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PERIODIC SOLUTIONS FOR A COUPLED VAN DER POL TYPE EQUATION WITH A FORCING TERM

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Abstract. In the present paper, we show the existence of periodic solutions for a coupled Van der Pol type equation with a forcing term of the form

$$\ddot{u} + f(u)\dot{u} + Au = e(t) \qquad t \in \mathbb{R},$$

where $f : \mathbb{R}^N \to \mathbb{R}^N$, A is a $N \times N$ constant matrix and $e : \mathbb{R} \to \mathbb{R}^N$ is a periodic function of t.

1. Introduction

Let $F : \mathbb{R} \to \mathbb{R}$, $g : \mathbb{R} \to \mathbb{R}$ be continuous functions and $e : \mathbb{R} \to \mathbb{R}$ be a T-periodic function with $T > 0$. The problem of the existence of a T-periodic solution of the forced Liénard equation

$$\text{(L)} \qquad \ddot{x}(t) + \frac{d}{dt}[F(x(t))] + g(x(t)) = e(t)$$

has been extensively discussed under various conditions on F and g in the literature. It is often assumed that $g(s)s$ does not change sign for large $|s|$, e.g., there exists $d > 0$ such that

$$g(s)s > 0 \quad \text{for } |s| \geq d.$$

For the existence of a periodic solution, additional conditions is needed to be imposed. Roughly speaking, one of the two type of conditions is assumed:

$$\lim_{|s|\to\infty} F(s) \operatorname{sgn} s = \infty \qquad \text{or} \qquad \lim_{|s|\to\infty} \frac{g(s)}{s} < 1.$$

For works dealing with (L), we refer the reader to, e.g., [1], [2], [7] and their lists of references. The results for the scalar equations were extended to the N-dimensional cases, e.g., $F : \mathbb{R}^N \to \mathbb{R}^N$ and $g : \mathbb{R}^N \to \mathbb{R}^N$ are continuous functions and $e : \mathbb{R} \to \mathbb{R}^N$ is a periodic function. In [11], Zanolin established an existence result for N-dimensional cases under the condition that $g = g_1 + g_2$ and $|\nabla g_1| \leq 2\pi/T$ and g_2 is bounded. In case of $g(u) = Au$, Mawhin [9] has given an interesting result with a remarkably weak restriction upon F. In the present paper, we consider the existence of solutions of Liénard equations with a forcing term. The typical system of equations our result is adopted is the system of N-dimensional Van der Pol type equations of the form

$$\text{(VDP)} \qquad \begin{cases} \ddot{u}_1 + \varepsilon_1(u_1^2 - a_1)\dot{u}_1 + c_{11}u_1 + c_{12}u_2 + \ldots + c_{1n}u_n = e_1(t), \\ \ddot{u}_2 + \varepsilon_2(u_2^2 - a_2)\dot{u}_2 + c_{21}u_2 + c_{22}u_2 + \ldots + c_{2n}u_n = e_2(t), \\ \cdots \\ \ddot{u}_n + \varepsilon_n(u_n^2 - a_n)\dot{u}_n + c_{n1}u_n + c_{n2}u_2 + \ldots + c_{nn}u_n = e_n(t), \end{cases}$$

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where $n \geq 1, \varepsilon_i > 0$ for each $i = 1, \cdots, n$, and $c_{ij} \in \mathbb{R}$ for all $1 \leq i, j \leq n$. The equation we treat in this paper has the form

$$\ddot{u} + f(u)\dot{u} + Au = e(t) \qquad t \in \mathbb{R}, \tag{V}$$

where

$$f(x) = \begin{pmatrix} f_1(x_1) & 0 & \dots & 0 \\ 0 & f_2(x_2) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & f_N(x_N) \end{pmatrix} \quad \text{for } x = (x_1, \cdots, x_N),$$

$f_i \in C(\mathbb{R}; \mathbb{R})$ for each $i = 1, \cdots, N$, A is a $N \times N$ constant matrix and $e : \mathbb{R} \to \mathbb{R}^N$ is a periodic function with period $T > 0$. In (V), Au is called a coupling term, where each element a_{ij} of A represents the interaction between u_i and u_j.

In the following section, we shall state our main existence result for (V) and give the proof of it by using Leray-Schauder degree theory (cf.[8]). In section 3, we will show that if e is sufficiently weak, then the periodic solution exists near the origin and it is repellent. In the final section, we give a concrete example of repellor and some observations.

