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APPLICATION OF SPLITTING METHODS TO A CLASS OF EQUILIBRIUM PROBLEMS

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Abstract. We consider a system of equilibrium type problems which can be viewed as an extension of general primal-dual equilibrium problems. This problem has a great number of applications including inverse optimization problems, network equilibrium problems and economic equilibrium problems. We suggest this problem to be solved with the help of the splitting method with respect to the dual variables and give its specializations for the applications above.

1. Introduction

Let Ω and X be nonempty, closed and convex sets in a real n-dimensional Euclidean space R^n such that $\Omega \subseteq X$ and let $\Phi : X \times X \to R$ be an equilibrium bifunction, i.e., $\Phi(x, x) = 0$ for all $x \in X$. Then one can define the *equilibrium problem* (EP for short) which is to find an element $x^* \in \Omega$ such that

$$\Phi(x^*, x) \geq 0 \quad \forall x \in \Omega. \tag{1}$$

EP is known to represent rather a common and suitable format for various problems arising in Economics, Mathematical Physics, Operations Research and other fields. Besides, it is closely related with other general problems in Nonlinear Analysis, such as fixed point problems, saddle point problems and variational inequalities; see e.g. [1]–[4].

In this paper, we shall consider a somewhat more general formulation of EP which is based upon the inverse EP proposed in [5]. Namely, the problem is to find a pair $(x^*, y^*) \in X \times Y$ such that

$$\Phi(x^*, x) + \langle y^*, H(x) - H(x^*)\rangle \geq 0 \quad \forall x \in X, \tag{2}$$

$$\langle G(y^*) - H(x^*), y - y^*\rangle \geq 0 \quad \forall y \in Y; \tag{3}$$

where Y is a nonempty, closed and convex subset of $R^m_+ = \{y \in R^m \mid y_i \geq 0 \quad i = 1, \dots, m\}$, $G : Y \to R^m$ and $H : X \to R^m$ are continuous mappings with $H_i : X \to R$ being convex for each $i = 1, \dots, m$. It is easy to see that a particular case of problem (2), (3) with $Y = R^m_+$, $G(y) \equiv b \in R^m$ is nothing but an analogue of the Karush-Kuhn-Tucker saddle point optimality conditions for EP (1) where

$$\Omega = \{x \in X \mid H(x) \leq b\}.$$

However, we shall consider the more general case where the right-hand side of constraints depends on the optimal value of the dual variables, hence our system (2), (3) in fact belongs to inverse problems. Our motivation for studying just this very general problem stems from the fact that its format is very suitable for formulation

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of many equilibrium type problems arising in complex systems. In this paper, we shall give examples of such applications in optimization, network equilibrium and economic equilibrium. Our main goal is to suggest an effective iteration method to find solutions of these problems.

Since system (2), (3) is an extension of the Lagrangian saddle point problem in constrained optimization, it is natural to apply some extensions of the known multiplier methods to this problem. Note that primal-dual methods do not seem too useful here because of their additional storage requirements and of difficulties related to usual essential difference in properties of primal and dual variables. So, we turn to the dual methods. The most popular of them is the Uzawa method [6], which is the gradient projection method for the dual optimization problem. However, to obtain convergence of its extension applied to system (2), (3), we have to impose additional strong monotonicity assumptions on the perturbation mapping G. Usually, this mapping is rather simple in the sense that it admits implicit iterations, but it may not be even strictly monotone in general. For this reason, we propose to solve system (2), (3) with the help of the dual iterative method of a splitting type, which enables us to weaken conditions on the perturbation mapping. We specialize the convergence result for the case of system (2), (3) under rather mild assumptions and describe applications of this method to some classes of inverse optimization problems, network equilibrium problems, and economic equilibrium problems.

2. Dual Variational Inequality

In this section, we replace the initial system (2), (3) with a variational inequality problem with respect to the dual variables, which involves a sum of two monotone mappings. First we recall some definitions of monotonicity properties for mappings and bifunctions; see e.g. [2, 7, 4].

Let V be a nonempty convex subset of a finite-dimensional space E. A mapping $Q : V \to E$ is said to be

(i) *monotone*, if, for all $u', u'' \in V$, we have

$$\langle Q(u') - Q(u''), u' - u'' \rangle \geq 0;$$

(ii) *strongly monotone* with modulus $\tau' > 0$, if, for all $u', u'' \in V$, we have

$$\langle Q(u') - Q(u''), u' - u'' \rangle \geq \tau' \left\| u' - u'' \right\|^2;$$

(iii) *co-coercive* with modulus $\tau'' > 0$, if, for all $u', u'' \in V$, we have

$$\langle Q(u') - Q(u''), u' - u'' \rangle \geq \tau'' \left\| Q(u') - Q(u'') \right\|^2.$$

Note that each strongly monotone or co-coercive mapping is monotone, each co-coercive mapping with modulus τ'' is Lipschitz continuous with constant $1/\tau''$, and each strongly monotone and Lipschitz continuous mapping is co-coercive, but the reverse assertions are not true in general. Clearly, co-coercivity is equivalent to strong monotonicity of the inverse mapping.

