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CLASSIFICATIONS OF IRRATIONAL NUMBERS AND RECURRENT DIMENSIONS OF QUASI-PERIODIC ORBITS

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Dedicated to Professor Wataru Takahashi for his 60th Birthday

Abstract. Diophantine conditions in the famous KAM theorem and also Liouville conditions(not Diophantine conditions) in the converse KAM theorem are deeply related to classifications of irrational numbers (of winding numbers or frequencies) according to badness or goodness levels of approximation by rational numbers. In our previous paper we introduced the gaps between the upper and the lower recurrent dimensions as the index parameters, which measure unpredictability levels of the orbits. In this paper we show that the gaps of recurrent dimensions of quasi-periodic orbits take positive values when the irrational frequencies are weak Liouville numbers with sufficiently large orders of goodness levels of approximation by rational numbers.

1. Introduction

The celebrated KAM theorem ignited so many mathematicians to study about stability or instability of the KAM tori and to proceed (or to be drowned) in analysis of chaotic systems. Showing various critical conditions for destructions of KAM quasi-periodic orbits is the most interesting and exciting problem in the analysis of " chaos". According to the KAM theorem (cf. [2] or [3]), if the irrational frequencies of integrable quasi-periodic Hamiltonian dynamical systems satisfy the Diophantine conditions (badly approximable by rationals), then the quasi-periodic tori are persistent (stable for small perturbations). On the other hand, we can see in the converse KAM theorem that, if the irrational frequencies are Liouville numbers (extremely good approximative numbers by rationals), then any small perturbations of the quasi-periodic systems contain the destructions of the quasi-periodic tori (see [9]).

In our previous papers ([4], [5], [6]) we estimated correlation dimensions or recurrent dimensions of discrete quasi-periodic orbits according to algebraic properties of the irrational frequencies, introducing some classes of irrational numbers, α-order Roth numbers (Diophantine numbers), which contains the class of Roth numbers, and α-order Liouville or α-order weak Liouville numbers, which have good approximation by rational numbers. These irrational numbers are classified according to badness or goodness levels of approximation by rational numbers. In our previous paper ([7]) we also introduced the gaps between the upper and the lower recurrent dimensions as the index parameters, which measure unpredictability levels of the orbits. In this paper we show that the gaps of recurrent dimensions of quasi-periodic

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orbits take positive values when the irrational frequencies are weak Liouville numbers with sufficiently large orders of goodness levels of approximation by rational numbers.

The plan of this paper is as follows. In section 2 we give definitions of recurrent dimensions and the gaps of these dimensions. In section 3 we review classifications of irrational numbers, following our pervious papers. In section 4 we estimate the gaps of recurrent dimensions of quasi-periodic orbits. In Appendixes we give examples of Roth or Liouville type numbers, following the classifications of irrational numbers.

2. Definitions of recurrent dimensions

We have introduced the definitions of recurrent dimensions in our previous papers ([6], [7]):

Let T be a nonlinear operator on a Banach space X. For an element $x \in X$ we consider a discrete dynamical system given by

$$x_n = T^n x, \quad n \in \mathbf{N}_0 := \mathbf{N} \cup \{0\}$$

and its orbit is denoted by

$$\Sigma_x = \{T^n x : n \in \mathbf{N}_0\}.$$

For a small $\varepsilon > 0$, define upper and lower first ε-recurrent times by

$$\overline{M}_\varepsilon = \sup_{n \in \mathbf{N}_0} \min\{m : T^{m+n}x \in V_\varepsilon(T^n x),\ m \in \mathbf{N}\},$$
$$\underline{M}_\varepsilon = \inf_{n \in \mathbf{N}_0} \min\{m : T^{m+n}x \in V_\varepsilon(T^n x),\ m \in \mathbf{N}\},$$

respectively, where $V_\varepsilon(z) = \{y \in X : \|y - z\| < \varepsilon\}$. Then upper and lower recurrent dimensions are defined as follows:

$$\overline{D}_r(\Sigma_x) = \limsup_{\varepsilon \to 0} \frac{\log \overline{M}_\varepsilon}{-\log \varepsilon},$$
$$\overline{d}_r(\Sigma_x) = \limsup_{\varepsilon \to 0} \frac{\log \underline{M}_\varepsilon}{-\log \varepsilon},$$
$$\underline{d}_r(\Sigma_x) = \liminf_{\varepsilon \to 0} \frac{\log \overline{M}_\varepsilon}{-\log \varepsilon},$$
$$\underline{D}_r(\Sigma_x) = \liminf_{\varepsilon \to 0} \frac{\log \underline{M}_\varepsilon}{-\log \varepsilon}.$$

On the other hand, we can define the gaps between the upper and the lower recurrent dimensions by

$$G_r(\Sigma_x) = \overline{D}_r(\Sigma_x) - \underline{D}_r(\Sigma_x), \quad g_r(\Sigma_x) = \overline{d}_r(\Sigma_x) - \underline{d}_r(\Sigma_x). \tag{2.1}$$

Since

$$\overline{D}_r(\Sigma_x) \geq \overline{d}_r(\Sigma_x), \quad \underline{d}_r(\Sigma_x) \geq \underline{D}_r(\Sigma_x),$$

we have $G_r(\Sigma_x) \geq g_r(\Sigma_x)$.

Instead of considering the whole orbit we can treat a local point in the orbit, say, $T^{n_0}x,\ n_0 \in \mathbf{N}$. Define the first ε-recurrent time by

$$M_\varepsilon(n_0) = \min\{m \in \mathbf{N} : T^{m+n_0}x \in V_\varepsilon(T^{n_0}x)\}$$

and the upper and lower recurrent dimensions by

$$\overline{D}_r(n_0) = \limsup_{\varepsilon \to 0} \frac{\log M_\varepsilon(n_0)}{-\log \varepsilon},$$
$$\underline{D}_r(n_0) = \liminf_{\varepsilon \to 0} \frac{\log M_\varepsilon(n_0)}{-\log \varepsilon}.$$

The gaps of the recurrent dimensions can be defined by

$$G_r(n_0) = \overline{D}_r(n_0) - \underline{D}_r(n_0)$$

and, obviously, we have

$$G_r(\Sigma_x) \geq G_r(n_0) \geq g_r(\Sigma_x)$$

for $n_0 \in \mathbf{N}$.