2. Existence of Periodic Solutions

In this section, we establish an existence result for periodic solutions of (V). To state our result, we need some preliminaries. Let us define that

$$|x| = \left(\sum_{i=1}^{N} x_i^2 \right)^{1/2}, \quad \langle x, y \rangle = \sum_{i=1}^{N} x_i y_i$$

for $x = (x_1, \cdots, x_N), y = (y_1, \cdots, y_N) \in \mathbb{R}^N$, and

$$\|u\| = \left(\int_0^T |u(t)| \, dt \right)^{1/2}, \quad \langle\langle u, v \rangle\rangle = \int_0^T \langle u(t), v(t) \rangle \, dt$$

for $u, v \in L^2([0,T]; \mathbb{R}^N)$. We write $\dot{u} = du/dt$ for each function $u : \mathbb{R} \to \mathbb{R}^N$. We put

$$H = \left\{ u \in L^2([0,T]; \mathbb{R}^N) : u(0) = u(T), \dot{u} \in L^2([0,T]; \mathbb{R}^N) \right\}.$$

H is a Hilbert space with the norm $\| \cdot \|_H$ defined by

$$\|u\|_H^2 = \|u\|^2 + \|\dot{u}\|^2 \quad \text{for each } u \in H.$$

We also put

$$\widetilde{H} = \left\{ u \in H : \int_0^T u(t) \, dt = 0 \right\}.$$

Then by standard arguments, we note that for each $u \in \widetilde{H}$, the following inequality holds.

$$\|u\| \leq \left(\frac{T}{2\pi} \right) \|\dot{u}\|. \tag{2.1}$$

For a matrix A, its norm is defined by Frobenius type

$$\|A\| = \left(\sum_{i=1}^{N} \sum_{j=1}^{N} |a_{ij}|^2 \right)^{1/2}.$$

Further, for a given set Ω, its closure is written by $\overline{\Omega}$, its boundary $\partial\Omega$. Throughout this paper, unless otherwise explicitly states, $c_0, c_1, \cdots$ and $C_0, C_1, \cdots$ denote various positive constants.

Let X be Banach space. Suppose that D is a bounded open subset of X and $\mathcal{F} : \overline{D} \to X$ is a continuous mapping of the form $\mathcal{F} = I - \mathcal{G}$, where I is identity and $\mathcal{G} : \overline{D} \to X$ is a compact mapping. If $\mathcal{F}x \neq p$ for any $x \in \partial D$, then we denote by $\deg(\mathcal{F}, D, p)$ Leray-Schauder degree of $\mathcal{F}$ at $p \in X$ relative to D.

We can now state our main results.

Theorem 2.1. *Let f and A satisfy the following conditions:*

(F) $$\liminf_{|s| \to \infty} \frac{f_i(s)}{s^2} > 0 \quad \text{for each } i;$$

(A) $$\det A \neq 0.$$

Then for each $e \in \widetilde{H} \setminus \{0\}$, problem (V) *has at least one solution $u \in \widetilde{H}$.*

Throughout the rest of this paper, we assume that (F) and (A) hold. To prove Theorem 2.1, we need a few lemmata.

Lemma 2.2. *If u is a T-periodic solution of* (V)*, then $u \in \widetilde{H}$.*

Proof. Let u be a possible T-periodic solution of (V). By integrating (V) over $[0, T]$, we obtain by the periodicity of u that

$$\int_0^T Au = 0.$$

Since A is invertible, $\int_0^T u = 0$. Therefore $u \in \widetilde{H}$. □

Lemma 2.3. *Let $\eta_0 \in (0, 1)$. Then the set*

$$S_1 = \left\{ u \in \widetilde{H} : \ddot{u} + f(u)\dot{u} + \eta Au = e(t) \quad \text{for some } \eta \in [\eta_0, 1] \right\}$$

is bounded in $\widetilde{H}$.

Proof. Let $\eta_0 \in (0, 1)$ and fix $\eta \in [\eta_0, 1]$. Let u be a possible T-periodic solution of the auxiliary equation

$$\ddot{u} + f(u)\dot{u} + \eta Au = e(t). \tag{2.2}$$

Multiplying (2.2) by u and integrating over $[0, T]$, we find

$$\|\dot{u}\|^2 = \eta \langle\langle Au, u \rangle\rangle - \langle\langle e, u \rangle\rangle \leq \eta \|A\| \|u\|^2 + \|e\| \|u\|. \tag{2.3}$$

On the other hand, multiplying (2.2) by $\dot{u}$ and integrating over $[0,T]$, we find by (2.1) that

$$\begin{aligned}\langle\!\langle f(u)\dot{u},\dot{u}\rangle\!\rangle &= -\eta\langle\!\langle Au,\dot{u}\rangle\!\rangle + \langle\!\langle e,\dot{u}\rangle\!\rangle \\ &\le \eta\|A\|\|u\|\|\dot{u}\| + \|e\|\|\dot{u}\| \\ &\le \left(\frac{\eta T}{2\pi}\right)\|A\|\|\dot{u}\|^2 + \|e\|\|\dot{u}\|.\end{aligned} \tag{2.4}$$

Here we claim that there exists $m>0$ and $\gamma>0$ such that

$$\int_0^T \sum_{i=1}^N u_i^2\dot{u}_i{}^2 \le \frac{1}{\gamma}\langle\!\langle f(u)\dot{u},\dot{u}\rangle\!\rangle + m^2\|\dot{u}\|^2. \tag{2.5}$$

