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UNILATERAL ELLIPTIC PROBLEMS IN L^1 WITH NATURAL GROWTH TERMS

A. ELMAHI AND D. MESKINE

ABSTRACT. We prove an existence result for solutions of nonlinear elliptic unilateral problems having natural growth terms and L^1 data.

1. INTRODUCTION

Let Ω be an open bounded subset of $\mathbb{R}^N$, $N \geq 2$. Let $f \in L^1(\Omega)$. Consider the following nonlinear Dirichlet problem:

$$A(u) + g(x,u,\nabla u) = f \tag{1.1}$$

where $A(u) = -\text{div}(a(x,u,\nabla u))$ is a Leray-Lions operator defined on $W_0^{1,p}(\Omega)$, with $1 < p < \infty$, and where g is a nonlinearity having "natural growth" with respect to $|\nabla u|$ (of order p) and which satisfies the classical "sign condition" with respect to u.

In the variational case (i.e., where $f \in W^{-1,p'}(\Omega)$), it is well known that the following obstacle problems corresponding to (1.1) have at least one solution (see [2], and [9] for $f \equiv 0$)):

$$\begin{cases} u \in K_\phi, \ g(x,u,\nabla u) \in L^1(\Omega), \ g(x,u,\nabla u)u \in L^1(\Omega) \\ \langle A(u), u-v\rangle + \displaystyle\int_\Omega g(x,u,\nabla u)(u-v)dx \leq \int_\Omega f(u-v)dx \\ \forall v \in K_\phi \cap L^\infty(\Omega), \end{cases} \tag{1.2}$$

where K_ϕ is a convex subset in $W_0^{1,p}(\Omega)$ defined by

$$K_\phi = \{v \in W_0^{1,p}(\Omega) : v \geq \phi \text{ a.e. in } \Omega\}$$

where ϕ is a measurable function with $\phi^+ \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$.

In the general case where f belongs to $L^1(\Omega)$, formulation (1.2) is not adequat since the term $\displaystyle\int_\Omega f(u-v)dx$ may not have a meaning. Many results have been obtained in this case, for example see [10], [11] if $g \equiv 0$ and [1] if $g \equiv g(x,u,\nabla u)$ satisfying further the following coercivity condition:

$$|g(x,s,\zeta)| \geq \beta|\zeta|^p \quad \text{for } |s| \geq \gamma. \tag{1.3}$$

The purpose of the paper is to prove an existence result for unilateral problems corresponding to (1.1) without assuming the coercivity condition (1.3). For the equation case the reader is referred to [2]-[7] and [12]. In the case where $1 < p \leq$

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$2-\frac{1}{N}$ we cannot expect the solutions to be in $W^{1,1}(\Omega)$ but in $T_0^{1,p}(\Omega)$, see [3] and Remark 2.3 below. The case $p > N$ is easier since the solutions turn out to be continuous.

2. The main result

Let Ω be an open bounded subset of $I\!R^N$, $N \geq 2$ and let $2-\frac{1}{N} < p \leq N$.
Let $A(u) = -\text{div}(a(x,u,\nabla u))$ be a Leray-Lions operator defined on $W_0^{1,p}(\Omega)$ into its dual where $a : \Omega \times I\!R \times I\!R^N \to I\!R^N$ is a Carathéodory function satisfying for a.e. $x \in \Omega$ and for all $\zeta, \zeta' \in I\!R^N, (\zeta \neq \zeta')$ and all $s \in I\!R$:

$$|a(x,s,\zeta)| \leq k(x) + k_1|s|^{p-1} + k_2|\zeta|^{p-1} \tag{2.1}$$

$$(a(x,s,\zeta) - a(x,s,\zeta'))(\zeta - \zeta') > 0 \tag{2.2}$$

$$a(x,s,\zeta)\zeta \geq \alpha|\zeta|^p \tag{2.3}$$

with $\alpha > 0,\ k_1 \geq 0,\ k_2 \geq 0$ and $k \in L^{p'}(\Omega)$, where p' denotes the conjugate exponent of p.

Furthermore, let $g : \Omega \times I\!R \times I\!R^N \to I\!R$ be a Carathéodory function such that for a.e. $x \in \Omega$ and for all $s \in I\!R$ and all $\zeta \in I\!R^N$:

$$g(x,s,\zeta)s \geq 0 \tag{2.4}$$

$$|g(x,s,\zeta)| \leq b(|s|)(c(x) + |\zeta|^p) \tag{2.5}$$

where $b : I\!R_+ \to I\!R$ is a continuous and nondecreasing function and $c(x)$ is a given nonnegative function in $L^1(\Omega),\ c(x) \geq 0$.

Let

$$K_\psi = \{v \in W_0^{1,p}(\Omega) : v \geq \psi \text{ a.e. in } \Omega\}$$

where $\psi : \Omega \to \bar{I\!R}$ is a measurable function on Ω such that

$$\psi^+ \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega). \tag{2.6}$$

Finally, we assume that

$$f \in L^1(\Omega). \tag{2.7}$$

We define, for s and k in $I\!R,\ k \geq 0,\ T_k(s) = \max(-k, \min(k,s))$.

We shall prove the following existence theorem.

Theorem 2.1. *Under assumptions* (2.1)-(2.7) *there exists at least one solution of the following obstacle problem*

$$\begin{cases} u \geq \psi \ \textit{a.e. in}\ \Omega \\ u \in W_0^{1,q}(\Omega), \forall\ 1 < q < \frac{N(p-1)}{N-1},\ g(x,u,\nabla u) \in L^1(\Omega), \\ T_k(u) \in W_0^{1,p}(\Omega), \forall k > 0 \\ \displaystyle\int_\Omega a(x,u,\nabla u)\nabla T_k(u-v)dx + \int_\Omega g(x,u,\nabla u)T_k(u-v)dx \\ \qquad\qquad \leq \displaystyle\int_\Omega fT_k(u-v)dx \\ \qquad\qquad \forall v \in K_\psi \cap L^\infty(\Omega). \end{cases} \tag{P_ψ}$$

Remark 2.1 We obtain the same result if we assume only that the sign condition (2.4) is verified at infinity, or if the data is the form $f - \mathrm{div}(F)$, with $f \in L^1(\Omega)$ and $F \in (L^{p'}(\Omega))^N$.

