Journal of Nonlinear and Convex Analysis
Volume 5, Number 2, 2004, 157–168

Yokohama Publishers
ISSN 1880-5221 ONLINE JOURNAL

SUBLEVEL SETS AND GLOBAL MINIMA OF COERCIVE FUNCTIONALS AND LOCAL MINIMA OF THEIR PERTURBATIONS

BIAGIO RICCERI

1. Introduction

In [24], we have identified a general variational principle of which the following theorem is a by-product:

Theorem A. *If Φ and Ψ are two sequentially weakly lower semicontinuous functionals on a reflexive real Banach space and if Ψ is also continuous and coercive, then the functional $\Psi + \lambda\Phi$ has at least one local minimum for each $\lambda > 0$ small enough.*

The variational principle of [24] has already been widely applied to nonlinear differential equations and Hammerstein integral equations as well (see, for instance, [1]-[6], [8]-[16], [18], [20], [25]-[28]).

The aim of the present paper is essentially to point out that, under the same assumptions as those of Theorem A, the following more precise conclusion holds: if, for some $r > \inf_X \Psi$, the weak closure of the set $\Psi^{-1}(]-\infty, r[)$ has at least k connected components in the weak topology, then, for each $\lambda > 0$ small enough, the functional $\Psi + \lambda\Phi$ has at least k local minima lying in $\Psi^{-1}(]-\infty, r[)$.

This, in particular, holds (for any $r > \inf_X \Psi$) when the set of all global minima of Ψ has at least k connected components in the weak topology.

This more precise conclusion can be used in a twofold way.

In a direct way, when we have, *a priori*, a sufficient information about the set of all global minima or, more generally, about the sublevel sets of Ψ, it just provides an information on the number of the local minima of suitable perturbations of Ψ.

Otherwise, when our primary objective is to get some information on the structure of the set of all global minima and of the sublevel sets of Ψ, we can try to use it in an indirect way.

For instance, if we are interested in knowing whether the sublevel sets of Ψ are connected in the weak topology (an important issue in minimax theory (see [22], [23])), then we could try to find a sequentially weakly lower semicontinuous functional Φ and a sequence of positive numbers $\{\mu_n\}$ converging to 0 in such a way that, for each $n \in \mathbf{N}$, the functional $\Psi + \mu_n\Phi$ has at most one local minimum.

We develop this point of view in the third section, when Ψ is the energy functional related to a Dirichlet problem.

In the next section, we first establish our basic results in full generality and then we formulate them in the setting of reflexive and separable real Banach spaces.

2. Basic Results

If (X,τ) is a topological space, for any $\Psi : X \to]-\infty,+\infty]$, with $\mathrm{dom}(\Psi) \neq \emptyset$, we denote by τ_Ψ the smallest topology on X which contains both τ and the family of sets $\{\Psi^{-1}(]-\infty, r[)\}_{r\in\mathbf{R}}$.

Our main abstract result is as follows.

Theorem 1. *Let (X,τ) be a Hausdorff topological space, and $\Phi, \Psi : X \to]-\infty,+\infty]$ two functions. Assume that there is $\rho > \inf_X \Psi$ such that the set $\overline{\Psi^{-1}(]-\infty,\rho[)}$ is compact and sequentially compact, has at least k connected components and each of them intersects the interior of* $\mathrm{dom}(\Phi)$*. Moreover, suppose that the function Φ is bounded below in $\overline{\Psi^{-1}(]-\infty,\rho[)}$ and that the function $\Psi+\lambda\Phi$ is sequentially lower semicontinuous for each $\lambda>0$ small enough.*

Then, there exists $\lambda^*>0$ such that, for each $\lambda \in]0,\lambda^*[$, the function $\Psi+\lambda\Phi$ has at least k τ_Ψ-local minima lying in $\mathrm{dom}(\Phi)\cap\Psi^{-1}(]-\infty,\rho[)$.

Proof. Denote by $\mathcal{C}$ the family of all connected components of $\overline{\Psi^{-1}(]-\infty,\rho[)}$. Note that these sets are closed in X since they are closed in $\overline{\Psi^{-1}(]-\infty,\rho[)}$ which is, in turn, closed in X. We now observe that there are k pairwise disjoint closed non-empty sets $C_1,...,C_k$ such that

$$\overline{\Psi^{-1}(]-\infty,\rho[)} = \bigcup_{i=1}^{k} C_i \ .$$

We distinguish two cases. First, assume that $\mathcal{C}$ is finite. Let h be its cardinality. Let $B_1,..,B_h$ be the members of $\mathcal{C}$. Then, if we choose $C_i = B_i$ for $i=1,...,k-1$ and $C_k = \cup_{i=k}^{h} B_i$, we are clearly done. Now, assume that $\mathcal{C}$ is infinite. In this case, we prove our claim by induction. The claim is true, of course, if $k=1$. Assume that it is true if $k=p$. So, we are assuming that there are p pairwise disjoint closed non-empty sets $D_1,...,D_p$, such that

$$\overline{\Psi^{-1}(]-\infty,\rho[)} = \bigcup_{i=1}^{p} D_i \ .$$

Notice that at least one of the sets D_i must be disconnected, since, otherwise, we would have $\{D_1,...,D_p\} = \mathcal{C}$, contrary to the assumption that $\mathcal{C}$ is infinite. Then, if D_{i^*} is disconnected, there are two disjoint closed non-empty sets E_1, E_2 such that $D_{i^*} = E_1 \cup E_2$. So, $D_1,...,D_{i^*-1}, D_{i^*+1},...,D_p, E_1, E_2$ are $p+1$ pairwise disjoint closed non-empty sets whose union is $\overline{\Psi^{-1}(]-\infty,\rho[)}$. So, our claim is true for $k=p+1$, and hence, by induction, for any k.