By using condition (F), we have that for each i, there exists $m_i>0$ and $\gamma_i>0$ such that

$$f_i(s) > \gamma_i s^2 \quad \text{for each } |s|>m_i. \tag{2.6}$$

We put $m=\max\{m_i : i=1,\cdots,N\}$, $\gamma=\min\{\gamma_i : i=1,\cdots,N\}$ and $D_i=\{t\in[0,T] : |u_i(t)|>m\}$. Then we have from (2.6) that

$$\begin{aligned}\int_0^T \sum_{i=1}^N u_i^2\dot{u}_i{}^2 &= \sum_{i=1}^N \int_{D_i} u_i^2\dot{u}_i{}^2 + \sum_{i=1}^N \int_{[0,T]\setminus D_i} u_i^2\dot{u}_i{}^2 \\ &\le \sum_{i=1}^N \int_{D_i} \frac{1}{\gamma_i} f_i(u_i)\dot{u}_i{}^2 + m^2\sum_{i=1}^N \int_{[0,T]\setminus D_i} \dot{u}_i{}^2 \\ &\le \frac{1}{\gamma}\sum_{i=1}^N \int_0^T f_i(u_i)\dot{u}_i{}^2 + m^2\sum_{i=1}^N \int_0^T \dot{u}_i{}^2.\end{aligned}$$

Hence (2.5) holds as claimed. While, we have

$$u_i^2(t) = 2\int_0^t u_i\dot{u}_i \le 2T^{1/2}\left(\int_0^T u_i^2\dot{u}_i{}^2\right)^{1/2} \qquad \text{for all } t\in[0,T].$$

Summing up with respect to i and integrating over $[0,T]$, we have that there exists a positive constant c_0 independent of u such that

$$\|u\|^2 \le 2T^{3/2}\sum_{i=1}^N \left(\int_0^T u_i^2\dot{u}_i{}^2\right)^{1/2} \le c_0T^{3/2}\left(\int_0^T \sum_{i=1}^N u_i^2\dot{u}_i{}^2\right)^{1/2}. \tag{2.7}$$

By (2.1) and (2.3), we have that

$$\|\dot{u}\|^2 - \left(\frac{\eta T}{2\pi}\right)\|e\|\|\dot{u}\| \le \eta\|A\|\|u\|^2.$$

Then it follows from (2.4), (2.5) and (2.7) that

$$\|\dot{u}\|^2 - \left(\frac{\eta T}{2\pi}\right)\|e\|\|\dot{u}\| \le c_0\eta T^{3/2}\|A\|\left(\int_0^T \sum_{i=1}^N u_i^2\dot{u}_i{}^2\right)^{1/2}$$

$$\leq c_0\eta T^{3/2}\|A\|\left(\frac{1}{\gamma}\langle\langle f(u)\dot{u},\dot{u}\rangle\rangle + m^2\|\dot{u}\|^2\right)^{1/2}$$
$$\leq c_0\eta T^{3/2}\|A\|\left(\left(\frac{\eta T\|A\|}{2\pi\gamma}+m^2\right)\|\dot{u}\|^2+\frac{\|e\|}{\gamma}\|\dot{u}\|\right)^{1/2}.$$

This implies that there exists C_0 and $C_1 > 0$ such that

$$\|\dot{u}\| \leq C_0 + C_1\|e\|. \tag{2.8}$$

By using (2.1) again, we obtain that there exists $M > 0$ independent of u such that

$$\|u\|_H \leq \sqrt{\left(\frac{T}{2\pi}\right)^2+1}\,\|\dot{u}\| \leq M \quad \text{for all } u \in S_1,$$

this completes the proof. □

We note that M depends on $\|e\|$ as shown in (2.8). We next see that the following lemma holds.

Lemma 2.4. *The set*

$$S_2 = \left\{u \in \widetilde{H} : \ddot{u} + \zeta f(u)\dot{u} + \eta_0 Au = \zeta e(t) \quad \textit{for some } \zeta \in [0,1]\right\}$$

is bounded in $\widetilde{H}$.

Proof. The proof is a slight modification of that of Lemma 2.3. In fact, by replacing f and e with ζf and ζe respectively, we can see that the assertion holds. □

Proof of Theorem 2.1. Let $B_r(a)$ be an open ball in $\widetilde{H}$ centered at a with radius $r > 0$. Let $\mathcal{L} : \widetilde{H} \to \widetilde{H}$ be the operator defined by the inverse of the mapping $u \mapsto -\ddot{u}$. Then we can write a solution u of (V) as

$$u = -\mathcal{L}\ddot{u} = \mathcal{L}\left[f(u)\dot{u} + Au - e(t)\right].$$

Here we define mappings $\mathcal{T}(\eta,\zeta) : \widetilde{H} \to \widetilde{H}$ by

$$\mathcal{T}(\eta,\zeta)u = \mathcal{L}\left[\zeta f(u)\dot{u} + \eta Au - \zeta e(t)\right]$$