Remark 2.2 If we assume that a satisfies $a(x, s, \zeta) = a(x, \zeta)$ and

$$[a(x,\zeta) - a(x,\zeta')][\zeta - \zeta'] \geq \alpha|\zeta - \zeta'|^p \qquad \text{if } p \geq 2$$

$$[a(x,\zeta) - a(x,\zeta')][\zeta - \zeta'] \geq \alpha\frac{|\zeta - \zeta'|^2}{(h(x) + |\zeta| + |\zeta'|)^{2-p}} \qquad \text{if } p < 2$$

where $\alpha > 0$ and $h \in L^p(\Omega)$. We can replace in (P_ψ), $K_\psi \cap L^\infty(\Omega)$ by K_ψ.

Proof of Remark 2.2. Let $v \in K_\psi$. By taking $T_n(v), n \geq \|\psi^+\|_\infty$, as test function in (P_ψ), we obtain

$$\int_\Omega a(x,\nabla u)\nabla T_k(u - T_n(v))dx + \int_\Omega g(x,u,\nabla u)T_k(u - T_n(v))dx \leq \int_\Omega fT_k(u - T_n(v))dx \quad (2.8)$$

then if $p \geq 2$, we have

$$\begin{aligned} \alpha & \int_\Omega |\nabla T_k(u - T_n(v))|^p dx \\ & \leq \int_\Omega (a(x,\nabla u) - a(x,\nabla T_n(v)))\nabla T_k(u - T_n(v))dx \\ & \leq 2C.k + 2\left(\|k\|_{p'}^{p'} + k_2^{p'}\|\nabla v\|_p^p\right)^{1/p'}\left(\int_\Omega |\nabla T_k(u - T_n(v))|^p dx\right)^{1/p} \end{aligned}$$

which implies

$$\int_\Omega |\nabla T_k(u - T_n(v))|^p dx \leq C_{k,v}$$

where $C_{k,v}$ is a constant which can depend on k and v but not on n.

If $p < 2$, we have

$$\begin{aligned} & \int_\Omega |\nabla T_k(u - T_n(v))|^p dx \\ & \leq \int_\Omega \frac{|\nabla T_k(u - T_n(v))|^p}{(h(x) + |\nabla u| + |\nabla T_n(v)|)^{(2-p)p/2}}(h(x) + |\nabla u| + |\nabla T_n(v)|)^{(2-p)p/2}dx \end{aligned}$$

and by using Hölder's inequality, we obtain

$$\begin{aligned} & \int_\Omega |\nabla T_k(u - T_n(v))|^p dx \\ & \leq \left[\int_\Omega \frac{|\nabla T_k(u - T_n(v))|^2}{(h(x) + |\nabla u| + |\nabla T_n(v)|)^{2-p}}dx\right]^{p/2} \\ & \times \left[\int_{\{|u - T_n(v)| \leq k\}} (h(x) + |\nabla u| + |\nabla T_n(v)|)^p dx\right]^{(2-p)/2} \\ & \leq \left[\int_\Omega (a(x,\nabla u) - a(x,\nabla T_n(v)))\nabla T_k(u - T_n(v))dx\right]^{p/2} \\ & \times 3^{p(2-p)/2}\left[\|h\|_p^p + \|\nabla T_k(u - T_n(v))\|_p^p + 2\|\nabla v\|_p^p\right]^{(2-p)/2} \end{aligned}$$

which gives, thanks to (2.8) and Hölder's inequality again,

$$\begin{aligned}&\int_\Omega |\nabla T_k(u-T_n(v))|^p dx\\ &\leq \Big[2C.k+2(\|k\|_{p'}^{p'}+k_2^{p'}\|\nabla v\|_p^p)^{1/p'}\|\nabla T_k(u-T_n(v))\|_p\Big]^{p/2}\\ &\times 3^{p(2-p)/2}\,[\|h\|_p^p+\|\nabla T_k(u-T_n(v))\|_p^p+2\|\nabla v\|_p^p]^{(2-p)/2}\end{aligned}$$

and so that

$$\int_\Omega |\nabla T_k(u-T_n(v))|^p dx \leq C_{k,v}, \quad \forall n.$$

We deduce that, in all cases,

$$\nabla T_k(u-T_n(v)) \rightharpoonup \nabla T_k(u-v) \text{ weakly in } (L^p(\Omega))^N$$

(for a subsequence).

We will now to pass to the limit in (2.8). Remark that,

$$\begin{aligned}&\int_\Omega a(x,\nabla u)\nabla T_k(u-T_n(v))dx\\ &\quad=\int_\Omega (a(x,\nabla u)-a(x,\nabla T_n(v)))\nabla T_k(u-T_n(v))dx\\ &\qquad+\int_\Omega a(x,\nabla T_n(v))\nabla T_k(u-T_n(v))dx\end{aligned}$$

then, by using Fatou's Lemma in the first term of the right hand side of the last equality and Lebesgue's Theorem in the second, we obtain

$$\begin{aligned}&\liminf_{n\to+\infty}\int_\Omega a(x,\nabla u)\nabla T_k(u-T_n(v))dx\\ &\qquad\geq \int_\Omega a(x,\nabla u)\nabla T_k(u-v)dx.\end{aligned}$$

Finally, we apply Lebesgue's Theorem on other terms in (2.8) to complete the proof. ■

First, we give some technical lemmas which we use throughout the paper.