If the gaps between the upper and the lower recurrent dimensions take positive values, we cannot exactly determine or predict the ε-recurrent time of the orbits. Thus we propose the value $G_r(\Sigma_x)$ or $g_r(\Sigma_x)$ as the parameter, which measures the unpredictability level of the orbit.

3. Roth numbers and Liouville numbers

Let $f(t) : [0, +\infty) \to X$ be a continuous periodic function such that $f(t+1) = f(t),\ t \geq 0$. For an irrational number $\tau : 0 < \tau < 1$, define a discrete quasi-periodic orbit by

$$T^n x = f(\tau n), \quad n \in \mathbf{N}, \qquad x = f(0).$$

Then our purpose is to estimate the recurrent dimensions of the orbits under some algebraic conditions on the frequency τ.

Consider the following continued fraction of the number τ:

$$\tau = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ddots}}} \qquad (a_i \in \mathbf{N}) \tag{3.1}$$

and take the rational approximation as follows. Let $m_0 = 1,\ n_0 = 0,\ m_{-1} = 0,\ n_{-1} = 1$ and define the pair of sequences of natural numbers

$$m_i = a_i m_{i-1} + m_{i-2}, \tag{3.2}$$
$$n_i = a_i n_{i-1} + n_{i-2}, \quad i \geq 1, \tag{3.3}$$

then the elementary number theory gives the Diophantine approximation $\{n_i/m_i\}$, which satisfies

$$\frac{1}{m_i(m_{i+1} + m_i)} < |\tau - \frac{n_i}{m_i}| < \frac{1}{m_i m_{i+1}} < \frac{1}{m_i^2}. \tag{3.4}$$

Here $\{n_i/m_i\}$ is the best approximation in the sense that

$$|\tau - \frac{n_i}{m_i}| \leq |\tau - \frac{r}{p}| \tag{3.5}$$

holds for every rational $r/p:\ p \leq m_i$, and, furthermore,

$$\inf_{r \in \mathbf{N}} |\tau m - r| \geq |\tau m_i - n_i| \tag{3.6}$$

holds for every $m : 1 \leq m < m_{i+1}$.

We can classify irrational numbers according to badness or goodness levels of approximation by rational numbers. First we treat badly approximable irrational numbers by rational numbers.

We call a irrationla number τ an α-order Roth number if there exists a constant $\alpha \geq 0$ such that, for every $\beta > \alpha$, there exists $c_\beta > 0$, which satisfies

$$|\tau - \frac{q}{p}| \geq \frac{c_\beta}{p^{2+\beta}}$$

for every rational q/p.

Roth numbers are 0-order Roth numbers, that is, for every $\varepsilon > 0$, there exists $c_\varepsilon > 0$, which satisfies

$$|\tau - \frac{q}{p}| \geq \frac{c_\varepsilon}{p^{2+\varepsilon}}$$

for every rational q/p.

Furthermore, for the most badly approximable numbers we say that τ is constant type or badly approximable if there exists a constant $c > 0$ such that

$$|\tau - \frac{q}{p}| \geq \frac{c}{p^2}$$

holds for every rational q/p.

The set of α-order Roth numbers for each $\alpha \geq 0$ is of full measure in the Lebesgue sense, while the class of constant type is dense in the space of real numbers, but it is of null measure.

In our previous papers ([5], [6], [8]) we have shown some equivalent conditions to these numbers by using the growth rates of $\{m_i\}$. However, the definitions of α-Roth numbers and the equivalent conditions in these papers are slightly different. For completeness we prove the equivalence relation.

Lemma 3.1. *τ is a Roth number with its order $\alpha_0 \geq 0$ if and only if for every $\beta : \beta > \alpha_0$ there exists a constant $K_\beta > 0$, which satisfies*

$$m_{j+1} \leq K_\beta m_j^{1+\beta}, \quad \forall j. \tag{3.7}$$

Proof. First we assume that τ is a Roth number with its order $\alpha_0 \geq 0$. It follows from the definition and (3.4) that, for every $\beta > \alpha_0$, there exists $K_\beta > 0$:

$$\frac{K_\beta^{-1}}{m_j^{2+\beta}} \leq |\tau - \frac{n_j}{m_j}| \leq \frac{1}{m_{j+1} m_j}. \tag{3.8}$$

Thus we obtain (3.7).

Next we assume (3.7) where we can assume that $K_\beta \geq 1$. Then we show that

$$|\tau m - l| \geq \frac{1}{2K_\beta m^{\beta+1}}$$

holds for every integer l, m. In the case where $m = m_k$ it is obvious from (3.4), (3.5) and (3.7).

Assume that there exist some positive integers $l, m : m_k < m < m_{k+1}$, which satisfies

$$|\tau m - l| < \frac{1}{K_\beta m^{\beta+1}}.$$

It follows from (3.4) and (3.6) that we have

$$\frac{1}{K_\beta m^{1+\beta}} > |\tau m - l| \geq |\tau m_k - n_k| \geq \frac{1}{m_{k+1} + m_k}.$$

Thus we have

$$\frac{1}{K_\beta (m_k + 1)^{1+\beta}} > \frac{1}{K_\beta m_k^{1+\beta} + m_k},$$

which yields the contradiction, since

$$K_\beta (x+1)^{1+\beta} > K_\beta x^{\beta+1} + x$$

holds for all $x \geq 1$. □

The equivalent condition of a badly approximable number is the boundedness of the partial quotients $\{a_i\}$ (cf. [10]). Thus, τ is badly approximable if and only if there exists a constant $K > 0$ such that

$$m_{j+1} \leq K m_j, \quad \forall j.$$

Next we consider Liouville(not Diophantine) numbers. An irrational number τ, which has extremely good approximable property by rational numbers, is called a Liouville number if

$$|\tau - \frac{n_i}{m_i}| \leq \frac{1}{m_i^i}, \quad \forall i.$$

In our previous paper [6] we introduced a class of irrational numbers which contains Liouville numbers. We state that an irrational number τ is an α-order Liouville number, or a Liouville number with its order α if there exist constants $c, \alpha > 0$ such that

$$|\tau - \frac{n_i}{m_i}| \leq \frac{c}{m_i^{2+\alpha}}, \quad \forall i. \tag{3.9}$$

Furthermore, considering some subsequece of the Diophantine approximation, we can extend the definitions of α-order Liouville numbers as follows. τ is called an α-order weak Liouville number if there exists a subsequence $\{m_{k_j}\} \subset \{m_j\}$, which satisfies

$$|\tau - \frac{n_{k_j}}{m_{k_j}}| < \frac{c}{m_{k_j}^{2+\alpha}}, \quad \forall j$$

for some constants $c, \alpha > 0$ (see [8]).