for each $\eta \in [0,1]$ and each $\zeta \in [0,1]$. One can see from the definition of $\mathcal{L}$ that the mappings $u \mapsto \mathcal{L}f(u)\dot{u}$ and $u \mapsto \mathcal{L}Au$ are compact. Then it follows that $\mathcal{T}(\eta,\zeta)$ is a compact operator. Here we choose $\eta_0 \in (0,1)$ such that

$$\eta_0\|A\|\left(\frac{T}{2\pi}\right)^2 < 1. \tag{2.9}$$

Then by putting

$$\mathcal{H}(\xi)u = \begin{cases} \mathcal{T}(1-3(1-\eta_0)\xi, 1)u & \text{for } \xi \in [0,\frac{1}{3}] \text{ and } u \in \widetilde{H}, \\ \mathcal{T}(\eta_0, 2-3\xi)u & \text{for } \xi \in [\frac{1}{3},\frac{2}{3}] \text{ and } u \in \widetilde{H}, \\ \mathcal{T}(3\eta_0(1-\xi), 0)u & \text{for } \xi \in [\frac{2}{3},1] \text{ and } u \in \widetilde{H}, \end{cases}$$

we can rewrite S_1 and S_2 as

$$S_1 = \left\{u \in \widetilde{H} : u = \mathcal{H}(\xi)u \quad \text{for some } \xi \in [0,\tfrac{1}{3}]\right\}$$

and

$$S_2 = \left\{ u \in \widetilde{H} : u = \mathcal{H}(\xi)u \quad \text{for some } \xi \in [\tfrac{1}{3}, \tfrac{2}{3}] \right\}$$

respectively. It is obvious from the defiinition that $\mathcal{H}(\xi)$ is a homotopy of compact mappings on $\widetilde{H}$. It also follows from the definition of $\mathcal{H}(\xi)$ that u is a fixed points of $\mathcal{H}(0)$ if and only if u is a solution of (V). Now we have that, as claimed in Lemma 2.3 and 2.4, there exists large $M_0 > 0$ such that

$$u \neq \mathcal{H}(\xi)u \qquad \text{for any } u \in \partial B_{M_0}(0) \text{ and any } \xi \in [0, \tfrac{2}{3}].$$

For $\xi \in [\frac{2}{3}, 1]$, we can see that each fixed point $u \in \widetilde{H}$ of $\mathcal{H}(\xi)$ satisfies

$$u = (3\eta_0(1-\xi))\mathcal{L}Au.$$

That is u satisfies

$$\ddot{u} + (3\eta_0(1-\xi))Au = 0. \tag{2.10}$$

Multiplying (2.10) by u and integrating over $[0, T]$, we have $\|\dot{u}\|^2 \leq 3\eta_0(1-\xi)\|A\|\|u\|^2$. Then if $\|\dot{u}\| \neq 0$, we have by (2.1) and (2.9) that

$$\|\dot{u}\|^2 \leq 3\eta_0(1-\xi)\|A\|\|u\|^2 \leq \eta_0\|A\| \left(\frac{T}{2\pi}\right)^2 \|\dot{u}\|^2 < \|\dot{u}\|^2.$$

This is a contradiction. Thus we find that (2.10) has no nontrivial solution for each $\xi \in [\frac{2}{3}, 1]$. This means that

$$u \neq \mathcal{H}(\xi)u \qquad \text{for any } u \in \partial B_{M_0}(0) \text{ and any } \xi \in [\tfrac{2}{3}, 1].$$

Therefore we have that $\deg(I - \mathcal{H}(\xi), B_{M_0}(0), 0)$ is well defined for $\xi \in [0, 1]$ on $B_{M_0}(0)$. As mentioned above, since $u \neq \mathcal{H}(\xi)u$ for any $u \in \partial B_{M_0}(0)$ and any $\xi \in [0, 1]$, we have by homotopy invariance that

$$\begin{aligned} \deg(I - \mathcal{H}(0), B_{M_0}(0), 0) &= \deg(I - \mathcal{H}(1), B_{M_0}(0), 0) \\ &= \deg(I, B_{M_0}(0), 0) \\ &= 1. \end{aligned}$$

Then we obtain that $\mathcal{H}(0)$ has at least one fixed point in $\widetilde{H}$, and Theorem 2.1 is proved. □

3. Repellor

In the previous section, we showed existence of periodic solutions of (V) with period T. In this section, we assume that f satisfies (F) and

$$f_i(0) < 0 \qquad \text{for each } 1 \leq i \leq N \tag{F'}$$

and we will show that if the forcing term is sufficiently small, then (V) has an unstable periodic solution near the origin. To state our result precisely, we need some notations and definitions. For $x \in \mathbb{R}^N \times \mathbb{R}^N$ and $\mathcal{A} \subset \mathbb{R}^N \times \mathbb{R}^N$, we denote by $\mathrm{d}(x, \mathcal{A})$ the distance of x from $\mathcal{A}$. Let $u(t)$ be a periodic solution with period T of (V), and let $v(t)$ be a solution of initial value problem

$$\begin{cases} \ddot{v} + f(v)\dot{v} + Au = e(t) & t \in \mathbb{R}^N, \\ (v(0), \dot{v}(0)) = (v_0, \dot{v}_0) \in \mathbb{R}^N \times \mathbb{R}^N. \end{cases} \tag{3.1}$$