Next, an equilibrium bifunction $h : V \times V \to R$ is said to be

(i) *monotone*, if, for all $u', u'' \in V$, we have

$$h(u', u'') + h(u'', u') \leq 0;$$

(ii) *strongly monotone* with modulus $\tau' > 0$, if, for all $u', u'' \in V$, we have

$$h(u', u'') + h(u'', u') \leq -\tau' \left\| u' - u'' \right\|^2.$$

It is easy to see that in the case where

$$h(u, v) = \langle Q(u), v - u \rangle,$$

the bifunction h is monotone (strongly monotone with modulus τ') if and only if so is Q.

We now show that the system (2), (3) can be in principle regarded as optimality conditions for a quasi-equilibrium problem.

Proposition 2.1. *If a pair $(x^*, y^*) \in X \times Y$ solves system (2), (3) with $Y = R^m_+$, then x^* solves the quasi-equilibrium problem: Find $x^* \in \Omega(y^*)$ such that*

$$\Phi(x^*, x) \geq 0 \quad \forall x \in \Omega(y^*), \tag{4}$$

where

$$\Omega(y^*) = \{x \in X \mid H(x) \leq G(y^*)\}. \tag{5}$$

Proof. From (3) with $Y = R^m_+$ it follows that

$$G(y^*) - H(x^*) \geq 0,$$

i.e. $x^* \in \Omega(y^*)$. Next, the above inequality together with (3) yields

$$\langle y^*, G(y^*) - H(y^*) \rangle = 0.$$

Choose any point $x \in \Omega(y^*)$, then (2) and the above equality give

$$\begin{aligned} 0 &\leq \Phi(x^*, x) + \langle y^*, H(x) - H(x^*) \rangle \\ &= \Phi(x^*, x) + \langle y^*, H(x) - G(y^*) \rangle - \langle y^*, H(x^*) - G(y^*) \rangle \\ &= \Phi(x^*, x) + \langle y^*, H(x) - G(y^*) \rangle \leq \Phi(x^*, x), \end{aligned}$$

i.e. x^* solves (4), (5). The proof is complete. □

The quasi-equilibrium problem (4), (5) is a particular case of the inverse EP from [5], where a primal-dual approach was suggested, but here we consider the dual approach to find its solutions. To this end, we introduce the blanket assumptions of this paper.

(A1) *X is a nonempty, convex and closed subset of R^n, Y is a nonempty, convex and closed subset of R^m_+.*

(A2) *$\Phi : X \times X \to R$ is a continuous equilibrium bifunction, such that $\Phi(x, \cdot)$ is convex for each $x \in X$.*

(A3) *$H : X \to R^m$ is a Lipschitz continuous mapping with modulus L_H such that each component $H_i : X \to R$ is convex for $i = 1, \dots, m$.*

(A4) *$G : R^m_+ \to R^m$ is a monotone continuous mapping.*

(A5) *$\Phi : X \times X \to R$ is strongly monotone with modulus $\tau > 0$.*

Given a point $\tilde{y} \in Y$, one can define the sets

$$X(\tilde{y}) = \{\tilde{x} \in X \mid \Phi(\tilde{x}, x) + \langle \tilde{y}, H(x) - H(\tilde{x}) \rangle \geq 0 \quad \forall x \in X\} \tag{6}$$

and

$$F(\tilde{y}) = \{f \in R^m \mid f = -H(\tilde{x}), \tilde{x} \in X(\tilde{y})\}. \tag{7}$$

Proposition 2.2. *Suppose that* **(A1)** *and* **(A2)** *hold and* $X(y) \neq \emptyset$ *for every* $y \in Y$*. Then, for all* $y', y'' \in Y$*, we have*

$$\langle f' - f'', y' - y'' \rangle \geq -\left[\Phi(x', x'') + \Phi(x'', x')\right], \tag{8}$$

where $x' \in X(y')$*,* $x'' \in X(y'')$ *and* $f' = -H(x')$*,* $f'' = -H(x'')$*.*

Proof. By definition, we have

$$\Phi(x', x'') + \langle y', H(x'') - H(x') \rangle \geq 0$$

and

$$\Phi(x'', x') + \langle y'', H(x') - H(x'') \rangle \geq 0.$$

Adding these inequalities yields

$$\Phi(x', x'') + \Phi(x'', x') \geq \langle y'' - y', H(x'') - H(x') \rangle,$$

i.e. (8) holds, as desired. □

Inequality (8) enables us to obtain certain monotonicity properties of the mapping F, defined by (7). In fact, if Φ is monotone, then so is F, however, F may be set-valued in general. We now consider somewhat strengthened conditions on Φ and H.