For the α-order Liouville numbers we have given the equivalent condition in [6].

Lemma 3.2 ([6]). *τ is a Liouville number with its order α if and only if there exist constants $\alpha, L > 0$:*

$$m_{j+1} \geq L m_j^{1+\alpha}, \quad \forall j. \tag{3.10}$$

Obviously, (3.10) is equivalent to the following condition on the partial quotients in the continued fraction expansion of τ.

There exist constants $\alpha, L' > 0$:

$$a_{j+1} \geq L' m_j^{\alpha}, \quad \forall j. \tag{3.11}$$

For an α-order Liouville number we have shown the following lemma.

Lemma 3.3 ([6])**.** *If the partial quotients in the continued fraction expansion of τ satisfies*

$$a_{j+1} \geq L_0 a_j^{\beta+1}, \quad \forall j$$

for some $\beta > 0$ and $L_0 \geq 2^{\beta+1}$, then τ is a Liouville number with its order β.

For the weak Liouville numbers we can show the equivalent condition.

Lemma 3.4 ([8])**.** *τ is a weak Liouville number with its order α if and only if there exist constants $\alpha, L > 0$:*

$$m_{k_j+1} \geq L m_{k_j}^{1+\alpha}, \quad \forall j. \tag{3.12}$$

Next we consider some relations between Roth numbers and Liouville numbers.

We can prove that the class of weak Liouville numbers is equivalent to the complement (in the space of all irrational numbers) of the class of Roth numbers.

Lemma 3.5. *For a constant $\alpha_0 > 0$, let*

$$\alpha_0 = \inf\{\alpha : \tau \textit{ is an } \alpha\textit{-order Roth number}\},$$

then τ is a β-order weak Liouville number for every $\beta : \beta < \alpha_0$. On the contrary, let τ be a β-order weak Liouville number, then τ is not an α-order Roth number for every $\alpha : \alpha < \beta$.

Proof. Since we have

$$\alpha_0 = \sup\{\alpha : \tau \text{ is not an } \alpha\text{-order Roth number}\},$$

let τ be not an α-order Roth number for $\alpha < \alpha_0$. Then there exists a constant $\beta : \alpha < \beta \leq \alpha_0$ such that for every $d_j : 0 < d_j < 1,\ d_j \to 0$ as $j \to \infty$, there exist positive integers l_j, r_j, which satisfies

$$|l_j \tau - r_j| \leq \frac{d_j}{l_j^{1+\beta}}. \tag{3.13}$$

We can take an infinite sequence $\{l_j\} : l_j \to \infty$. In fact, let $d_j \to 0$, then the existence of a finite set or a bounded infinite sequence $\{l_j\}$, which has a convergent subsequence, gives a contradiction by (3.13) that τ is a rational number. It follows from (3.6) that, for each l_j, there exists $m_{k_j} \in \{m_j\} : m_{k_j} \leq l_j < m_{k_j+1}$, which satisfies

$$|m_{k_j}\tau - n_{k_j}| \leq |l_j \tau - r_j| \leq \frac{d_j}{l_j^{1+\beta}} \leq \frac{1}{m_{k_j}^{1+\beta}}.$$

Thus τ is a β-order weak Liouville number. Since for every $\alpha < \alpha_0$ there exists $\beta : \alpha < \beta \leq \alpha_0$ such that τ is a β-order weak Liouville number, we can conclude that τ is a β-order weak Liouville number for every $\beta : \beta < \alpha_0$.

On the other hand, let τ be a β-order weak Liouville number τ, that is, there exists a subsequence $\{m_{k_j}\}$, which satisfies

$$|m_{k_j}\tau - n_{k_j}| \leq \frac{c}{m_{k_j}^{1+\beta}},$$

and assume that τ is a Roth number with its order $\alpha : \alpha < \beta$. Then, for every $\gamma : \alpha < \gamma < \beta$, there exists a constant c_γ such that

$$|m\tau - n| \geq \frac{c_\gamma}{m^{1+\gamma}}$$

holds for every positive integers m, n. Putting $m = m_{k_j}, n = n_{k_j}$ gives

$$\frac{c_\gamma}{m_{k_j}^{1+\gamma}} \leq \frac{c}{m_{k_j}^{1+\beta}},$$

which has the contradiction if $j \to \infty$. Thus τ is not an α-order Roth number for every $\alpha : \alpha < \beta$. □

Denote the set of α-order Roth numbers by $R(\alpha)$ and the set of β-order weak Liouville numbers by $wL(\beta)$, then it follows from the proof of Lemma 3.5 that the following inclusion relations hold.

$$R(\alpha)^c \subset \bigcap_{\beta<\alpha} wL(\beta),$$

$$wL(\beta) \subset \bigcap_{\beta>\alpha} R(\alpha)^c$$

where the complements are considered in the set of all irrational numbers. That is,

$$\text{(3.14)} \quad \inf\{\alpha : \tau \text{ is an } \alpha\text{-order Roth number}\} = \sup\{\beta : \tau \text{ is a } \beta\text{-order weak Liouville number}\} := d_0.$$

Thus, for each irrational number τ, there exists a constant d_0, which specifies the badness or goodness levels of rational approximations. In our previous paper ([7]) we introduced a d_0-(D) condition for a pair of irrational numbers. For a single irrational case, let us say that τ satisfies a d_0-(D) condition if (3.14) holds.