If there exists a neighborhood U of $\Gamma = \{(u(t), \dot{u}(t)) : t \in [0, T]\}$ such that for each $(v_0, \dot{v}_0) \in U$, the solution $v(t)$ of problem (3.1) satisfies

$$\lim_{t\to-\infty} \mathrm{d}\left(\left(v(t), \dot{v}(t)\right), \Gamma\right) = 0,$$

then u is said to be repellor. If there exists a neighborhood U of $\Gamma = \{(u(t), \dot{u}(t)) : t \in [0, T]\}$ such that for each $(v_0, \dot{v}_0) \in U$, the solution $v(t)$ of problem (3.1) satisfies

$$\lim_{t\to+\infty} \mathrm{d}\left(\left(v(t), \dot{v}(t)\right), \Gamma\right) = 0,$$

then u is said to be attractor.

Proposition 3.1. *Suppose that $f_i \in C^1(\mathbb{R};\mathbb{R})$ satisfies* (F) *and* (F') *for each i. Suppose in addition that the following condition holds:*

(A') *A is a positive definite Hermitean matrix.*

Then there exists $\varepsilon > 0$ such that for each $e \in \widetilde{H} \setminus \{0\}$ with $\|e\| < \varepsilon$, (V) *has a T-periodic solution which is a repellor.*

To prove Proposition 3.1, we need the following lemma. Let $\delta > 0$ such that $\kappa = \max\{f_i(s) : |s| \le \delta, i = 1, \cdots, N\} < 0$. Then we can choose $\rho > 0$ so that for each $u \in \overline{B_\rho(0)} \subset \widetilde{H}$, $\sup\{|u_i(t)| : t \in [0, T], i = 1, \cdots, N\} \le \delta$.

Lemma 3.2. *There exists $\varepsilon > 0$ such that for any $\eta \in [0, 1]$ and each $e \in \widetilde{H} \setminus \{0\}$ with $\|e\| < \varepsilon$, problem*

$$\ddot{u} + f(u)\dot{u} + \eta A u = e(t) \tag{2.2}$$

has no solution u in $\partial B_\rho(0)$.

Proof. Fix $\eta \in [0, 1]$ and let $u \in \overline{B_\rho(0)}$ be a solution of (2.2). Multiplying (2.2) by $\dot{u}$ and integrating over $[0, T]$, we find, by noting that A is a Hermitean matrix, that

$$\langle\langle f(u)\dot{u}, \dot{u}\rangle\rangle = \langle\langle e, \dot{u}\rangle\rangle.$$

Since $|f_i(u_i(t))| \ge |\kappa|$ for every i and any $t \in [0, T]$, we have

$$|\kappa|\|\dot{u}\|^2 \le |\langle\langle f(u)\dot{u}, \dot{u}\rangle\rangle| = |\langle\langle e, \dot{u}\rangle\rangle| \le \|e\|\|\dot{u}\|,$$

hence $\|\dot{u}\| \le |\kappa|^{-1}\|e\|$. Thus we obtain that if $\|e\|$ is sufficiently small, then $\|u\|_H < \rho$. Therefore we can choose $\varepsilon > 0$ satisfying the assertion. □

Proof of Proposition 3.1. First, we shall compute $\deg(I - \mathcal{H}(0), B_\rho(0), 0)$ in order to show the existence of the T-periodic solutions near the origin of $\widetilde{H}$, where $\mathcal{H}$ is the compact operator defined in the proof of Theorem 2.1. Let $e \in \widetilde{H} \setminus \{0\}$ with $\|e\| < \varepsilon$. We define a set $\{\mathcal{K}(\mu, \nu) : \mu, \nu \in [0, 1]\}$ of compact mappings by

$$\mathcal{K}(\mu, \nu)u = \mathcal{L}\left[\nu f(u)\dot{u} + (\eta_0 + (1 - \eta_0)\mu)Au - \mu e(t)\right]$$

for each $\mu \in [0, 1]$ and each $\nu \in [0, 1]$, where $\eta_0 \in (0, 1)$ is the constant satisfying (2.9). We note that $\mathcal{K}(1, 1) = \mathcal{H}(0)$ and $\mathcal{K}(0, 0) = \mathcal{H}(\frac{2}{3})$. We also have from Lemma 3.2 that $\mathcal{K}(\mu, 1)u \ne u$ for $u \in \partial B_\rho(0)$ and $\mu \in [0, 1]$. Therefore we find that

$$\deg(I - \mathcal{K}(\mu, 1), B_\rho(0), 0) = \deg(I - \mathcal{K}(0, 1), B_\rho(0), 0) \quad \text{for any } \mu \in [0, 1].$$