Lemma 2.1. *If (x_n) is a sequence of real numbers then*

$$|\limsup_{n\to+\infty} x_n| \leq \limsup_{n\to+\infty} |x_n|.$$

Proof. Since $-|x_n| \leq x_n \leq |x_n|$ then

$$\limsup_{n\to+\infty} x_n \leq \limsup_{n\to+\infty} |x_n|.$$

On the other hand

$$\liminf_{n\to+\infty}(-|x_n|) \leq \liminf_{n\to+\infty} x_n \leq \limsup_{n\to+\infty} x_n$$

wich gives

$$-\limsup_{n\to+\infty} |x_n| \leq \limsup_{n\to+\infty} x_n.$$

Consequently

$$|\limsup_{n\to+\infty} x_n| \leq \limsup_{n\to+\infty} |x_n|.$$

Lemma 2.2. *Let $(f_{n,m})$ and $(g_{n,m})$ be two sequences in $L^1(\Omega)$ such that*

$$\begin{cases} i)\ |f_{n,m}| \leq g_{n,m}, \quad \text{for all } n, m \\ ii)\ \lim\limits_{n\to+\infty} f_{n,m} = f_m \text{ and } \lim\limits_{m\to+\infty} f_m = f \text{ a.e.} \\ iii)\ \lim\limits_{n\to+\infty} g_{n,m} = g_m \text{ and } \lim\limits_{m\to+\infty} g_m = g \text{ a.e. with } g \text{ and } g_m \text{ belong to } L^1(\Omega) \\ iv)\ \limsup\limits_{m\to+\infty} \limsup\limits_{n\to+\infty} \displaystyle\int_\Omega g_{n,m}dx = \int_\Omega g dx. \end{cases}$$

Then

$$f \in L^1(\Omega) \text{ and } \lim_{m\to+\infty} \limsup_{n\to+\infty} \int_\Omega |f_{n,m} - f|dx = 0.$$

By the last lemma, we deduce that

$$\lim_{m\to+\infty} \limsup_{n\to+\infty} \int_\Omega f_{n,m}dx = \int_\Omega f dx.$$

Proof. Put $h_{n,m} = g_{n,m} + g - |f_{n,m} - f| \geq 0$. Fatou's Lemma applied on n implies that

$$\int_\Omega \liminf_{n\to+\infty} h_{n,m}dx \leq \liminf_{n\to+\infty} \int_\Omega h_{n,m}dx,$$

which gives

$$\begin{aligned} \int_\Omega (g_m + g - |f_m - f|)dx &\leq \liminf_{n\to+\infty} \left[-\int_\Omega |f_{n,m} - f|dx + \int_\Omega g dx + \int_\Omega g_{n,m}dx \right] \\ &\leq -\limsup_{n\to+\infty} \int_\Omega |f_{n,m} - f|dx + \int_\Omega g dx + \limsup_{n\to+\infty} \int_\Omega g_{n,m}dx. \end{aligned}$$

Using Fatou's Lemma again, but now on m, we obtain

$$\begin{aligned} &\int_\Omega \liminf_{m\to+\infty} (g_m + g - |f_m - f|)dx \\ &\quad \leq \liminf_{m\to+\infty} \left[-\limsup_{n\to+\infty} \int_\Omega |f_{n,m} - f|dx + \int_\Omega g dx + \limsup_{n\to+\infty} \int_\Omega g_{n,m}dx \right] \\ &\quad \leq -\limsup_{m\to+\infty} \limsup_{n\to+\infty} \int_\Omega |f_{n,m} - f|dx + \int_\Omega g dx + \limsup_{m\to+\infty} \limsup_{n\to+\infty} \int_\Omega g_{n,m}dx, \end{aligned}$$

which yields

$$2\int_\Omega g dx \leq 2\int_\Omega g dx - \limsup_{m\to+\infty} \limsup_{n\to+\infty} \int_\Omega |f_{n,m} - f|dx$$

hence

$$\limsup_{m\to+\infty} \limsup_{n\to+\infty} \int_\Omega |f_{n,m} - f|dx \leq 0$$

and finally

$$\lim_{m\to+\infty} \limsup_{n\to+\infty} \int_\Omega |f_{n,m} - f|dx = 0.$$

Proof of Theorem 2.1.
Step1: A priori estimates.
Consider the approximate unilateral problems:

$$\begin{cases} u_n \in K_\psi,\ g(x,u_n,\nabla u_n) \in L^1(\Omega),\ g(x,u_n,\nabla u_n)u_n \in L^1(\Omega) \\ \langle A(u_n), u_n - v\rangle + \int_\Omega g(x,u_n,\nabla u_n)(u_n - v)dx \le \int_\Omega f_n(u_n - v)dx \\ \qquad \forall v \in K_\psi \cap L^\infty(\Omega), \end{cases} \tag{2.9}$$

where f_n is a sequence of smooth functions which converges strongly to f in $L^1(\Omega)$. By Theorem 3.1 of [2], there exists at least one solution u_n of (2.9).
Thanks to Remark 2.2 of [1], we have:

$$\begin{cases} \langle A(u_n), T_k(u_n - v)\rangle + \int_\Omega g(x,u_n,\nabla u_n)T_k(u_n - v)dx, \\ \qquad\qquad \le \int_\Omega f_n T_k(u_n - v)dx \\ \qquad \forall v \in K_\psi,\ \forall k > 0. \end{cases} \tag{P_n}$$

Taking $v = \psi^+$ as test function in (P_n) gives

$$\int_\Omega a(x,u_n,\nabla u_n)\nabla T_k(u_n - \psi^+)dx + \int_\Omega g(x,u_n,\nabla u_n)T_k(u_n - \psi^+)dx \le \int_\Omega f_n T_k(u_n - \psi^+)dx$$

and by using the fact that $g(x,u_n,\nabla u_n)T_k(u_n - \psi^+) \ge 0$ we obtain

$$\int_{\{|u_n - \psi^+| \le k\}} a(x,u_n,\nabla u_n)\nabla(u_n - \psi^+)dx \le C.k.$$