Now, fix i $(1 \leq i \leq k)$. By compactness and Hausdorffness, it is clear that there exists an open set $A_i \subset X$ such that $C_i \subset A_i$ and $A_i \cap \cup_{j=1, j\neq i}^{k} C_j = \emptyset$. Furthermore, it is easily seen that, if we put

$$G_i = \{x \in A_i : \Psi(x) < \rho\} \ ,$$

we have

$$\overline{G_i} = C_i \ .$$

Clearly, C_i contains at least one member of $\mathcal{C}$, and hence, by assumption, $\mathrm{int}(\mathrm{dom}(\Phi)) \cap C_i \neq \emptyset$. This implies that $\mathrm{dom}(\Phi) \cap G_i \neq \emptyset$. Taken into account that, by assumption, $\inf_{C_i} \Phi$ is finite, put

$$\mu_i = \inf_{x \in \mathrm{dom}(\Phi) \cap G_i} \frac{\Phi(x) - \inf_{C_i} \Phi}{\rho - \Psi(x)} \ .$$

Let $\lambda' > 0$ be such that $\Psi + \lambda\Phi$ is sequentially lower semicontinuous for each $\lambda \in]0, \lambda']$. Fix $\mu > \max\{\mu_i, \frac{1}{\lambda'}\}$. Then, there exists $y \in \mathrm{dom}(\Phi) \cap G_i$ such that

$$\mu\rho > \mu\Psi(y) + \Phi(y) - \inf_{C_i} \Phi \ .$$

Moreover, since C_i is sequentially compact, there exists $x_i^* \in \mathrm{dom}(\Phi) \cap C_i$ such

$$\Phi(x_i^*) + \mu\Psi(x_i^*) \leq \Phi(x) + \mu\Psi(x)$$

for all $x \in C_i$. We claim that $x_i^* \in G_i$. Arguing by contradiction, assume that $\Psi(x_i^*) \geq \rho$. We then have

$$\Phi(x_i^*) + \mu\Psi(x_i^*) \geq \Phi(x_i^*) + \mu\rho > \Phi(x_i^*) + \Phi(y) + \mu\Psi(y) - \inf_{C_i} \Phi \geq \Phi(y) + \mu\Psi(y)$$

which is absurd. Now, let i vary. Put $\mu^* = \max\{\mu_1, ..., \mu_k, \frac{1}{\lambda'}\}$. Clearly, each set G_i is τ_Ψ-open, and hence each x_i^* is a τ_Ψ-local minimum of $\Phi + \mu\Psi$ for all $\mu > \mu^*$. Consequently, the points $x_1^*, ..., x_k^*$ satisfy the conclusion, taking $\lambda^* = \frac{1}{\mu^*}$, and the proof is complete. $\square$

The next result provides a reasonable way to check the key assumption of Theorem 1.

Proposition 1. *Let X be a Hausdorff topological space and $\Psi : X \to]-\infty, +\infty]$ a sequentially lower semicontinuous function. Assume that there is $r > \inf_X \Psi$ such that the set $\overline{\Psi^{-1}(]-\infty, r[)}$ is compact and first-countable. Moreover, assume that the set of all global minima of Ψ has at least k connected components.*

Then, there exists $\rho^* \in]\inf_X \Psi, r]$ such that, for each $\rho \in]\inf_X \Psi, \rho^*]$, the set $\overline{\Psi^{-1}(]-\infty, \rho[)}$ has at least k connected components.

Proof. Arguing by contradiction, assume that there is a decreasing sequence $\{\rho_n\}$ in $]\inf_X \Psi, r[$, coverging to $\inf_X \Psi$, such that for each $n \in \mathbf{N}$, the set $\overline{\Psi^{-1}(-\infty, \rho_n[)}$ has at most $k-1$ connected components. Clearly, we have

$$\Psi^{-1}(\inf_X \Psi) = \bigcap_{n=1}^{\infty} \overline{\Psi^{-1}(]-\infty, \rho_n[)} \ .$$