Let $u \in \overline{B_\rho(0)}$ satisfy $u = \mathcal{K}(0,\nu)u$ for some $\nu \in [0,1]$. That is

$$\ddot{u} + \nu f(u)\dot{u} + \eta_0 Au = 0. \tag{3.2}$$

Multiplying the equation above by u and integrating over $[0,T]$, we find

$$\|\dot{u}\|^2 = \eta_0 \langle\!\langle Au, u\rangle\!\rangle \leq \eta_0 \|A\|\|u\|^2 \leq \eta_0 \left(\frac{T}{2\pi}\right)^2 \|A\|\|\dot{u}\|^2.$$

Then again by the definition of η_0, we find that $u = 0$. Thus $u = 0$ is the unique solution of (3.2), and this implies that

$$\deg(I - \mathcal{K}(0,\nu), B_\rho(0), 0) = \deg(I - \mathcal{K}(0,0), B_\rho(0), 0) \quad \text{for any } \nu \in [0,1].$$

It follows from the same argument that $u \neq \mathcal{H}(\xi)u$ for any $u \in \partial B_\rho(0)$ and any $\xi \in [\frac{2}{3}, 1]$. Therefore, we obtain

$$\deg(I - \mathcal{H}(0), B_\rho(0), 0) = \deg(I - \mathcal{H}(1), B_\rho(0), 0) = 1.$$

This implies that problem (V) has a T-periodic solution $u \in B_\rho(0)$ for each $e \in \widetilde{H} \setminus \{0\}$ with $\|e\| < \varepsilon$.

Next, we show that each solution $u \in B_\rho(0)$ is a repellor. The initial value problem (3.1) can be rewirtten as a system of equations

$$\begin{cases} \dot{u} = v \\ \dot{v} = -f(u)v - Au + e \end{cases} \tag{3.3}$$

with $(u(0), v(0)) = (u_0, v_0) \in \mathbb{R}^N \times \mathbb{R}^N$. Then the linearized equation of (3.3) is given by the form

$$\begin{pmatrix} \dot{\Phi} \\ \dot{\Psi} \end{pmatrix} = \begin{pmatrix} 0 & I_N \\ -f'(u)\dot{u} - A & -f(u) \end{pmatrix} \begin{pmatrix} \Phi \\ \Psi \end{pmatrix} \stackrel{\text{def}}{=} \mathcal{M}(u) \begin{pmatrix} \Phi \\ \Psi \end{pmatrix}, \tag{S}$$

To prove that each solution $u \in B_\rho(0)$ is repellor, it is sufficient to show that $|\Phi_i(t)| \to \infty$ and $|\Psi_i(t)| \to \infty$ as $t \to \infty$. We denote by $\Theta(t)$ a fundamental matrix of (S). Then by Floquet's theorem (e.g., see [6]), we have that $\Theta(t)$ can be written by the form

$$\Theta(t) = Q(t) \exp(Rt), \tag{P}$$

where $Q = \{q_{kl}\}$ is a matrix such that $q_{kl}(t)$ is a continuous T-periodic function for each $k, l = 1, \cdots, 2N$, and R is a $2N \times 2N$ constant matrix. Noting that $f_i(0) < 0$ for each i, we have that the real part of each eigenvalue of the matrix

$$\begin{pmatrix} 0 & I_N \\ -A & -f(0) \end{pmatrix}$$

is positive. In fact, each eigenvalue λ of the matrix above satisfies

$$\lambda(\lambda + f_i(0_i)) = -\alpha_i \quad \text{for each } i,$$

where α_i is an eigenvalue of A, and then the real part of λ is positive. Recalling that $u \in B_\rho(0)$, we can choose $\rho > 0$ sufficiently small that the positive part of each eigenvalue of R is positive. This implies that $|\Phi_i(t)| \to \infty$ and $|\Psi_i(t)| \to \infty$ as $t \to \infty$ for every i. This completes the proof. □

4. Discussion

In this section, we give an example of a coupled Van der Pol oscillator with a forcing term and illustrate the results of its numerical simulations.

Example 4.1. *Consider the 2-dimensional coupled model of Van der Pol oscillator*

$$\begin{aligned} \ddot{x} + \left(x^2 - 1\right)\dot{x} + x + 0.3y &= \beta_1 \cos t, \\ \ddot{y} + \left(y^4 + y^3 - 2y^2 - 1\right)\dot{y} + 0.3x + y &= \beta_2 \cos\left(t + \frac{\pi}{2}\right). \end{aligned} \tag{4.1}$$

Putting $e_1(t) = \beta_1 \cos t$, $e_2(t) = \beta_2 \cos\left(t + \frac{\pi}{2}\right)$,

$$f(x,y) = \begin{pmatrix} x^2 - 1 & 0 \\ 0 & y^4 + y^3 - 2y^2 - 1 \end{pmatrix}, \quad \text{and} \quad A = \begin{pmatrix} 1 & 0.3 \\ 0.3 & 1 \end{pmatrix},$$