As in [1], we deduce then by Young's inequality

$$\int_{\{|u_n - \psi^+| \le k\}} a(x,u_n,\nabla u_n)\nabla u_n \le C_1 + \frac{\alpha}{2}\int_{\{|u_n - \psi^+| \le k\}} |\nabla u_n|^p dx$$

where C_1 is a constant which doesn't depend on n (but which can depend on $k, \psi^+, c(x)$,
k_1, k_2, α).
Thus by using (2.3) we obtain

$$\frac{\alpha}{2}\int_{\{|u_n - \psi^+| \le k\}} |\nabla u_n|^p dx \le C_1.$$

Finally, we have for any $h > 0$

$$\int_{\{|u_n| \le h\}} |\nabla u_n|^p dx \le \int_{\{|u_n - \psi^+| \le h + \|\psi^+\|_\infty\}} |\nabla u_n|^p dx \le C_h \tag{2.10}$$

where C_h is a constant which depends on h but not on n.
The choice of the test function $v = T_h(u_n), h \ge \|\psi^+\|_\infty$ with $k = 1$ in (P_n) yields

$$\int_{\{h \le |u_n| < h+1\}} |\nabla u_n|^p dx \le C$$

and $\int_{\{|u_n|\geq h+1\}} |g(x,u_n,\nabla u_n)|dx \leq C.$

Consequently, as in [5], for every q such that $1 < q < \frac{N(p-1)}{N-1}$

$$\int_\Omega |\nabla u_n|^q dx \leq C_q \tag{2.11}$$

where C_q is a constant which doesn't depend on n (but which depends on q and meas(Ω)).

On the other hand, it is easy to see that

$$\int_\Omega |g(x,u_n,\nabla u_n)|dx \leq C. \tag{2.12}$$

Thanks to (2.11) there exists some $u \in W_0^{1,q}(\Omega)$ such that

$$u_n \rightharpoonup u \text{ weakly in } W_0^{1,q}(\Omega)$$

and by (2.10)

$$T_k(u_n) \rightharpoonup T_k(u) \text{ weakly in } W_0^{1,p}(\Omega),\ \forall k > 0.$$

Step2: Almost everywhere convergence of the gradients.
Fix q such that $1 < q < \bar{q}$. Consider

$$I_n = \int_\Omega \{[a(x,u_n,\nabla u_n) - a(x,u_n,\nabla u)]\nabla(u_n - u)\}^\theta dx$$

where $0 < \theta < \frac{q}{p}$. Let $k \geq \|\psi^+\|_\infty$. The use of the test function $T_k(u)$ in (P_n) gives for any $\eta > 0$:

$$\langle A(u_n), T_\eta(u_n - T_k(u))\rangle + \int_\Omega g(x,u_n,\nabla u_n)T_\eta(u_n - T_k(u))dx \leq \int_\Omega f_n T_\eta(u_n - T_k(u))dx. \tag{2.13}$$

Thanks to (2.12) and (2.13), we prove , as in [8] (see Remark 2.6), that I_n converges to zero, and so that

$$\nabla u_n \to \nabla u \text{ a.e. in } \Omega. \tag{2.14}$$

Step3: Passage to the limit.
Let now k such that $k \geq \|\psi^+\|_\infty$ and let $\gamma = (\frac{b(k)}{2\alpha})^2$ and let $\phi(s) = s\exp(\gamma s^2)$.
It is well known that

$$\phi'(s) - \frac{b(k)}{\alpha}|\phi(s)| \geq \frac{1}{2},\ \forall s \in I\!R. \tag{2.15}$$

Consider now the function $h_m, m > 0$ defined by:

$$h_m(t) = \begin{cases} 1 & \text{if } |t| \leq m \\ -\frac{t}{m}sgn(t) + 2 & \text{if } m \leq |t| \leq 2m \\ 0 & \text{if } |t| > 2m. \end{cases}$$

Let $v_{n,m} = u_n - \eta h_m(u_n)\phi(z_n)$, with $\eta = \exp(-4\gamma k^2), z_n = T_k(u_n) - T_k(u)$.
The use of $v_{n,m}$ as test function in (P_n) gives, for all $h > 0$,

$$\langle A(u_n), T_h(\eta h_m(u_n)\phi(z_n))\rangle + \int_\Omega g(x,u_n,\nabla u_n)T_h(\eta h_m(u_n)\phi(z_n))dx \leq \int_\Omega f_n T_h(\eta h_m(u_n)\phi(z_n))dx,$$

and by taking $h > 2k$ we obtain

$$\langle A(u_n), h_m(u_n)\phi(z_n)\rangle + \int_\Omega g(x,u_n,\nabla u_n)h_m(u_n)\phi(z_n)dx \leq \int_\Omega f_n h_m(u_n)\phi(z_n)dx.$$

which gives

$$\begin{aligned}&\int_\Omega a(x,u_n,\nabla u_n)[\nabla T_k(u_n)-\nabla T_k(u)]h_m(u_n)\phi'(z_n)dx\\ &\quad+\int_\Omega a(x,u_n,\nabla u_n)\nabla u_n h_m'(u_n)\phi(z_n)dx\\ &\quad+\int_\Omega g(x,u_n,\nabla u_n)h_m(u_n)\phi(z_n)dx \leq \int_\Omega f_n h_m(u_n)\phi(z_n)dx.\end{aligned} \tag{2.16}$$