Proposition 2.3. *Suppose* **(A1)** – **(A3)** *and* **(A5)** *hold. Then the mappings* X *and* F*, defined by (6) and (7), are single-valued and have nonempty values on* Y*. Besides,* $F : Y \to R^m$ *is co-coercive with modulus* $\gamma = \tau / L_H^2$*.*

Proof. The fact that $X(y)$ is a nonempty singleton for each $y \in Y$ is rather standard; see e.g. [4, Proposition 2.1.16]. Hence, so is $F(y)$. Using (8), we obtain

$$\begin{aligned}\langle F(y') - F(y''), y' - y'' \rangle &\geq \tau \left\| x' - x'' \right\|^2 \\ &\geq (\tau / L_H^2) \left\| H(x') - H(x'') \right\|^2 = \gamma \left\| F(x') - F(x'') \right\|^2,\end{aligned}$$

i.e. F is co-coercive with the prescribed constant γ. The proof is complete. □

It is easy to see that the mapping X, defined by (6), is an analogue of the dual mapping in constrained optimization, so that the assertions of Propositions 2.2 and 2.3 represent some extensions of those in optimization and variational inequalities; see e.g. [8, Chapter 7] and [9] and references therein.

We are now ready to define the *dual variational inequality* of the system (2), (3), which is to find a point $y^* \in Y$ such that

$$\langle G(y^*) + F(y^*), y - y^* \rangle \geq 0 \quad \forall y \in Y. \tag{9}$$

Proposition 2.4. *Suppose* **(A1)** – **(A3)** *and* **(A5)** *hold.*

(i) If (x^*, y^*) *solves (2), (3), then* y^* *solves (9).*

(ii) If y^* *solves (9) and* $x^* = X(y^*)$*, then* (x^*, y^*) *solves the system (2), (3).*

Proof. Assertion (i) follows from the definition of the mapping F in (7). Conversely, if y^* solves (9), then due to Proposition 2.3 there exists the unique point $x^* = X(y^*)$ which clearly satisfies (2), i.e., (x^*, y^*) solves the system (2), (3). □

Thus, we can replace the system (2), (3) with the variational inequality (9) whose cost mapping will possess certain monotonicity assumptions under **(A1)** – **(A5)**.

3. Dual Splitting Method

There exist a great number of various iterative methods for solving variational inequalities; see e.g. [7, 4]. Nevertheless, taking into account **(A4)** and Proposition 2.3, we can suppose that the splitting method by Lions and Mercier from [10], which is explicit with respect to F and implicit with respect to G, can be a reasonable choice for problem (9).

Given a starting point $y^0 \in Y$, the splitting method, applied to (9), consists in sequential finding for each $k = 0, 1, \ldots,$ a solution $y^{k+1} \in Y$ of the problem

$$\langle G(y^{k+1}) + \frac{1}{\gamma_k}(y^{k+1} - y^k) + F(y^k), y - y^{k+1} \rangle \geq 0 \quad \forall y \in Y, \tag{10}$$

where y^k is a current iterate, $\gamma_k > 0$ is a chosen stepsize parameter. Note that the cost mapping in (10) is continuous and strongly monotone because of **(A4)**, hence (10) always has a unique solution. Set

$$N(Y, y) = \{q \in R^m \mid \langle q, z - y \rangle \leq 0 \quad \forall z \in Y\}$$

and

$$N_+(y) = \begin{cases} N(Y, y) & \text{if} \quad y \in Y, \\ \emptyset & \text{if} \quad y \notin Y. \end{cases}$$

It is well known that $N_+(y)$ is the subdifferential of the indicator function of the set Y. Then, problem (9) can be rewritten equivalently as the inclusion

$$0 \in F(y^*) + \tilde{G}(y^*),$$

where $\tilde{G}(y) = G(y) + N_+(y)$, and (10) is equivalent to

$$0 \in F(y^k) + \tilde{G}(y^{k+1}) + \frac{1}{\gamma_k}(y^{k+1} - y^k), \tag{11}$$

which is the classical form of the splitting method. The convergence results for this method were obtained by Gabay [11] and Tseng [12].

We first give an auxiliary result of continuity properties of the mapping X in (6).