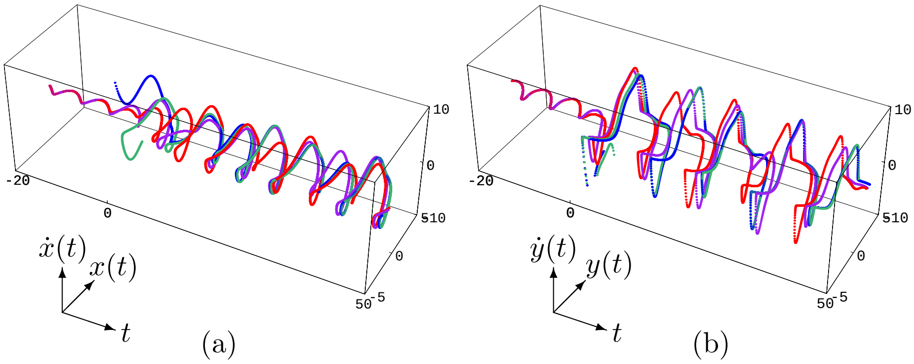
one can see that (4.1) satisfies $e = (e_1, e_2) \in \widetilde{H} \setminus \{0\}$, condition (F) and (A'). It is certain from Theorem 3.1 that (4.1) has a repellent periodic solution with period 2π for sufficiently small $\|e\|$. We show the results of numerical simulations for two cases where one is β_1 and β_2 are small and another is β_1 and β_2 are large. Both in Case 1 and in Case 2, setting four initial values at $(t_0, x(t_0), y(t_0), \dot{x}(t_0), \dot{y}(t_0)) = (0,0,0,0,0)$, $(0,0,0,-1,-1)$, $(0,-2,-1,-5,-10)$ and $(0,-2,-2.5,5,-10)$ reasonably, we have the following results. Here, let us call a orbit of a solution for $t > 0$ and for $t < 0$ by positive orbit and negative orbit respectively. If we can find a closed invariant set $(x(t), y(t), \dot{x}(t), \dot{y}(t))$ with respect to t in a negative orbit, the solution is repellent. By contrast, if we can find a closed invariant set in a positive orbit, the solution is attractive.

Case 1. $\beta_1 = 0.7$ *and* $\beta_2 = 1$.

We first give an instance of a repellent periodic solution with period 2π. Figure 1 (a) and (b) show the time series $(t, x(t), \dot{x}(t))$ and $(t, y(t), \dot{y}(t))$ for $t \in [-20, 50]$ respectively. One can see on Figure 1 that the two negative orbits synchronize near the origin with the amplitude small. Figure 2 (a) and (b) plot out the projections $(x(t), \dot{x}(t))$ and $(y(t), \dot{y}(t))$ of the invariant set of the negative orbits for sufficiently large $-t$ respectively. This means that the closed orbit is a repellor. In case that $\beta_1 = 0$ and $\beta_2 = 0$, $(x(t), y(t)) = (0,0)$ is clearly a trivial and unstable solution of (4.1). We can regard a repellent periodic solution as a small perturbation of this trivial solution. Figure 3 are drawn for the purpose of making sure that the period of the orbit is 2π, where (a) and (b) are the plots of $(t \bmod 2\pi, x(t))$ and $(t \bmod 2\pi, y(t))$ for sufficiently large $-t$ respectively.

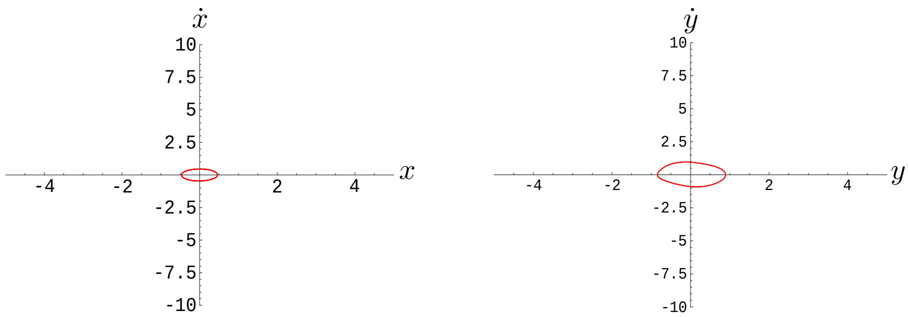
Case 2. $\beta_1 = 7$ *and* $\beta_2 = 3$.

We next give an instance of an attractive periodic solution with period 2π. In this case, we can find the invariant set only for the positive orbits of (4.1). Figure 4 (a) and (b) show the projections $(x(t), \dot{x}(t))$ and $(y(t), \dot{y}(t))$ of the invariant set respectively. By plotting out $(t \bmod 2\pi, x(t))$ and $(t \bmod 2\pi, y(t))$ for sufficiently large t as Figure 3, we have that the period of the attractor is 2π. One can see on Figure 4 that (a) is symmetrical with respect to the origin but (b) is not. This is caused by the symmetry of f_1 and the asymmetry of f_2. In other words, a distant orbit from the origin depends on the symmetry of f_i on each dimension.