Denote by $\epsilon_m^1(n), \epsilon_m^2(n), ...,$ various sequences of real numbers which converge to zero when n tends to infinity with any fixed value of m. Since $g(x,u_n,\nabla u_n)h_m(u_n)\phi(z_n) \geq 0$ on the subset $\{x \in \Omega : |u_n(x)| > k\}$, we deduce from (2.15) that

$$\begin{aligned}&\int_\Omega a(x,u_n,\nabla u_n)[\nabla T_k(u_n)-\nabla T_k(u)]h_m(u_n)\phi'(z_n)dx\\ &\quad+\int_\Omega a(x,u_n,\nabla u_n)\nabla u_n h_m'(u_n)\phi(z_n)dx\\ &\quad+\int_{\{|u_n|\leq k\}} g(x,u_n,\nabla u_n)h_m(u_n)\phi(z_n)dx \leq \int_\Omega f_n h_m(u_n)\phi(z_n)dx = \epsilon_m^1(n).\end{aligned} \tag{2.17}$$

The first term of the last inequality can be written as:

$$\begin{aligned}&\int_\Omega a(x,u_n,\nabla u_n)[\nabla T_k(u_n)-\nabla T_k(u)]h_m(u_n)\phi'(z_n)dx\\ &\quad=\int_{\{|u_n|\leq k\}} a(x,u_n,\nabla u_n)[\nabla T_k(u_n)-\nabla T_k(u)]h_m(u_n)\phi'(z_n)dx\\ &\qquad-\int_{\{|u_n|>k\}} a(x,u_n,\nabla u_n)\nabla T_k(u)h_m(u_n)\phi'(z_n)dx.\end{aligned} \tag{2.18}$$

It's easy to observe that we have

$$\begin{aligned}&\left|\int_{\{|u_n|>k\}} a(x,u_n,\nabla u_n)\nabla T_k(u)h_m(u_n)\phi'(z_n)dx\right|\\ &\qquad\leq C_k\int_\Omega |a(x,T_{2m}(u_n),\nabla T_{2m}(u_n))||\nabla T_k(u)|\chi_{\{|u_n|>k\}}dx\end{aligned}$$

where $C_k = \phi'(2k)$. The right hand side of the last inequality tends to 0 as n tends to infinity. Indeed, the sequence $(a(x,T_{2m}(u_n),\nabla T_{2m}(u_n)))_n$ is bounded in $(L^{p'}(\Omega))^N$ while $\nabla T_k(u)\chi_{\{|u_n|>k\}}$ tends to 0 strongly in $(L^p(\Omega))^N$.

For the first term of the right hand side of (2.17), one can write

$$\begin{aligned}
&\int_{\{|u_n|\le k\}} a(x,u_n,\nabla u_n)[\nabla T_k(u_n)-\nabla T_k(u)]h_m(u_n)\phi'(z_n)dx\\
&\quad=\int_\Omega [a(x,T_k(u_n),\nabla T_k(u_n))-a(x,T_k(u_n),\nabla T_k(u))]\\
&\qquad\times[\nabla T_k(u_n)-\nabla T_k(u)]h_m(u_n)\phi'(z_n)dx\\
&\qquad+\int_\Omega a(x,T_k(u_n),\nabla T_k(u))[\nabla T_k(u_n)-\nabla T_k(u)]h_m(u_n)\phi'(z_n)dx.
\end{aligned}\tag{2.19}$$

The second term of the right hand side of (2.19) tends to 0 since

$$a(x,T_k(u_n),\nabla T_k(u))\to a(x,T_k(u),\nabla T_k(u)) \text{ strongly in } (L^{p'}(\Omega))^N$$

and

$$[\nabla T_k(u_n)-\nabla T_k(u)]h_m(u_n)\phi'(z_n)\rightharpoonup 0 \text{ weakly in } (L^p(\Omega))^N.$$

Consequently, from (2.17) we have

$$\begin{aligned}
&\int_\Omega a(x,u_n,\nabla u_n)[\nabla T_k(u_n)-\nabla T_k(u)]h_m(u_n)\phi'(z_n)dx\\
&\quad=\int_\Omega [a(x,T_k(u_n),\nabla T_k(u_n))-a(x,T_k(u_n),\nabla T_k(u))]\\
&\qquad\times[\nabla T_k(u_n)-\nabla T_k(u)]h_m(u_n)\phi'(z_n)dx\\
&\qquad+\epsilon_m^2(n).
\end{aligned}\tag{2.20}$$

On the other hand

$$\left|\int_\Omega a(x,u_n,\nabla u_n)\nabla u_n h_m'(u_n)\phi(z_n)dx\right|\le\frac{2\phi(2k)}{m}\int_{\{m\le|u_n|\le 2m\}} a(x,u_n,\nabla u_n)\nabla u_n dx$$

and by using $T_m(u_n), m\ge\|\psi^+\|_\infty$, as test function in (P_n), we obtain

$$\left|\int_\Omega a(x,u_n,\nabla u_n)\nabla u_n h_m'(u_n)\phi(z_n)dx\right|\le 2\phi(2k)\int_{\{|u_n|\ge m\}}|f_n|dx.\tag{2.21}$$

If we denote by $J_{n,m}$ the third term of the left hand side of (2.17), one has

$$\begin{aligned}
|J_{n,m}| &\le \int_{\{|u_n|\le k\}} b(k)(c(x)+|\nabla u_n|^p h_m(u_n))|\phi(z_n)|dx\\
&\le b(k)\int_\Omega c(x)|\phi(z_n)|dx\\
&\quad+\frac{b(k)}{\alpha}\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u_n)h_m(u_n)|\phi(z_n)|dx\\
&\le \epsilon_m^3(n)+\frac{b(k)}{\alpha}\int_\Omega [a(x,T_k(u_n),\nabla T_k(u_n))-a(x,T_k(u_n),\nabla T_k(u))]\\
&\qquad\times[\nabla T_k(u_n)-\nabla T_k(u)]h_m(u_n)|\phi(z_n)|dx
\end{aligned}\tag{2.22}$$

indeed, we have

$$\begin{aligned}
&\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u_n)h_m(u_n)|\phi(z_n)|dx\\
&\quad=\int_\Omega [a(x,T_k(u_n),\nabla T_k(u_n))-a(x,T_k(u_n),\nabla T_k(u))]\\
&\qquad\times[\nabla T_k(u_n)-\nabla T_k(u)]h_m(u_n)|\phi(z_n)|dx\\
&\qquad+\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u)h_m(u_n)|\phi(z_n)|dx\\
&\qquad+\int_\Omega a(x,T_k(u_n),\nabla T_k(u))[\nabla T_k(u_n)-\nabla T_k(u)]h_m(u_n)|\phi(z_n)|dx.
\end{aligned}$$