Applying the argument in the above proof, we can show the relation:

$$R(0)^c = \bigcup_{\beta>0} wL(\beta).$$

Lemma 3.6 ([7]). *If an irrational number is not a Roth number, then it is an α-order weak Liouville number for some $\alpha > 0$. On the contrary, if an irrational number is an α-order weak Liouville number for some $\alpha > 0$, then it is not a Roth number.*

Remark 3.7. Since the set of 0-order Roth numbers is of full measure, the set of irrational numbers, which satisfy d_0-(D) conditions for $d_0 > 0$, is of null measure.

4. Recurrent dimensions of quasi-periodic orbits

In this section, considering a quasi-periodic orbit in a Banach space X with its irrational frequency given by a weak Liouville number:

$$\Sigma = \{\varphi(n) \in X : \varphi(n) = f(\tau n),\ n \in \mathbf{N}_0\},$$

we estimate the recurrent dimensions of Σ. Here, let $f : \mathbf{R} \to X$ be a nonlinear function, which satisfies the following Hölder conditions:

(H1) There exist constants $K_1 > 0$ and $\vartheta_1 :\ 0 < \vartheta_1 \leq 1$, which satisfy

$$\|f(t_1) - f(t_2)\| \leq K_1|t_1 - t_2|^{\vartheta_1}, \quad t_1, t_2 \in \mathbf{R} :\ |t_1 - t_2| \leq \varepsilon_0$$

for a small constant $\varepsilon_0 > 0$.

(H2) There exist constants $K_2 > 0$ and $\vartheta_2 :\ 0 < \vartheta_2 \leq 1$, which satisfy

$$\|f(t_1) - f(t_2)\| \geq K_2|t_1 - t_2|^{\vartheta_2}, \quad t_1, t_2 \in \mathbf{R} :\ |t_1 - t_2| \leq \frac{1}{2}.$$

We can obtain the following estimate for the upper bounds of the lower recurrent dimensions.

Theorem 4.1. *Under the condition* **(H1)***, assume that the irraional frequency τ is an α-order weak Liouville number. Then we have*

$$\underline{d}_r(\Sigma) \leq \frac{1}{(1+\alpha)\vartheta_1}. \tag{4.1}$$

Proof. It follows from **(H1)**, (3.4) and Lemma 3.4 that we have

$$\begin{aligned}
\|\varphi(m_{k_j} + n) - \varphi(n)\| &= \|f(\tau(m_{k_j} + n)) - f(\tau n + n_{k_j})\| \\
&\leq K_1|\tau m_{k_j} - n_{k_j}|^{\vartheta_1} \\
&\leq K_1(\frac{1}{m_{k_j+1}})^{\vartheta_1} \\
&\leq c m_{k_j}^{-(1+\alpha)\vartheta_1} \\
&:= \varepsilon_j.
\end{aligned}$$

Here and hereafter c denotes a suitable constant in each estimate. Thus we can estimate

$$\begin{aligned}
\underline{d}_r(\Sigma) &= \lim_{j\to\infty} \inf_{0<\varepsilon\leq\varepsilon_j} \frac{\log \overline{M}_\varepsilon}{-\log \varepsilon} \\
&\leq \lim_{j\to\infty} \frac{\log \overline{M}_{\varepsilon_j}}{-\log \varepsilon_j} \\
&\leq \lim_{j\to\infty} \frac{\log m_{k_j}}{-\log \varepsilon_j} \\
&= \lim_{j\to\infty} \frac{\log m_{k_j}}{-\log c m_{k_j}^{-(1+\alpha)\vartheta_1}} \\
&\leq \lim_{j\to\infty} \frac{\log m_{k_j}}{-\log c + (1+\alpha)\vartheta_1 \log m_{k_j}}
\end{aligned}$$

$$= \frac{1}{(1+\alpha)\vartheta_1}. \qquad \square$$

Furthermore, by using the Hölder exponents and simple calculations we can estimate the lower bounds of the recurrent dimensions without any algebraic conditions on the irrational frequencies.

Theorem 4.2. *Assume that the function f satisfies* **(H2)**. *Then we have*

$$\overline{d}_r(\Sigma) \geq \frac{1}{\vartheta_2}.$$

Proof. Since, for every integer $m : m < m_{j+1}$, there exists an integer n' which satisfies $|\tau m - n'| < 1/2$, it follows from **(H2)**, (3.4) and (3.6) that we have

$$\begin{aligned} \|\varphi(m+n) - \varphi(n)\| &= \|f(\tau(m+n)) - f(\tau n + n')\| \\ &\geq K_2 |\tau m - n'|^{\vartheta_2} \\ &\geq K_2 |\tau m_j - n_j|^{\vartheta_2} \\ &\geq K_2 (\frac{1}{2m_{j+1}})^{\vartheta_2} = c(\frac{1}{m_{j+1}})^{\vartheta_2} := \varepsilon_j. \end{aligned}$$

Thus we can estimate

$$\begin{aligned} \overline{d}_r(\Sigma) &= \lim_{j\to\infty} \sup_{0<\varepsilon\leq\varepsilon_j} \frac{\log \underline{M}_\varepsilon}{-\log\varepsilon} \\ &\geq \lim_{j\to\infty} \frac{\log \underline{M}_{\varepsilon_j}}{-\log\varepsilon_j} \\ &\geq \lim_{j\to\infty} \frac{\log m_{j+1}}{\log c^{-1} m_{j+1}^{\vartheta_2}} \\ &= \frac{1}{\vartheta_2}. \qquad \square \end{aligned}$$

By using Theorem 4.1 and 4.2 we obtain the gap of the recurrent dimension.

Corollary 4.3. *Under Hypotheses* **(H1)** *and* **(H2)**, *assume the same assumptions as those of Theorem 4.1. Then we have*

$$g_r(\Sigma) \geq \frac{1}{\vartheta_2} - \frac{1}{(1+\alpha)\vartheta_1}. \tag{4.2}$$

Remark 4.4. It follows from (4.2) that the gaps of the recurrent dimensions become positive if $\vartheta_1 \simeq \vartheta_2$ or the order α is sufficiently large.

5. Appendix

In this section we summarize our previous results on the classifications of irrational numbers and examples. Before showing examples of Roth numbers we prepare some Lemmas.