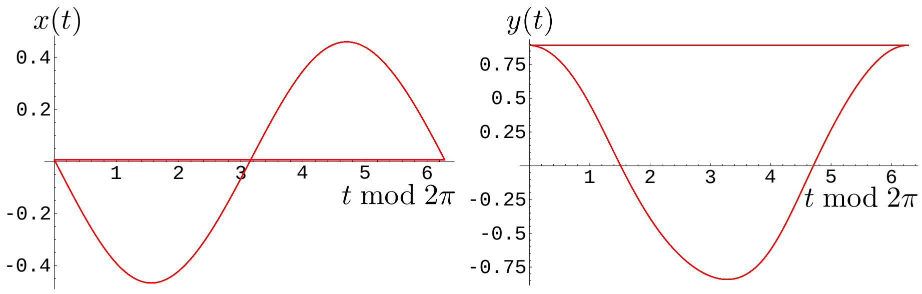
(a) (b)

FIGURE 1



(a) (b)

FIGURE 2



(a) (b)

FIGURE 3

Moreover, changing parameters on (4.1), we make a guess at the following remarks.

Remark 4.2. *In Theorem 3.1, condition* (A') *is reducible to*

(A") *A is a positive definite matrix i.e.* $\langle Ax, x\rangle > 0$ *for* $x \in \mathbb{R}^N \setminus \{0\}$.

Remark 4.3. *Under conditions* (F) *and* (A"), (V) *has an unique and attractive* T*-periodic solution for sufficiently large* $e \in \widetilde{H} \setminus \{0\}$.

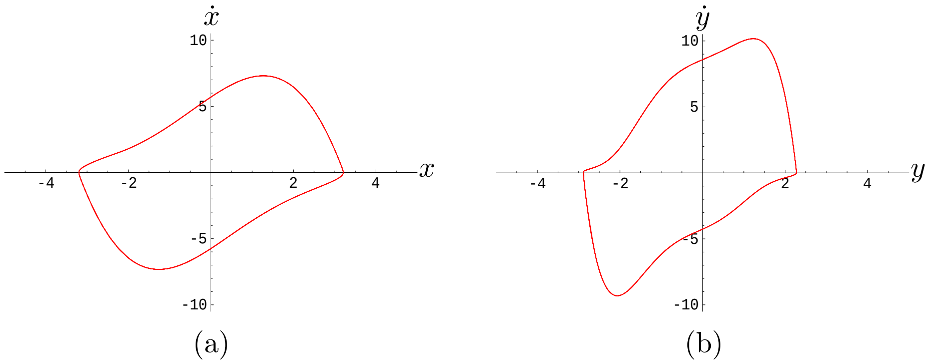


FIGURE 4

It is suggested about the remarks above that these proofs is more or less complicated for lack of the symmetry of A. Furthermore, it is also well-known that (V) has a subharmonic solution with odd number times T for medium $\|e\|$, and then the line symmetry of f_i may be required for each i. We expect to argue such problems in the forthcoming paper.

REFERENCES

1. Nguyen Phuong Các, *Periodic solutions of a liénard equation with forcing term*, Nonlinear Anal. **43** (2001), no. 3, Ser. A: Theory Methods, 403–415.
2. M. L. Cartwright and J. E. Littlewood, *Some fixed point theorems*, Ann. of Math. **54** (1951), no. 1, 1–37.
3. J. E. Flaherty and F. C. Hoppensteadt, *Frequency entrainment of a forced van der pol oscillator*, Studies in Appl. Math. **58** (1978), no. 1, 5–15.
4. John Guckenheimer and Philip Holmes, *Nonlinear oscillations, dynamical systems, and bifurcations of vector fields*, Applied Mathematical Sciences, vol. 42, Springer-Verlag, 1983.
5. Chaitan P. Gupta, Juan J. Nieto, and Luis Sanchez, *Periodic solutions of some lienard and duffing equations*, J. Math. Anal. Appl. **140** (1989), no. 1, 67–82.
6. Frank C. Hoppensteadt, *Analysis and simulation of chaotic systems*, second ed., Applied Mathematical Sciences, vol. 94, Springer-Verlag, 2000.
7. Norman Levinson, *A second order differential equation with singular solutions*, Ann. of Math. **50** (1949), no. 1, 127–153.
8. N. G. Lloyd, *Degree theory*, Cambridge University Press, 1978.
9. Jean Mawhin, *An extension of a theorem of a.c.lazer on forced nonlinear oscillations*, J. Math. Anal. Appl. **40** (1972), 20–29.
10. Duane W. Storti and Per G. Reinhall, *Stability of in-phase and out-of-phase modes for a pair of linearly coupled van der Pol oscillators*, Nonlinear dynamics, Ser. Stab. Vib. Control Syst. Ser. B, vol. 2, World Sci. Publishing, 1997, pp. 1–23.
11. Fabio Zanolin, *Remarks on multiple periodic solutions for nonlinear ordinary differential systems of Liénard type*, Boll. Un. Mat. Ital. B (6) **1** (1982), no. 2, 683–698. MR **83i:**34048

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