It's easy to see that the second and third terms of the right hand side of the last equality tend to 0 as n tends to infinity, since $(a(x,T_k(u_n),\nabla T_k(u_n)))_n$ is bounded in $(L^{p'}(\Omega))^N$,

$$\nabla T_k(u)h_m(u_n)|\phi(z_n)|\to 0 \text{ strongly in } (L^p(\Omega))^N$$

and

$$a(x,T_k(u_n),\nabla T_k(u))\to a(x,T_k(u),\nabla T_k(u)) \text{ strongly in } (L^{p'}(\Omega))^N,$$

$$[\nabla T_k(u_n)-\nabla T_k(u)]h_m(u_n)|\phi(z_n)|\rightharpoonup 0 \text{ weakly in } (L^p(\Omega))^N.$$

Combining (2.20), (2.21) and (2.22) we obtain

$$\begin{aligned}
&\int_\Omega [a(x,T_k(u_n),\nabla T_k(u_n))-a(x,T_k(u_n),\nabla T_k(u))]\\
&\qquad\times[\nabla T_k(u_n)-\nabla T_k(u)]h_m(u_n)(\phi'(z_n)-\tfrac{b(k)}{\alpha}|\phi(z_n)|)dx\\
&\qquad\le \epsilon_m^4(n)+2\phi(2k)\int_{\{|u_n|\ge m\}}|f_n|dx
\end{aligned}$$

which implies, by using (2.15)

$$\begin{aligned}
&\int_\Omega [a(x,T_k(u_n),\nabla T_k(u_n))-a(x,T_k(u_n),\nabla T_k(u))]\\
&\qquad\times[\nabla T_k(u_n)-\nabla T_k(u)]h_m(u_n)dx\\
&\qquad\le 2\epsilon_m^4(n)+4\phi(2k)\int_{\{|u_n|\ge m\}}|f_n|dx
\end{aligned}$$

by passing to the limit sup over n, one has

$$\begin{aligned}
&\limsup_{n\to+\infty}\int_\Omega [a(x,T_k(u_n),\nabla T_k(u_n))-a(x,T_k(u_n),\nabla T_k(u))]\\
&\qquad\times[\nabla T_k(u_n)-\nabla T_k(u)]h_m(u_n)dx\\
&\qquad\le 4\phi(2k)\int_{\{|u|\ge m\}}|f|dx
\end{aligned}$$

pass again to the limit sup but now over m we obtain

$$\begin{aligned}
&\limsup_{m\to+\infty}\limsup_{n\to+\infty}\int_\Omega [a(x,T_k(u_n),\nabla T_k(u_n))-a(x,T_k(u_n),\nabla T_k(u))]\\
&\qquad\times[\nabla T_k(u_n)-\nabla T_k(u)]h_m(u_n)dx\le 0.
\end{aligned}$$

Finally, we claim that

$$\limsup_{m\to+\infty}\limsup_{n\to+\infty}\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u_n)h_m(u_n)dx = \int_\Omega a(x,T_k(u),\nabla T_k(u))\nabla T_k(u)dx. \tag{2.23}$$

Indeed, we have

$$\lim_{m\to+\infty}\lim_{n\to+\infty}\int_\Omega a(x,T_k(u_n),\nabla T_k(u))[\nabla T_k(u_n)-\nabla T_k(u)]h_m(u_n)dx=0$$

and

$$\lim_{m\to+\infty}\lim_{n\to+\infty}\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u)h_m(u_n)dx = \int_\Omega a(x,T_k(u),\nabla T_k(u))\nabla T_k(u)dx.$$

Let now $v\in K_\psi\cap L^\infty(\Omega)$. Then, by taking $u_n-h_m(u_n)T_k(u_n-v)$ as test function in (P_n) we obtain

$$\langle A(u_n),T_k(h_m(u_n)T_k(u_n-v))\rangle + \int_\Omega g(x,u_n,\nabla u_n)T_k(h_m(u_n)T_k(u_n-v))dx \leq \int_\Omega f_nT_k(h_m(u_n)T_k(u_n-v))dx$$

which gives

$$\langle A(u_n),h_m(u_n)T_k(u_n-v)\rangle + \int_\Omega g(x,u_n,\nabla u_n)h_m(u_n)T_k(u_n-v)dx \leq \int_\Omega f_nh_m(u_n)T_k(u_n-v)dx$$

and so that

$$\begin{aligned}\int_\Omega g(x,u_n,\nabla u_n)h_m(u_n)T_k(u_n-v)dx &\leq -\int_\Omega a(x,u_n,\nabla u_n)\nabla T_k(u_n-v)h_m(u_n)dx\\ &\quad+\int_\Omega -a(x,u_n,\nabla u_n)\nabla u_nh_m'(u_n)T_k(u_n-v)dx\\ &\quad+\int_\Omega f_nh_m(u_n)T_k(u_n-v)dx\end{aligned}$$