Reasoning as in the proof of Thoerem 1, we find k open and pairwise disjoint subsets of X, $\Omega_1, ..., \Omega_k$, such that $\Omega_i \cap \Psi^{-1}(\inf_X \Psi) \neq \emptyset$ for all $i = 1, ..., k$ and $\Psi^{-1}(\inf_X \Psi) \subseteq \bigcup_{i=1}^k \Omega_i$. Then, for each $n \in \mathbf{N}$, the set $\overline{\Psi^{-1}(]-\infty, \rho_n[)}$ cannot be contained in $\bigcup_{i=1}^k \Omega_i$, since, otherwise, it would have at least k connected components. So, $\left\{\overline{\Psi^{-1}(]-\infty, \rho_n[)} \cap (X \setminus \bigcup_{i=1}^k \Omega_i)\right\}$ is a non-increasing sequence of non-empty closed subsets of a compact one, and hence $\bigcap_{n=1}^\infty \overline{\Psi^{-1}(]-\infty, \rho_n[)} \cap (X \setminus \bigcup_{i=1}^k \Omega_i) \neq \emptyset$ which is absurd. $\square$

So, putting Theorem 1 and Proposition 1 together, we clearly get

Theorem 2. *Let (X, τ) be a Hausdorff topological space and $\Phi : X \to \mathbf{R}$, $\Psi : X \to]-\infty, +\infty]$ two functions. Assume that there is $r > \inf_X \Psi$ such that the set $\overline{\Psi^{-1}(]-\infty, r[)}$ is compact and first-countable. Moreover, suppose that the function Φ is bounded below in $\overline{\Psi^{-1}(]-\infty, r[)}$ and that the function $\Psi + \lambda\Phi$ is sequentially lower semicontinuous for each $\lambda \geq 0$ small enough. Finally, assume that the set of all global minima of Ψ has at least k connected components.*

Then, there exists $\lambda^* > 0$ such that, for each $\lambda \in]0, \lambda^*[$, the function $\Psi + \lambda\Phi$ has at least k τ_Ψ-local minima lying in $\Psi^{-1}(]-\infty, r[)$.

Arguing by contradiction, the use of Theorems 1 and 2 gives the following

Theorem 3. *Let (X, τ) be a Hausdorff topological space and $\Psi : X \to]-\infty, +\infty]$ a sequentially lower semicontinuous function such that, for some $r > \inf_X \Psi$, the set $\overline{\Psi^{-1}(]-\infty, r[)}$ is compact and first-countable.*

Suppose that there are a function $\Phi : X \to \mathbf{R}$, bounded below in $\overline{\Psi^{-1}(]-\infty, r[)}$, and a sequence $\{\mu_n\}$ in $\mathbf{R}^+$ converging to 0 such that, for each $\lambda > 0$ small enough, the function $\Psi + \lambda\Phi$ is sequentially lower semicontinuous, and, for each $n \in \mathbf{N}$, the function $\Psi + \mu_n\Phi$ has at most k τ_Ψ-local minima lying in $\Psi^{-1}(]-\infty, r[)$.

Then, for every $\rho \in]\inf_X \Psi, r]$, the sets $\overline{\Psi^{-1}(]-\infty, \rho[)}$ and $\Psi^{-1}(\inf_X \Psi)$ have at most k connected components. So, in particular, these sets are connected when $k = 1$.

Remark 1. When $k = 1$, Theorem 3 ensures that, for every $\rho \in]\inf_X \Psi, r[$, the set $\Psi^{-1}(]-\infty, \rho])$ is connected. This follows from the equality

$$\Psi^{-1}(]-\infty, \rho]) = \bigcap_{\rho < s < r} \overline{\Psi^{-1}(]-\infty, s[)}$$

and from the fact that, for every $s \in]\inf_X \Psi, r]$, the set $\overline{\Psi^{-1}(]-\infty, s[)}$ is connected and compact.

An interesting consequence of Theorem 3 is the following two local minima result.

Theorem 4. *Let (X, τ) be a Hausdorff topological space, and $\Phi : X \to \mathbf{R}$, $\Psi : X \to]-\infty, +\infty]$ two functions. Assume that there is $r > \inf_X \Psi$ such that the set $\overline{\Psi^{-1}(]-\infty, r[)}$ is compact and first-countable. Moreover, assume that there is a strict local minimum of Ψ, say x_0, such that $\inf_X \Psi < \Psi(x_0) < r$. Finally, suppose that the function Φ is bounded below in $\overline{\Psi^{-1}(]-\infty, r[)}$ and that the function $\Psi + \lambda\Phi$ is sequentially lower semicontinuous for each $\lambda \geq 0$ small enough.*

Then, there exists $\lambda^* > 0$ such that, for each $\lambda \in]0, \lambda^*[$, the function $\Psi + \lambda\Phi$ has at least two τ_Ψ-local minima lying in $\Psi^{-1}(]-\infty, r[)$.

Proof. Arguing by contradiction, assume that the conclusion does not hold. Then, by Theorem 3 and Remark 1, the set $\Psi^{-1}(]-\infty, \Psi(x_0)])$ is connected. But this set contains x_0 as an isolated point (since x_0 is a strict local minimum of Ψ) and does not reduce to it (since $\inf_X \Psi < \Psi(x_0)$), against connectedness. □

With the aim to apply them to nonlinear differential equations, we now establish some consequences of the previous general results in the setting of reflexive and separable real Banach spaces.

For a set A in a Banach space, we denote by $(\overline{A})_w$ its closure in the weak topology. We say that A is weakly connected if it is connected in the weak topology. The weakly connected components of A are its connected components in the weak topology.