Lemma 5.1 ([8]). *Assume that there exists a subsequence $\{m_{k_j}\}$ which satisfies*

$$m_{k_{j+1}} \leq K m_{k_j}^{1+\beta}, \quad \forall j. \tag{5.1}$$

for some constants $\beta, K > 0$. Then τ is a Roth number with its order $\beta(\beta+3)$.

Proof. For every positive integer q, there exists a number j:

$$m_{k_{j-1}} < q \leq m_{k_j} < K m_{k_{j-1}}^{\beta+1} < K q^{\beta+1}.$$

Since n_j/m_j is a best approximation of τ, we have

$$\begin{aligned}|\tau - \frac{r}{q}| &\geq |\tau - \frac{n_{k_j}}{m_{k_j}}| \\ &\geq \frac{1}{(m_{k_j+1} + m_{k_j})m_{k_j}} \\ &\geq \frac{1}{(m_{k_{j+1}} + m_{k_j})m_{k_j}} \\ &\geq \frac{1}{2m_{k_{j+1}}m_{k_j}} \geq \frac{c}{m_{k_j}^{\beta+2}} \\ &> \frac{c}{q^{(\beta+1)(\beta+2)}}\end{aligned}$$

where we denote by c a suitable constant in each term. Thus for every rational number r/q we have

$$|\tau - \frac{r}{q}| > \frac{c}{q^{2+\beta(\beta+3)}}.$$ □

In [5] we have given a sufficient condition for an α-order Roth number, using the partial quotients of the continued fraction expansion.

Lemma 5.2 ([5])**.** *Let $\{a_j\}$ be the partial quotients in the continued fraction expansion of τ. Assume that, for a given constant $\varepsilon > 0$, there exists a constant $C_\varepsilon > 0$, which satisfies*

$$a_{j+1}a_j^2 \leq C_\varepsilon (a_{j-1}a_{j-2}\cdots a_1)^\varepsilon, \quad \forall j.$$

Then we have

$$|\tau - \frac{r}{q}| \geq \frac{c_\varepsilon}{q^{2+\varepsilon}}, \quad \forall q, r \in N$$

where $c_\varepsilon = 1/(16C_\varepsilon)$.

Also we introduce another sufficient condition for α-order Roth numbers.

Lemma 5.3 ([8])**.** *Let $\{a_j\}$ be the partial quotients in the continued fraction expansion of τ. Assume that there exists a subsequence $\{a_{k_j}\}$, which satisfies that, for a given constant $\varepsilon > 0$, there exists a constant $C_\varepsilon > 0$ such that*

$$\begin{aligned}(a_{k_j+1}+1)(a_{k_j}+1)^2(a_{k_j-1}+1)^2\cdots \\ \cdots(a_{k_{j-1}+2}+1)^2(a_{k_{j-1}+1}+1)^2 \\ \leq C_\varepsilon(a_{k_{j-1}}a_{k_{j-1}-1}\cdots a_1)^\varepsilon, \quad \forall j.\end{aligned}$$

Then we have

$$|\tau - \frac{r}{q}| \geq \frac{c_\varepsilon}{q^{2+\varepsilon}}, \quad \forall q, r \in N.$$

Proof. Let q be a positive integer, then there exists a number $j : m_{k_{j-1}} < q \leq m_{k_j}$ and we have

$$\begin{aligned} m_{k_{j-1}} &< q \leq m_{k_j} \\ &\leq (a_{k_j}+1)(a_{k_j-1}+1)\cdots(a_{k_{j-1}+1}+1)m_{k_{j-1}} \\ &\leq (a_{k_j}+1)(a_{k_j-1}+1)\cdots(a_{k_{j-1}+1}+1)q. \end{aligned}$$

Since n_j/m_j is the best rational approximation, it follows from Hypothesis that we have

$$\begin{aligned} |\tau - \frac{r}{q}| &\geq |\tau - \frac{n_{k_j}}{m_{k_j}}| \\ &\geq \frac{1}{(m_{k_j+1}+m_{k_j})m_{k_j}} \geq \frac{1}{2(a_{k_j+1}+1)m_{k_j}^2} \\ &\geq \frac{1}{2(a_{k_j+1}+1)(a_{k_j}+1)^2\cdots(a_{k_{j-1}+1}+1)^2q^2} \\ &\geq \frac{1}{2C_\varepsilon(a_{k_{j-1}}a_{k_{j-1}-1}\cdots a_1)^\varepsilon q^2} \end{aligned}$$

for every $r \in N$. On the other hand, we can estimate

$$\begin{aligned} q > m_{k_{j-1}} &\geq a_{k_{j-1}}m_{k_{j-1}-1} \geq \cdots \\ &\geq a_{k_{j-1}}a_{k_{j-1}-1}\cdots a_1 m_0 \\ &= a_{k_{j-1}}a_{k_{j-1}-1}\cdots a_1. \end{aligned}$$

Thus we obtain the conclusion. □

Now we consider some examples of Roth numbers.

For two sequences $\{a_j\}, \{b_j\}$, we write $a_j \sim b_j$ if there exist constants $c_1, c_2 > 0$ such that

$$c_1 a_j < b_j < c_2 a_j.$$

Example 5.4. If $a_j \sim j^\alpha$, $\alpha > 0$, then τ is a Roth number (0-order Roth number).

In fact, for every $\varepsilon > 0$ there exists d_ε:

$$(j+1)^{\frac{3}{\varepsilon}} c_2^{\frac{3}{\alpha\varepsilon}} c_1^{-\frac{j-1}{\alpha\varepsilon}} \leq d_\varepsilon (j-1)!, \qquad \forall j. \tag{5.2}$$

It follows that

$$c_2^3(j+1)^{3\alpha} \leq d'_\varepsilon\{c_1^{j-1}(j-1)!\}^{\alpha\varepsilon}$$

and we have

$$a_{J+1}^3 < d'_\varepsilon(a_{j-1}a_{j-2}\cdots a_1)^\varepsilon.$$

Thus we can apply Lemma 5.2 for every $\varepsilon > 0$.

Example 5.5. If $a_j \sim K^j$, $K > 1$, then τ is also a Roth number.