then by passing to the limit sup on n

$$\begin{aligned}\limsup_{n\to+\infty}\int_\Omega g(x,u_n,\nabla u_n)h_m(u_n)T_k(u_n-v)dx &\leq -\liminf_{n\to+\infty}\int_\Omega a(x,u_n,\nabla u_n)\nabla T_k(u_n-v)h_m(u_n)dx\\ &\quad+\limsup_{n\to+\infty}\left|\int_\Omega a(x,u_n,\nabla u_n)\nabla u_nh_m'(u_n)T_k(u_n-v)dx\right|\\ &\quad+\int_\Omega fh_m(u)T_k(u-v)dx\end{aligned}$$

and thus

$$\begin{aligned}&\limsup_{m\to+\infty}\limsup_{n\to+\infty}\int_\Omega g(x,u_n,\nabla u_n)h_m(u_n)T_k(u_n-v)dx\\&\quad\le-\liminf_{m\to+\infty}\liminf_{n\to+\infty}\int_\Omega a(x,u_n,\nabla u_n)\nabla T_k(u_n-v)h_m(u_n)dx\\&\qquad+\limsup_{m\to+\infty}\limsup_{n\to+\infty}\left|\int_\Omega a(x,u_n,\nabla u_n)\nabla u_n h'_m(u_n)T_k(u_n-v)dx\right|\\&\qquad+\int_\Omega fT_k(u-v)dx.\end{aligned}\tag{2.24}$$

On the one hand

$$\begin{aligned}&\int_\Omega a(x,u_n,\nabla u_n)\nabla T_k(u_n-v)h_m(u_n)dx\\&\qquad=\int_\Omega[a(x,u_n,\nabla u_n)-a(x,u_n,\nabla v)]\nabla T_k(u_n-v)h_m(u_n)dx\\&\qquad\quad+\int_\Omega a(x,T_{k+\|v\|_\infty}(u_n),\nabla v)\nabla T_k(u_n-v)h_m(u_n)dx,\end{aligned}$$

thanks to Fatou's Lemma on n in the first term of the right hand side, and the fact that

$$a(x,T_{k+\|v\|_\infty}(u_n),\nabla v)\to a(x,T_{k+\|v\|_\infty}(u),\nabla v)\text{ strongly in }(L^{p'}(\Omega))^N$$

and

$$\nabla T_k(u_n-v)\rightharpoonup\nabla T_k(u-v)\text{ weakly in }(L^p(\Omega))^N$$

in the second term, we obtain

$$\begin{aligned}&\liminf_{n\to+\infty}\int_\Omega a(x,u_n,\nabla u_n)\nabla T_k(u_n-v)h_m(u_n)dx\\&\qquad\ge\int_\Omega[a(x,u,\nabla u)-a(x,u,\nabla v)]\nabla T_k(u-v)h_m(u)dx\\&\qquad\quad+\int_\Omega a(x,T_{k+\|v\|_\infty}(u),\nabla v)\nabla T_k(u-v)h_m(u)dx\end{aligned}$$

in which we can use Fatou's Lemma on m in the first term of the right hand side and Lebesgue's Theorem in the second one to obtain

$$\begin{aligned}&\liminf_{m\to+\infty}\liminf_{n\to+\infty}\int_\Omega a(x,u_n,\nabla u_n)\nabla T_k(u_n-v)h_m(u_n)dx\\&\qquad\ge\int_\Omega a(x,u,\nabla u)\nabla T_k(u-v)dx.\end{aligned}\tag{2.25}$$

On the other hand we have

$$\begin{aligned}&\left|\int_\Omega a(x,u_n,\nabla u_n)\nabla u_n h'_m(u_n)T_k(u_n-v)dx\right|\\&\quad\le\ 2\frac{k}{m}\int_{\{m\le|u_n|\le 2m\}}a(x,u_n,\nabla u_n)\nabla u_n dx\\&\quad\le\ 2k\int_{\{|u_n|\ge m\}}|f_n|dx\end{aligned}$$

which implies

$$\limsup_{n\to+\infty}\left|\int_\Omega a(x,u_n,\nabla u_n)\nabla u_n h_m'(u_n)T_k(u_n-v)dx\right| \le 2k\int_{\{|u|\ge m\}}|f|dx$$

and so that

$$\limsup_{m\to+\infty}\limsup_{n\to+\infty}|\int_\Omega a(x,u_n,\nabla u_n)\nabla u_n h_m'(u_n)T_k(u_n-v)dx| = 0. \tag{2.26}$$

About the left hand side of (2.24), we argue as follows: Let $l>0$ one can write

$$\begin{aligned}&\left|\int_\Omega g(x,u_n,\nabla u_n)T_k(u_n-v)h_m(u_n)dx - \int_\Omega g(x,u,\nabla u)T_k(u-v)dx\right|\\ &\le \int_\Omega |g(x,u_n,\nabla u_n)T_k(u_n-v)h_m(u_n)\chi_{\{|u_n|\le l\}} - g(x,u,\nabla u)T_k(u-v)\chi_{\{|u|\le l\}}|dx\\ &\quad+\int_\Omega |g(x,u_n,\nabla u_n)T_k(u_n-v)h_m(u_n)\chi_{\{|u_n|\ge l\}}|dx\\ &\quad+\int_\Omega |g(x,u,\nabla u)T_k(u-v)\chi_{\{|u|\ge l\}}|dx\\ &\le \int_\Omega |g(x,u_n,\nabla u_n)T_k(u_n-v)h_m(u_n)\chi_{\{|u_n|\le l\}} - g(x,u,\nabla u)T_k(u-v)\chi_{\{|u|\le l\}}|dx\\ &\quad+k\int_\Omega |g(x,u_n,\nabla u_n)|\chi_{\{|u_n|\ge l\}}dx + k\int_\Omega |g(x,u,\nabla u)|\chi_{\{|u|\ge l\}}dx.\end{aligned}$$

The use of test function $T_l(u_n), l\ge \|\psi^+\|_\infty$ as test function in (P_n) gives

$$\int_\Omega |g(x,u_n,\nabla u_n)|\chi_{\{|u_n|\ge l+1\}}dx \le \int_{\{|u_n|\ge l\}}|f_n|dx.$$