Theorem 5. *Let X be a sequentially weakly closed subset of a reflexive and separable real Banach space E, and $\Phi : X \to \mathbf{R}$, $\Psi : X \to]-\infty, +\infty]$ two functionals. Assume that there is $\rho > \inf_X \Psi$ such that the set $(\overline{\Psi^{-1}(]-\infty, \rho[)})_w$ is bounded and has at least k weakly connected components. Moreover, suppose that the functional Φ is bounded below in $(\overline{\Psi^{-1}(]-\infty, \rho[)})_w$ and that the functional $\Psi + \lambda\Phi$ is sequentially weakly lower semicontinuous for each $\lambda > 0$ small enough.*

Then, there exists $\lambda^* > 0$ such that, for each $\lambda \in]0, \lambda^*[$, the functional $\Psi + \lambda\Phi$ has at least k τ_Ψ-local minima lying in $\Psi^{-1}(]-\infty, \rho[)$, where τ is the relative weak topology of X.

Proof. Apply Theorem 1, τ just being the relative weak topology. In particular, observe that, since E is reflexive and separable, the weak closure of any bounded set is weakly compact and metrizable (and so first-countable). □

Analogously, from Theorem 2, we get

Theorem 6. *Let X be a sequentially weakly closed subset of a reflexive and separable real Banach space E, and $\Phi : X \to \mathbf{R}$, $\Psi : X \to]-\infty, +\infty]$ two functionals. Assume that there is $\rho > \inf_X \Psi$ such that the set $\Psi^{-1}(]-\infty, \rho[)$ is bounded. Moreover, suppose that the functional Φ is bounded below in $(\overline{\Psi^{-1}(]-\infty, \rho[)})_w$ and that the functional $\Psi + \lambda\Phi$ is sequentially weakly lower semicontinuous for each $\lambda \geq 0$ small enough. Finally, assume that the set $\Psi^{-1}(\inf_X \Psi)$ has at least k weakly connected components.*

Then, the conclusion of Theorem 5 holds.

Arguing by contradiction, from Theorems 5 and 6 we then get

Theorem 7. *Let X be a sequentially weakly closed subset of a reflexive and separable real Banach space, and $\Psi : X \to]-\infty, +\infty]$ a sequentially weakly lower semicontinuous functional such that, for some $r > \inf_X \Psi$, the set $\Psi^{-1}(]-\infty, r[)$ is bounded.*

Suppose that there are a functional $\Phi : X \to \mathbf{R}$, bounded below in $(\overline{\Psi^{-1}(]-\infty, r[)})_w$, and a sequence $\{\mu_n\}$ in $\mathbf{R}^+$ converging to 0 such that, for each $\lambda > 0$ small enough, the functional $\Psi + \lambda\Phi$ is sequentially weakly lower semicontinuous, and, for each $n \in \mathbf{N}$, the functional $\Psi + \mu_n\Phi$ has at most k τ_Ψ-local minima lying in $\Psi^{-1}(]-\infty, r[)$, where τ is relative weak topology of X.

Then, for every $\rho \in]\inf_X \Psi, r]$, the sets $\overline{\Psi^{-1}(]-\infty, \rho[)}$ and $\Psi^{-1}(\inf_X \Psi)$ have at most k weakly connected components. So, in particular, these sets are weakly connected when $k = 1$.

The next result is an application of Theorems 5 and 6 to critical point theory. If J is a Gâteaux differentiable functional on a Banach space X, the critical points of J are the zeros of its derivative, J'. Moreover, J is said to satisfy the Palais-Smale condition if each sequence $\{x_n\}$ in X such that $\sup_{n\in\mathbf{N}} |J(x_n)| < +\infty$ and $\lim_{n\to+\infty} \|J'(x_n)\|_{X^*} = 0$ admits a strongly converging subsequence.

Theorem 8. *In addition to the assumptions of either Theorem 5 or Theorem 6, suppose that $X = E$, that the functionals $\Psi, \Phi : X \to \mathbf{R}$ are continuously Gâteaux differentiable, and that $k \geq 2$.*

Then, there exists $\lambda^* > 0$ such that, for each $\lambda \in]0, \lambda^*[$ for which the functional $\Psi + \lambda\Phi$ satisfies the Palais-Smale condition, the same functional has at least $k+1$ critical points, k of which are lying in $\Psi^{-1}(]-\infty, \rho[)$.

Proof. By either Theorem 5 or Theorem 6, there exists $\lambda^* > 0$ such that, for each $\lambda \in]0, \lambda^*[$, the functional $\Psi + \lambda\Phi$ has at least k τ_Ψ-local minima lying in $\Psi^{-1}(]-\infty, \rho[)$, where τ is the weak topology of X. Note that Ψ, being C^1, is (norm) continuous. Consequently, the topology τ_Ψ is weaker than the strong topology, and so the above mentioned τ_Ψ-local minima of $\Psi + \lambda\Phi$ are local minima of this functional in the strong topology. Now, assuming that $\Psi + \lambda\Phi$ satisfies the Palais-Smale condition, the conclusion follows from Theorem (1.ter) of [17]. □