In fact, for every $\varepsilon > 0$ there exists j_ε:

$$c_2^3 K^{(3+\frac{\log c_1^{-\varepsilon}}{\log K})j_\varepsilon+1} < c_1^{-\varepsilon} K^{\frac{(j_\varepsilon-1)j_\varepsilon}{2}\varepsilon}.$$

Put

$$d_\varepsilon = c_2^3 K^{(3+\frac{\log c_1^{-\varepsilon}}{\log K})j_\varepsilon+1},$$

then we have

$$c_2^3 K^{3j+1} < d_\varepsilon (c_1^{j-1} K^{j-1} \cdots K^2 K^1)^\varepsilon, \quad \forall j,$$

which yields Hypothesis of Lemma 5.2.

Example 5.6. If $a_{j+1} \sim m_j^\beta, \quad \beta > 0$, then it follows from Lemma 3.1 that τ is a β-order Roth number.

Now we consider the case that the growth rate of a_j has the order M^{κ^j} for some constans $M, \kappa > 1$.

Lemma 5.7 ([5]). *For constants $c_1, c_2, M, \kappa, \alpha : \ M, \kappa > 1, \quad \alpha \geq 1$, assume that the partial quotients in the continued fraction expansion of τ satisfies*

$$c_1 M^{\kappa^j} < a_j < c_2 (M^\alpha)^{\kappa^j}. \tag{5.3}$$

Then τ is a Roth number with its order $(\kappa - 1)(\kappa + 2)\alpha$.

Next we consider some examples of Liouville type numbers. Obviously, (3.12) is equivalent to the following condition on the partial quotients in the continued fraction expansion of τ.

There exist constants $\alpha, L' > 0$:

$$a_{k_j+1} \geq L' m_{k_j}^\alpha, \quad \forall j. \tag{5.4}$$

Furthermore, for a weak Liouville number, we can show the following lemma.

Lemma 5.8 ([8]). *Assume that the partial quotients $\{a_j\}$ in the continued fraction expansion of τ has a subsequence $\{a_{k_j}\}$, which satisfies*

$$a_{k_{j+1}+1} \geq (a_{k_{j+1}} + 1)^\beta (a_{k_{j+1}-1} + 1)^\beta \cdots (a_{k_j+1} + 1)^\beta a_{k_j+1} \tag{5.5}$$

for some $\beta > 0$, then τ is a weak Liouville number with its order β.

Proof. By induction we can show that the condition (5.4) is satisfied. It is obvious that there exists a constant $L > 0$: Let $k_0 = 0$, then

$$a_{k_0+1} = a_1 \geq L m_0^\beta = L \cdot 1.$$

Assume that

$$a_{k_j+1} \geq L m_{k_j}^\beta.$$

It follows from Hypothesis that we have

$$\begin{aligned} a_{k_{j+1}+1} &\geq (a_{k_{j+1}} + 1)^\beta (a_{k_{j+1}-1} + 1)^\beta \cdots (a_{k_j+1} + 1)^\beta a_{k_j+1} \\ &\geq L (a_{k_{j+1}} + 1)^\beta (a_{k_{j+1}-1} + 1)^\beta \cdots (a_{k_j+1} + 1)^\beta m_{k_j}^\beta \\ &\geq L (a_{k_{j+1}} + 1)^\beta (a_{k_{j+1}-1} + 1)^\beta \cdots (a_{k_j+2} + 1)^\beta m_{k_j+1}^\beta \\ &\ \vdots \\ &\geq L m_{k_{j+1}}^\beta. \end{aligned}$$

□

Example 5.9. For some positive numbers $\kappa, M > 1$, let

$$a_j \sim M^{\kappa^j},$$

that is, there exist constants $d_1 > d_2 > 0$:

$$d_1 M^{\kappa^j} \geq a_j \geq d_2 M^{\kappa^j}. \tag{5.6}$$

Assume that

$$M^{\kappa^2-\kappa} \geq \frac{2d_1}{d_2}, \tag{5.7}$$

then τ is a Liouville number with its order β:

$$\beta \leq \frac{\log d_2 + \kappa^2 \log M}{\log 2d_1 + \kappa \log M} - 1. \tag{5.8}$$

In fact, (5.7) yields

$$\log d_2 + \kappa^2 \log M \geq \log 2d_1 + \kappa \log M$$

and (5.8) gives

$$\beta + 1 \leq \frac{\log d_2 + \kappa^{j+1} \log M}{\log 2d_1 + \kappa^j \log M} \tag{5.9}$$

for $j \geq 1$. It follows that

$$a_{j+1} \geq d_2 M^{\kappa^{j+1}} \geq (2d_1)^{\beta+1} \cdot (M^{\kappa^j})^{\beta+1} \geq 2^{\beta+1} a_j^{\beta+1},$$

which satisfies Hypothesis of Lemma 3.3.

Example 5.10. Let $\{k_j\}$ be a sequence of integers which is increasing and goes to infinity such that

$$k_j - k_{j-1} \leq C\kappa^j \tag{5.10}$$

for some $C > 0$ and $\kappa > 1$. For constants $M, M' > 1$, to simplify the argument, let

$$a_{k_j+1} = M^{\kappa^j}, \qquad a_l \leq M', \quad l \notin \{k_j + 1 : j \in \mathbf{N}\}. \tag{5.11}$$

Then the irrational number, which has the partial quotients above, is a weak Liouville number with its order β, which satisfies

$$\beta \leq \frac{\kappa - 1}{1 + \frac{C\kappa \log(M'+1)}{\log M} + \frac{\kappa^{-1} \log 2}{\log M}}. \tag{5.12}$$

In fact, the inequality above implies

$$\kappa \geq \frac{\log(M'+1)}{\log M} \beta C\kappa + \beta + 1 + \frac{\log 2}{\log M} \beta \kappa^{-1}.$$

It follows from Hypotheses that the estimates

$$\begin{aligned} M^{\kappa^{j+1}} &\geq M^{\frac{\log(M'+1)}{\log M}\beta C\kappa^{j+1}} M^{\kappa^j(\beta+1) + \frac{\log 2}{\log M}\beta} \\ &\geq M^{\frac{\log(M'+1)}{\log M}\beta(k_{j+1}-k_j-1)} 2^\beta M^{\kappa^j(\beta+1)} \\ &\geq (M'+1)^{\beta(k_{j+1}-k_j-1)} (M^{\kappa^j}+1)^\beta M^{\kappa^j} \end{aligned}$$

hold for every positive integer j. Thus we have the condition (5.5) in Lemma 5.8.

