K$,*

(a) *$D(x_0) \subset \operatorname{int} C(x_0)$,*

(b) *for every $t < t^* \in \mathbf{R}$, $k \in B(x_0)$, $x, y \in K$ and $z \in F(x_0, y)$ satisfying with $F(x,y) \cap [z + tk - \operatorname{int} C(x_0)] \neq \emptyset$, $F(x,y) \cap [z + t^*k - \operatorname{cone} D(x_0)] \neq \emptyset$.*

Then, the solutions set

$$S = \{x \in K \mid F(x,y) \cap (-\operatorname{int} C(x)) = \emptyset, \text{ for every } y \in K\}$$

is a nonempty and compact subset of P.

Proof. Put

$$a(x,y) := -\inf_{k \in B(y)} \inf_{z \in F(y,x)} h_C(y,z;k), \quad b(x,y) := \inf_{k \in B(x)} \inf_{z \in G(x,y)} h_C(x,z;k).$$

It is easy to check that

$$a(x,y) > 0 \text{ if and only if } F(y,x) \cap (-\operatorname{int} C(y)) \neq \emptyset,$$

$$b(y,x) < 0 \text{ if and only if } G(y,x) \cap (-\operatorname{int} C(y)) \neq \emptyset,$$

$$a(x,x) \leq 0, \quad b(x,x) \geq 0.$$

Lemmas 2.4, 2.6 and (b) of the condition (iv) show that the conditions of Lemma 3.1 are satisfied. Further the proof is the same as that of Theorem 3.1, but in this case $S_y := \{x \in P \mid F(x,y) \cap (-\operatorname{int} C(x)) = \emptyset\}$. □

Remark 3.2. The above theorem is an improvement of the theorem that it is replaced F and G in [4, Theorem 4.2] by $-F$ and $-G$, respectively. The main difference between our result and [4, Theorem 4.2] is the condition (vi), which is more relaxed with respect to compactness assumption of the images of F in the condition (iii) of [4, Theorem 4.2]. However, (d) of the condition (iv) is added in comparison with [4, Theorem 4.2], because we want to use Lemma 2.4 in the proof directly.

4. Conclusions

We have established new inherited properties of convexity for set-valued maps. By using one of those new inherited properties and applying to set-valued Fan's inequality in [4, 5], we have generalized the existence theorem in [1]. We have also presented an existence theorem for a different type of the generalized vector equilibrium problem in [1].

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