From Theorem 8, arguing by contradiction, we then obtain the following

Theorem 9. *Let X be a reflexive and separable real Banach space and $\Psi : X \to \mathbf{R}$ a continuously Gâteaux differentiable and sequentially weakly lower semicontinuous functional such that, for some $r > \inf_X \Psi$, the set $\Psi^{-1}(]-\infty, r[)$ is bounded. Let $k \in \mathbf{N}$ with $k \geq 2$.*

Suppose that there are a continuously Gâteaux differentiable functional $\Phi : X \to \mathbf{R}$, which is bounded below in $(\overline{\Psi^{-1}(]-\infty, r[)})_w$, and a sequence $\{\mu_n\}$ in $\mathbf{R}^+$ converging to 0 such that the functional $\Psi + \lambda\Phi$ is sequentially weakly lower semicontinuous for each $\lambda > 0$ small enough and, for each $n \in \mathbf{N}$, the functional $\Psi + \mu_n\Phi$ satisfies the Palais-Smale condition and has at most k critical points in X.

Then, for every $\rho \in]\inf_X \Psi, r]$, the sets $(\overline{\Psi^{-1}(]-\infty, \rho[)})_w$ and $\Psi^{-1}(\inf_X \Psi)$ have at most $k-1$ weakly connected components. So, in particular, these sets are weakly connected when $k = 2$.

Remark 2. When $k = 2$, Theorem 9 ensures that, for every $\rho \in]\inf_X \Psi, r[$, the set $\Psi^{-1}(]-\infty, \rho])$ is weakly connected (see Remark 1).

Here is an application of Theorem 4.

Theorem 10. *Let X be a uniformly convex and separable real Banach space, $g : [0, +\infty[\to \mathbf{R}$ a strictly increasing continuous function, and $J : X \to \mathbf{R}$ a sequentially weakly lower semicontinuous functional. For every $x \in X$, put*

$$\Psi(x) = g(\|x\|) + J(x) \ .$$

Assume that the functional Ψ is coercive and has a strict, not global, local minimum, say x_0.

Then, for every $r > \Psi(x_0)$ and every functional $\Phi : X \to \mathbf{R}$ which is bounded below in $(\overline{\Psi^{-1}(]-\infty, r[)})_w$ and such that $\Psi + \lambda\Phi$ is sequentially weakly lower semicontinuous for each $\lambda > 0$ small enough, there exists $\lambda^* > 0$ such that, for each $\lambda \in]0, \lambda^*[$, the functional $\Psi + \lambda\Phi$ has at least two τ_Ψ-local minima lying in $\Psi^{-1}(]-\infty, r[)$, where τ is the weak topology of X.

Proof. From Theorem 1 of [21], it follows that x_0 is a τ-strict local minimum of Ψ. More precisely, the statement of the above quoted result deals with local minima, but exactly the same proof shows that the same is true for strict local minima. Now, the conclusion follows from Theorem 4, taking as τ just the weak topology. $\square$

We conclude this section with an application of Theorem 7 in the setting of Hilbert spaces.

Theorem 11. *Let X be a separable real Hilbert space and $J : X \to \mathbf{R}$ a continuous, Gâteaux differentiable, and sequentially weakly upper semicontinuous functional. For every $x \in X$, put*

$$\Psi(x) = \frac{1}{2}\|x\|^2 - J(x) \ .$$

Assume that, for some $r > \inf_X \Psi$, the set $\Psi^{-1}(]-\infty, r[)$ is bounded. Moreover, suppose that the restriction of J' to $\Psi^{-1}(]-\infty, r[)$ is nonexpansive.

Then, for every $\rho \in]\inf_X \Psi, r[$, the sets $\Psi^{-1}(]-\infty, \rho])$ and $\Psi^{-1}(\inf_X \Psi)$ are weakly connected.

Proof. Let us apply Theorem 7 taking as $\{\mu_n\}$ any sequence in $]0, 1[$ coverging to zero, and $\Phi(x) = \frac{1}{2}\|x\|^2$. Thus, Φ and Ψ are two continuous, Gâteaux differentiable and sequentially weakly lower semicontinuous functionals, and for each $n \in \mathbf{N}$, $\Psi + \mu_n\Phi$ admits at most one τ_Ψ-local minimum in $\Psi^{-1}(]-\infty, r[)$. This follows from the fact that the strong topology is stronger than τ_Ψ and that the retriction of $\frac{1}{1+\mu_n}J'$ to $\Psi^{-1}(]-\infty, r[)$ is a contraction, and so the equation $\Psi'(x) + \mu_n\Phi'(x) = 0$ has at most one solution in $\Psi^{-1}(]-\infty, r[)$. Hence, the hypotheses of Theorem 9 are satisfied, and the conclusion follows from it. $\square$

The following proposition is an useful complement to both Theorems 9 and 11.

Proposition 2. *Let X be a real Hilbert space and $J : X \to \mathbf{R}$ a continuously Gâteaux differentiable functional whose derivative is compact. For every $x \in X$, put*

$$\Psi(x) = \frac{1}{2}\|x\|^2 - J(x) \ .$$

Assume that, for some $r > \inf_X \Psi$, the set $\Psi^{-1}(]-\infty, r[)$ is bounded.