The number, which satisfies (5.10) and (5.11), is also a Roth number with its order α such that

$$\alpha \geq (\kappa - 1)(2C\kappa\frac{\log(M'+1)}{\log M} + 2 + \kappa).$$

In fact, for every $\varepsilon > 0$, which satisfies

$$\varepsilon\frac{\kappa}{\kappa - 1} \geq \frac{\log(M'+1)}{\log M}2C\kappa^2 + 2\kappa + \kappa^2,$$

there exists a constant C_ε such that

$$\begin{aligned} C_\varepsilon M^{\frac{\kappa(\kappa^{j-1}-1)}{\kappa-1}\varepsilon} &\geq M^{\frac{\log(M'+1)}{\log M}2C\kappa^{j+1}} 2^2 M^{2\kappa^j} 2M^{\kappa^{j+1}} \\ &\geq M^{\frac{\log(M'+1)}{\log M}2(k_{j+1}-k_j-1)}(M^{\kappa^j}+1)^2(M^{\kappa^{j+1}}+1). \end{aligned}$$

Thus we have

$$C_\varepsilon M^{(\kappa^{j-1}+\kappa^{j-2}+\cdots+\kappa^1)\varepsilon} \geq (M^{\kappa^{j+1}}+1)(M'+1)^{2(k_{j+1}-k_j-1)}(M^{\kappa^j}+1)^2,$$

which implies the condition in Lemma 5.3.

6. Appendix II: Khinchine-Lévi class

In this section, using the growth rate of the denominators $\{m_j\}$ of the Diophantine approximation, we introduce another class of irrational numbers, which possibly has the full Lebesgue measure.

Let us call an irrational number τ a Khinchin-Lévy class number or (KL) class number if, for the denominators $\{m_j\}$ of the Diophantine approximation of τ, there exist constants $C_1, C_2 > 1$, which satisfy

$$C_1^j \leq m_j \leq C_2^j, \qquad \forall j \geq j_0 \tag{6.1}$$

for some $j_0 \in \mathbf{N}$.

In [1] Khinchin proved that almost all irrational numbers satisfy (6.1) and furthermore, he had shown that there exists a constant γ_0, which satisfies

$$\lim_{k\to\infty}(m_k)^{\frac{1}{k}} = \gamma_0$$

for almost all irrational numbers. By Lévy this constant was estimated:

$$\gamma_0 = e^{\frac{\pi^2}{12\log 2}} \sim 3.27582...$$

Obviously, the intersection of the class of (KL) and the class of Roth numbers has the full measure. We can show that there exist Roth numbers, which are not in the class of (KL). In fact, for an irrational number τ, let a_j be the partial quotients of its continued fraction expansions. If

$$A_1^j \leq a_j \leq A_2^j, \quad \forall j$$

holds for some constants $A_1, A_2 > 0$, then we can see that $m_j > C^{j^2}$ holds and τ is a Roth number (see Eample 5.5).

Also, we can show that there exist numbers in the class of (KL), which are not Roth numbers, that is, weak Liouville numbers.

Lemma 6.1 ([7]). *For an irrational number τ, let $\{a_j\}$ be the partial quotients of the continued fraction expansion of τ and $\{n_j/m_j\}$ be its Diophantine approximation. For a constant $\beta > 0$ and some subsequence $\{m_{k_j}\}$ we assume that*

$$a_{k_j+1} = (K-1)m_{k_j}^{\beta}, \tag{6.2}$$

$$a_l \le K-1, \quad l \notin \{k_j+1 : j \in \mathbf{N}\} \tag{6.3}$$

for some positive integer $K \ge 2$. Then, τ is not a Roth number. Furthermore, if $\beta \ge 1$ and the growth rate of k_j is sufficiently large such as

$$(1+\beta)^{2^j} < k_j < c(1+\beta)^{2^j}, \qquad c > 1 \tag{6.4}$$

then τ belongs to the class of (KL).

Next we consider the weak Liouville numbers, which are not in (KL) class.

Lemma 6.2 ([7]). *Let τ be a β-order weak Liouville number and assume that*

$$k_j - k_{j-1} - 1 \ge c_1(1+\beta)^{j-1}, \tag{6.5}$$

$$k_j < c_2(1+\beta)^j, \qquad \forall j \tag{6.6}$$

hold for some constants $c_1, c_2 > 0$. Then τ is not in the class of (KL).

It is obvious from Theorem 3.6 that the class of α-order Liouville numbers is contained in the complement of the class of Roth numbers. Furthermore, the class of α-order Liouville numbers is also contained in the complement of the class (KL). In fact, since it follows from Lemma 3.2 that we have

$$m_{j+1} \ge K m_j^{\alpha+1} \ge K^{1+(\alpha+1)} m_{j-1}^{(\alpha+1)^2} \cdots \ge K^{1+(\alpha+1)+\cdots+(\alpha+1)^{j-1}} m_1^{(\alpha+1)^j},$$

the sequence m_j cannot be majorized by C^j for any $C > 0$.

References

[1] Y.A.Khinchin, "Continued Fractions", the University of Chicago Press 1964. 28 # 5037

[2] F.V.Lazutkin, "KAM theory and semiclassical approximations to eigenfunctions", Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], 24, Springer-Verlag, Berlin, 1993.

[3] A.J. Lichtenberg and M.A. Lieberman, "Regular and Stochastic Motion", Applied Mathematical Sciences, Springer, New York 1983.

[4] K.Naito, *Dimension estimate of almost periodic attractors by simultaneous Diophantine approximation*, J. Differential Equations, **141** (1997), 179–200.

[5] ———, *Correlation dimensions of quasi-periodic orbits with frequencies given by quasi Roth numbers*, J. Korean Math. Soc. **37** (2000), 857–870.