Let $\epsilon>0$ be arbitrary, then there exists $l=l(\epsilon)>1$ such that:

$$\int_\Omega |g(x,u_n,\nabla u_n)|\chi_{\{|u_n|\ge l\}}dx \le \frac{\epsilon}{2k},\quad \forall n,$$

and by using Fatou's Lemma

$$\int_\Omega |g(x,u,\nabla u)|\chi_{\{|u|\ge l\}}dx \le \frac{\epsilon}{2k}.$$

Then

$$\begin{aligned}&\left|\int_\Omega g(x,u_n,\nabla u_n)T_k(u_n-v)h_m(u_n)dx - \int_\Omega g(x,u,\nabla u)T_k(u-v)dx\right|\\ &\le \int_\Omega |g(x,u_n,\nabla u_n)T_k(u_n-v)h_m(u_n)\chi_{\{|u_n|\le l\}} - g(x,u,\nabla u)T_k(u-v)\chi_{\{|u|\le l\}}|dx\\ &\quad+\tfrac{\epsilon}{2}+\tfrac{\epsilon}{2}.\end{aligned}$$

However,

$$\begin{aligned}&|g(x,u_n,\nabla u_n)T_k(u_n-v)h_m(u_n)\chi_{\{|u_n|\le l\}}|\\ &\qquad\le kb(l)\left[c(x)+\frac{1}{\alpha}a(x,T_l(u_n),\nabla T_l(u_n))\nabla T_l(u_n))h_m(u_n)\right]\end{aligned}$$

by setting

$$f_{n,m} = g(x,u_n,\nabla u_n)T_k(u_n-v)h_m(u_n)\chi_{\{|u_n|\le l\}}$$

and

$$g_{n,m} = kb(l)\left[c(x) + \frac{1}{\alpha}a(x, T_l(u_n), \nabla T_l(u_n))\nabla T_l(u_n))h_m(u_n)\right]$$

it's obvious, in view of (2.14) and (2.23), that our sequences satisfy the hypotheses of Lemma 2.2 and thus

$$\limsup_{m\to+\infty}\limsup_{n\to+\infty}\int_\Omega |g(x,u_n,\nabla u_n)T_k(u_n-v)h_m(u_n)\chi_{\{|u_n|\le l\}} - g(x,u,\nabla u)T_k(u-v)\chi_{\{|u|\le l\}}|dx = 0$$

and finally, we have for all $\epsilon > 0$

$$\limsup_{m\to+\infty}\limsup_{n\to+\infty}\left|\int_\Omega g(x,u_n,\nabla u_n)T_k(u_n-v)h_m(u_n)dx - \int_\Omega g(x,u,\nabla u)T_k(u-v)dx\right| \le \epsilon$$

which yields in vertue of Lemma 2.1

$$\limsup_{m\to+\infty}\limsup_{n\to+\infty}\int_\Omega g(x,u_n,\nabla u_n)T_k(u_n-v)h_m(u_n)dx = \int_\Omega g(x,u,\nabla u)T_k(u-v)dx. \tag{2.27}$$

Combining (2.25), (2.26) and (2.27), the inequality (2.24) becomes

$$\int_\Omega g(x,u,\nabla u)T_k(u-v)dx \le -\int_\Omega a(x,u,\nabla u)\nabla T_k(u-v)dx + \int_\Omega fT_k(u-v)dx$$

and so that

$$\int_\Omega a(x,u,\nabla u)\nabla T_k(u-v)dx + \int_\Omega g(x,u,\nabla u)T_k(u-v)dx \le \int_\Omega fT_k(u-v)dx.$$

■

Remark 2.3 In the case where $p \in]1, 2-\frac{1}{N}]$, the solutions of (P_ψ) belong only to $T_0^{1,p}(\Omega)$ where

$$T_0^{1,p}(\Omega) = \{v : \Omega \to I\!R \text{ measurable },\ T_k(v) \in W_0^{1,p}(\Omega),\ \forall k > 0\}.$$

Indeed, into account of the fact that

$$\int_\Omega |\nabla T_k(u_n - T_{h_0}(u_n))|^p dx \le C.k, \qquad \forall k > 0 \text{ and for some } h_0 \ge \|\psi^+\|_\infty$$

then as in [3], and in view of (2.10), there exists a measurable function u, finite almost everywhere, such that

$$T_h(u_n) \rightharpoonup T_h(u) \text{ weakly in } W_0^{1,p}(\Omega), \quad \forall h > 0$$

and by using weak lower semicontinuity in (2.10)

$$\int_\Omega |\nabla T_h(u)|^p dx \le C_h.$$

Thanks to Lemma 4.2 of [3], we have for any $q < \bar{q} = \frac{N(p-1)}{N-1}$:

$$\int_\Omega |\nabla(u_n - T_{h_0}(u_n))|^q dx \le \text{meas}(\Omega) + qC\int_1^\infty t^{q-1-\bar{q}}dt < \infty$$

and so that

$$\int_\Omega |\nabla u_n|^q dx \leq C$$

and similarly

$$\int_\Omega |\nabla u|^q dx \leq C.$$

On the other hand, we have

$$\nabla u - \nabla T_k(u) = \nabla(u - T_k(u))$$

since for any $\eta > 0, T_\eta(u - T_k(u)) \in W_0^{1,p}(\Omega)$ (see the definition of the gradient given in [3]). So that, we can argue as in the previous proof of Theorem 2.1.

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A. Elmahi
C. P. R de Fès, B.P 49, Fès, Maroc

D. Meskine
Département de Mathématiques et Informatique, Faculté des Sciences Dhar-Mahraz, B. P 1796 Atlas Fès, Maroc
E-mail address: meskinedriss@hotmail.com