Then, the set $\Psi^{-1}(\inf_X \Psi)$ is compact.

Proof. The functional Ψ is Gâteaux differentiable and its critical points are exactly the fixed points of J'. Let B be a closed ball in X containing $\Psi^{-1}(]-\infty, r[)$. Let $\{x_n\}$ be a sequence of fixed points of J' lying in B. Since this sequence is bounded and J' is compact, there is a subsequence $\{x_{n_k}\}$ such that $J'(x_{n_k})$ converges to some $z \in B$. Clearly, by continuity, z is a fixed point of J'. So, the set $\{x \in B : J'(x) = x\}$ is compact. Of course, it contains $\Psi^{-1}(\inf_X \Psi)$ which is closed since Ψ is continuous, and the conclusion follows. $\square$

Remark 3. Note that when we can apply Theorem 7 with $k = 1$ to a functional Ψ as in Proposition 2, then the set $\Psi^{-1}(\inf_X \Psi)$ is connected. This follows from the fact in any compact subset of a Banach space the relative strong and weak topologies coincide. This remark, in particular, applies to Theorem 9 (when $k = 2$) and to Theorem 11.

3. Applications

In this section, we intend to present some applications of Theorems 9 and 11 to the energy functional related to the Dirichlet problem for a semilinear elliptic equation.

In the sequel, Ω will denote an open connected subset of $\mathbf{R}^h$ with sufficiently smooth boundary.

Put $X = W_0^{1,2}(\Omega)$, and consider it with the usual norm $\|u\| = (\int_\Omega |\nabla u(x)|^2 dx)^{\frac{1}{2}}$. If $h \geq 2$, we denote by $\mathcal{A}$ the class of all Carathéodory functions $f : \Omega \times \mathbf{R} \to \mathbf{R}$ such that

$$\sup_{(x,\xi)\in\Omega\times\mathbf{R}} \frac{|f(x,\xi)|}{1+|\xi|^q} < +\infty ,$$

where $0 < q < \frac{h+2}{h-2}$ if $h > 2$ and $0 < q < +\infty$ if $h = 2$. While, when $h = 1$, we denote by $\mathcal{A}$ the class of all Carathéodory functions $f : \Omega \times \mathbf{R} \to \mathbf{R}$ such that, for each $r > 0$, the function $x \to \sup_{|\xi|\leq r} |f(x,\xi)|$ belongs to $L^1(\Omega)$.

For each $f \in \mathcal{A}$ and $u \in X$, we put

$$J_f(u) = \int_\Omega \left(\int_0^{u(x)} f(x,\xi) d\xi \right) dx$$

and

$$\Psi_f(u) = \frac{1}{2} \int_\Omega |\nabla u(x)|^2 dx - J_f(u) .$$

So, by classical results, the functional J_f is (well defined and) continuously Gâteaux differentiable on X, its derivative is compact, and one has

$$\Psi_f'(u)(v) = \int_\Omega \nabla u(x) \nabla v(x) dx - \int_\Omega f(x,u(x)) v(x) dx$$

for all $u, v \in X$. Hence, the critical points of Ψ_f in X are exactly the weak solutions of the Dirichlet problem

$$\begin{cases} -\Delta u = f(x,u) & \text{in } \Omega \\ u_{|\partial\Omega} = 0. \end{cases}$$

Recall also that if Ψ_f is coercive, then it satisfies the Palais-Smale condition (see, for instance, Example 38.25 of [29]). We denote by λ_1 the first eigenvalue of the problem

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega \\ u_{|\partial\Omega} = 0. \end{cases}$$

Reacall that $\|u\|_{L^2(\Omega)} \leq \lambda_1^{-\frac{1}{2}} \|u\|$ for all $u \in X$. So, if

$$\limsup_{|\xi|\to+\infty} \frac{\sup_{x\in\Omega} \int_0^\xi f(x,t)dt}{\xi^2} < \frac{\lambda_1}{2}$$

then the functional Ψ_f is coercive in X.

Let us state the following

Theorem 12. *Let $f : \Omega \times \mathbf{R} \to \mathbf{R}$ be a Carathéodory function belonging to $\mathcal{A}$ such that*

$$\sup_{(\xi,\eta)\in\mathbf{R}^2,\ \xi\neq\eta} \frac{\sup_{x\in\Omega} |f(x,\xi) - f(x,\eta)|}{|\xi - \eta|} \leq \lambda_1$$

and

$$\limsup_{|\xi|\to+\infty} \frac{\sup_{x\in\Omega} \int_0^\xi f(x,t)dt}{\xi^2} < \frac{\lambda_1}{2} .$$

Then, the sublevel sets of Ψ_f are weakly connected, and the set of all global minima of Ψ_f is compact and connected.