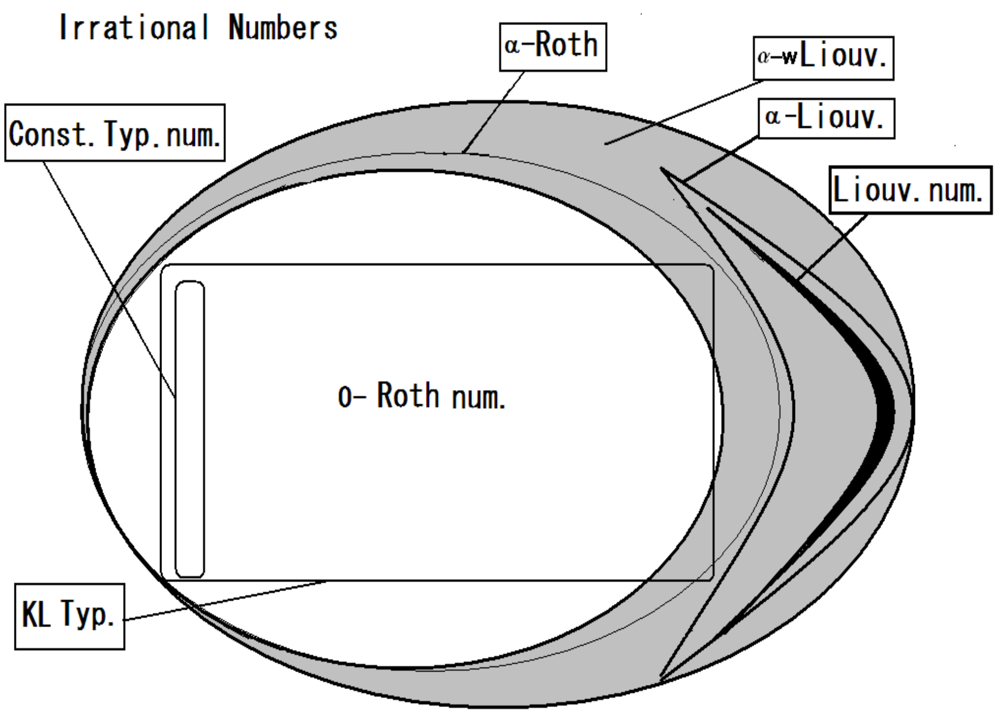
[6] ———, *Recurrent dimensions of quasi-periodic solutions for nonlinear evolution equations*, Trans. Amer. Math. Soc. **354** (2002) no. 3, 1137-1151.

[7] ———, *Recurrent dimensions of quasi-periodic solutions for nonlinear evolution equations II: Gaps of dimensions and Diophantine conditions*, Discrete and Continuous Dynamical Systems **11** (2004), 449-488.

[8] ———, *Recurrent dimensions of quasi-qeriodic orbits with frequencies given by weak Liouville numbers* to appear in International Journal of Mathematics, Game Theory and Algebra

[9] J.Mather, *Destruction of invariant circles*, Ergodic Theory Dynam. Systems **8** (1988), 199-214.

[10] W.M.Schmidt, "Diophantine Approximation", Springer Lecture Notes in Math. 785, 1980.



$R(\alpha)$: the set of α-order Roth numbers,
$wL(\beta)$: the set of β-order weak Liouville numbers

$$R(\alpha) \subset R(\alpha'), \quad \alpha \leq \alpha', \quad wL(\beta) \subset wL(\beta'), \quad \beta \geq \beta',$$

$$R(\alpha)^c \subset \bigcap_{\beta<\alpha} wL(\beta), \quad wL(\beta) \subset \bigcap_{\beta>\alpha} R(\alpha)^c,$$

$$R(0)^c = \bigcup_{\beta>0} wL(\beta).$$

τ satisfies d_0-(D):

$$d_0 = \inf\{\alpha : \tau \text{ is an } \alpha\text{-order Roth number}\}$$
$$= \sup\{\beta : \tau \text{ is a } \beta\text{-order weak Liouville number}\}.$$

FIGURE. Classifications of Irrationals

Table. Table of Irrationals

Type	Definition	m_j	a_j	meas.
Const.Typ.	$\exists c : \|\tau - q/p\| \geq c/p^2,\ \forall q/p$	$m_{j+1} \leq K m_j$	bounded	null
Roth.num	$\forall \varepsilon > 0, \exists c_\varepsilon : \|\tau - q/p\| \geq c_\varepsilon / p^{2+\varepsilon}$	$\forall \beta > \alpha, \exists K_\beta :$	$\sim \kappa^j$	full
α-Roth	$\forall \beta > \alpha, \exists c_\beta : \|\tau - q/p\| \geq c_\beta / p^{2+\beta}$	$m_{j+1} \leq K_\beta m_j^{1+\beta}$	$\sim m_j^\alpha$ (*1)	full
α-Liouv.	$\|\tau - n_j/m_j\| \leq c/m_j^{2+\alpha}$	$m_{j+1} \geq K m_j^{1+\alpha}$	$\sim M^{\kappa^j}$	null
α-w.Liouv.	$\exists \{m_{k_j}\} : \|\tau - n_j/m_{k_j}\| \leq c/m_{k_j}^{2+\alpha}$	$m_{k_j+1} \geq K m_{k_j}^{1+\alpha}$	$\sim M^{\kappa^{k_j}}$ (*2)	null
Liouv.	$\|\tau - n_j/m_j\| \leq 1/m_j^j$	—	—	null
(KL) class	$\exists C_1, C_2 : C_1^j < m_j < C_2^j$	—	$\sim m_{k_j}^\beta$ (*3)	full

$\{n_j/m_j\}$: Diophantine sequence
$\{a_j\}$: partial quotients of continued fraction expansions (examples)
(*1): $a_{j+1} \sim m_j^\alpha,\ \ a_j \sim M^{\kappa^j}$
(*2): $a_{k_j+1} \sim M^{\kappa^{k_j}},\ \{k_j\}$ is sparse: $k_j - k_{j-1} \sim \kappa^j$
(*3): $a_{k_j+1} \sim m_{k_j}^\beta,\ \{k_j\}$ is sparse: $k_j \sim (1+\beta)^{2^j}$

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