Proof. Fix $u, v, w \in X$, with $\|w\| = 1$. We have

$$|J_f'(u)(w) - J_f'(v)(w)| \leq \int_\Omega |f(x,u(x)) - f(x,v(x))||w(x)|dx$$
$$\leq \lambda_1 \|u - v\|_{L^2(\Omega)} \|w\|_{L^2(\Omega)} \leq \|u - v\| ,$$

and hence

$$\|J_f'(u) - J_f'(v)\| \leq \|u - v\| ,$$

that is J' is nonexpansive in X. Moreover, Ψ_f is coercive in X. Thus, the functionals J_f and Ψ_f satisfy all the assumptions of Theorem 11, and the conclusion follows from it, taking also into account Proposition 2 and Remark 3. $\square$

Let us segnalize an open question related to Theorem 12.

Problem 1. Is there some f satisfying the assumptions of Theorem 12 for which the set of all global minima of the functional Ψ_f is neither a singleton nor a segment ? In particular, what happens when $f(\xi) = \lambda_1(\sin\xi + a)$, with $a > 0$? Or when $f(\xi) = \lambda_1 \mathrm{dist}(\xi, A)$, where $A \subset \mathbf{R}$?

We now establish

Theorem 13. *Let $g : \overline{\Omega} \times [0, +\infty[\to \mathbf{R}$ be a locally Hölder continuous function belonging to $\mathcal{A}$ such that*

$$\limsup_{\xi\to+\infty} \frac{\sup_{x\in\Omega} g(x,\xi)}{\xi} < \lambda_1 .$$

Assume also that, for each $x \in \Omega$, $g(x,0) = 0$ and the function $\xi \to \frac{g(x,\xi)}{\xi}$ is non-increasing in $]0, +\infty[$.

Let $f : \overline{\Omega} \times \mathbf{R} \to \mathbf{R}$ be the function defined by

$$f(x,\xi) = \begin{cases} g(x,\xi) & \text{if } (x,\xi) \in \overline{\Omega} \times [0,+\infty[\\ 0 & \text{otherwise.} \end{cases}$$

Then, the conclusion of Theorem 12 holds.

Proof. For each $\lambda > 0$, $(x,\xi) \in \overline{\Omega} \times \mathbf{R}$, put

$$\alpha(\xi) = -(\xi + |\xi|)\xi$$

and

$$h_\lambda(x,\xi) = f(x,\xi) + \lambda\alpha(\xi) .$$

Clearly, $h_\lambda \in \mathcal{A}$. Since h_λ is locally Hölder continuous in $\overline{\Omega}\times\mathbf{R}$, the critical points of the functional Ψ_{h_λ} are continuous in $\overline{\Omega}$. Thus, since h_λ is zero in $\overline{\Omega}\times]-\infty,0]$, they are non-negative in Ω. Now, observe that, for each $x \in \Omega$, the function $\xi \to \frac{h_\lambda(x,\xi)}{\xi}$ is (strictly) decreasing in $]0,+\infty[$. So, by Theorem 1 of [7], the functional Ψ_{h_λ} has at most one non-zero crtitical point in X. Consequently, it has at most two critical points (note that 0 is one of them). Moreover, Ψ_{h_λ} satisfies the Palais-Smale condition, as it is coercive (as well as Ψ_f). Thus, since $\Psi_{h_\lambda} = \Psi_f - \lambda J_\alpha$, all the assumptions of Theorem 9 are satisfied, and the conclusion follows from it. □

The final result is as follows

Theorem 14. *Let $h = 1$, $\Omega =]0,1[$, and let $g \in C^2([0,+\infty[)$ be a convex and non-negative function, with $g(0) = 0$, such that that $\sup_{\xi>0}\frac{g(\xi)}{\xi} < \pi^2$. Let $f : \mathbf{R} \to \mathbf{R}$ be the function defined by*

$$f(\xi) = \begin{cases} g(\xi) & \text{if } \xi \geq 0 \\ 0 & \text{if } \xi < 0. \end{cases}$$

Then, the sublevel sets of Ψ_f are weakly connected.

Proof. Note that, in the present case, one has $\lambda_1 = \pi^2$. Let $0 < \lambda < \pi^2 - \sup_{\xi>0}\frac{g(\xi)}{\xi}$. Define $\alpha : \mathbf{R} \to \mathbf{R}$ by

$$\alpha(\xi) = \begin{cases} \xi - \log(\xi+1) & \text{if } \xi \geq 0 \\ 0 & \text{if } \xi < 0. \end{cases}$$

Then, the functional $\Psi_f - \lambda J_\alpha$ satisfies the Palais-Smale condition (it is coercive as well as Ψ_f) and its crtical points are non-negative. Clearly, $f + \lambda\alpha \in C^2([0,+\infty[)$, $f(0) + \lambda\alpha(0) = f'(0) + \lambda\alpha'(0) = 0$ and $f''(\xi) + \lambda\alpha''(\xi) > 0$ for all $\xi > 0$. Then, by Example 2 of [19], the functional $\Psi_f - \lambda J_\alpha$ has at most two critical points in X. The conclusion now follows from Theorem 9. □

We conclude with the following problem.

Problem 2. Is there a function g satisfying the hypotheses of Theorem 14 for which the functional Ψ_f has a non-absolute local minimum?

References

[1] G. ANELLO and G. CORDARO, *Existence of solutions of the Neumann problem involving the p-Laplacian via a variational principle of Ricceri*, Arch. Math. (Basel), **79** (2002), 274-287.

[2] G. ANELLO and G. CORDARO, *An existence theorem for the Neumann problem involving the p-Laplacian*, J. Convex Anal., **10** (2003), 185-198.

[3] G. ANELLO and G. CORDARO, *Positive infinitely many and arbitrarily small solutions for the Dirichlet problem involving the p-Laplacian*, Proc. Royal Soc. Edinburgh Sect. A, **132** (2002), 511-519.

[4] G. ANELLO, *A multiplicity theorem for critical points of functionals on reflexive Banach spaces*, Arch. Math. (Basel), **82** (2004), 172-179.

[5] G. ANELLO, *Existence of infinitely many weak solutions for a Neumann problem*, Nonlinear Anal., **57** (2004), 199-209.

[6] D. AVERNA and G. BONANNO, *A three critical points theorem and its applications to the ordinary Dirichlet problem*, Topol. Methods Nonlinear Anal., **22** (2003), 93-104.

[7] H. BREZIS and L. OSWALD, *Remarks on sublinear elliptic equations*, Nonlinear Anal., **10** (1986), 55-64.

[8] P. CANDITO, *Infinitely many solutions to the Neumann problem for elliptic equations involving the p-Laplacian and with discontinuous nonlinearities*, Proc. Edinb. Math. Soc., **45** (2002), 397-409.

[9] G. CORDARO, *Three periodic solutions to an eigenvalue problem for a class of second order Hamiltonian systems*, Abstr. Appl. Anal., **18** (2003), 1037-1045.

[10] G. CORDARO, *Existence and location of periodic solutions to convex and non coercive Hamiltonian systems*, Discrete Contin. Dyn. Syst, to appear.

[11] F. FARACI, *Bifurcation theorems for Hammerstein nonlinear integral equations*, Glasg. Math. J., **44** (2002), 471-481.

[12] F. FARACI, *Multiplicity results for a Neumann problem involving the p-Laplacian*, J. Math. Anal. Appl., **277** (2003), 180-189.

[13] F. FARACI, *Three periodic solutions for a second order nonautonomous system*, J. Nonlinear Convex Anal., **3** (2002), 393-399.

[14] F. FARACI and R. LIVREA, *Infinitely many periodic solutions for a second order nonautonomous system*, Nonlinear Anal., **54** (2003), 417-429.

[15] F. FARACI and R. LIVREA, *Bifurcation theorems for nonlinear problems with lack of compactness*, Ann. Polon. Math., **82** (2003), 77-85.

[16] F. FARACI and V. MOROZ, *Solutions of Hammerstein integral equations via a variational principle*, J. Integral Equations Appl., **15** (2003), 385-402.

[17] N. GHOUSSOUB and D. PREISS, *A general mountain pass principle for locating and classifying critical points*, Ann. Inst. H. Poincaré Anal. Non Linéaire, **6** (1989), 321-330.

[18] A. IANNIZZOTTO, *A sharp existence and localization theorem for a Neumann problem*, Arch. Math. (Basel), **82** (2004), 352-360.

[19] P. KORMAN and Y. LI, *Generalized averages for solutions of two-point Dirichlet problems*, J. Math. Anal. Appl., **239** (1999), 478-484.

[20] S. A. MARANO and D. MOTREANU, *Infinitely many critical points of non-differentiable functions and applications to a Neumann type problem involving the p-Laplacian*, J. Differential Equations, **182** (2002), 108-120.

[21] O. NASELLI, *A class of functionals on a Banach spaces for which strong and weak local minima do coincide*, Optimization, **50** (2001), 407-411.

[22] B. RICCERI, *Some topological mini-max theorems via an alternative principle for multifunctions*, Arch. Math. (Basel), **60** (1993), 367-377.

[23] B. RICCERI, *On a topological minimax theorem and its applications*, in "Minimax theory and applications", B. Ricceri and S. Simons eds., 191-216, Kluwer Academic Publishers, 1998.

[24] B. RICCERI, *A general variational principle and some of its applications*, J. Comput. Appl. Math., **113** (2000), 401-410.

[25] B. RICCERI, *Infinitely many solutions of the Neumann problem for elliptic equations involving the p-Laplacian*, Bull. London Math. Soc., **33** (2001), 331-340.

[26] B. RICCERI, *On a classical existence theorem for nonlinear elliptic equations*, in "Experimental, constructive and nonlinear analysis", M. Théra ed., 275-278, CMS Conf. Proc. **27**, Canad. Math. Soc., 2000.

[27] B. RICCERI, *A bifurcation theory for some nonlinear elliptic equations*, Colloq. Math., **95** (2003), 139-151.

[28] J. SAINT RAYMOND, *On the multiplicity of the solutions of the equation $-\Delta u = \lambda f(u)$*, J. Differential Equations, **180** (2002), 65-88.

[29] E. ZEIDLER, *Nonlinear functional analysis and its applications*, vol. III, Springer-Verlag, 1985.

Manuscript received January 14, 2004

Biagio Ricceri
Department of Mathematics, University of Catania, Viale A. Doria 6, 95125 Catania, Italy
E-mail address: ricceri@dmi.unict.it