Lemma 3.1. *Suppose that* **(A1)** – **(A3)** *and* **(A5)** *hold. Then the mapping X, defined by (6), is Lipschitz continuous with constant L_H/τ.*

Proof. Fix y' and y'' in Y. From Proposition 2.3 it follows that X has nonempty values and that it is single-valued, hence we can set $x' = X(y')$ and $x'' = X(y'')$. By definition, we have

$$\Phi(x', x'') + \langle y', H(x'') - H(x') \rangle \geq 0$$

and

$$\Phi(x'', x') + \langle y'', H(x') - H(x'') \rangle \geq 0.$$

Adding these inequalities yields

$$\langle y' - y'', H(x'') - H(x') \rangle \geq - \left[\Phi(x'', x') + \Phi(x', x'') \right].$$

Using now **(A3)** and **(A5)**, we obtain

$$L_H \left\| y' - y'' \right\| \left\| x'' - x' \right\| \geq \tau \left\| x' - x'' \right\|^2,$$

i.e.,

$$\|x' - x''\| \leq (L_H/\tau) \|y' - y''\|,$$

and the result follows. □

We are now ready to present a convergence result for method (10).

Theorem 3.1. *Let assumptions* **(A1)** – **(A5)** *hold and let the system (2), (3) be solvable. If a sequence* $\{y^k\}$ *is constructed by method (10), where*

$$0 < \gamma' \leq \gamma_k \leq \gamma'' < 2\tau/L_H^2, \tag{12}$$

then it converges to a solution y^* *of the problem (9). Besides, the sequence* $\{x^k\}$*, generated by the rule* $x^k = X(y^k)$*, is also convergent to a point* x^**, such that* (x^*, y^*) *solves the problem (2), (3).*

Proof. From the assumptions of this theorem and from the assertions of Proposition 2.3 and Lemma 3.1, we see that the mappings F and G are continuous, F is co-coercive with modulus $\gamma = \tau/L_H^2$, and G is monotone. From Proposition 1 in [12] it now follows that the sequence $\{y^k\}$ generated in conformity with (10), (12), converges to a solution y^* of (9). Next, from Lemma 3.1, we have

$$\left\|y^* - y^k\right\| \geq (\tau/L_H) \left\|x^* - x^k\right\|,$$

where $x^* = X(y^*)$. Hence, $\{x^k\}$ converges to x^* and the result follows now from Proposition 2.4. □

The dual variational inequality (9) can be in principle solved by the explicit projection method

$$y^{k+1} = \pi_Y \left[y^k - \lambda_k(G(y^k) + F(y^k))\right], \lambda_k > 0, \tag{13}$$

where $\pi_Y[\cdot]$ denotes the projection mapping onto Y. It is well known that (13) can be rewritten equivalently as follows

$$\langle G(y^k) + \frac{1}{\lambda_k}(y^{k+1} - y^k) + F(y^k), y - y^{k+1}\rangle \geq 0 \quad \forall y \in Y$$

(cf. (10)). Iteration (13) can be viewed as an extension of the Uzawa method for system (2), (3). If the mapping G is strongly monotone with modulus τ' and Lipschitz continuous with modulus L_G and **(A1)** – **(A3)** and **(A5)** hold, then from Proposition 2.3 it follows that the mapping $G+F$ is strongly monotone with modulus τ' and Lipschitz continuous with modulus $N = L_G + L_H^2/\tau$. It is well known that the algorithm (13) generates a sequence $\{y^k\}$ which converges to a solution of (9) if for instance

$$\lambda_k = \tilde{\lambda} \in (0, 2\tau'/N^2)$$

(see e.g. [7, p.24]). However, the splitting algorithm (10) converges under essentially weaker assumptions on G and its real rate of convergence is usually better than that of (13). In fact, if G is single-valued, but "almost discontinuous" in the sense that its Lipschitz constant is too large, then, unlike (10), the explicit method (13) will have low convergence. Moreover, algorithm (10) can be easily adjusted for each specific problem under consideration. For instance, suppose that G is a gradient mapping of a convex function μ, i.e.

$$G(y) = \mu'(y).$$

Then problem (9) becomes equivalent to the *mixed variational inequality* (MVI):

$$\langle F(y^*), y - y^* \rangle + \mu(y) - \mu(y^*) \geq 0 \quad \forall y \in Y$$

(see e.g. [7, Proposition 1.3]). Similarly, we can replace (10) with the equivalent MVI:

$$\langle F(y^k) + \frac{1}{\gamma_k}(y^{k+1} - y^k), y - y^{k+1} \rangle + \mu(y) - \mu(y^{k+1}) \geq 0 \quad \forall y \in Y,$$

which in turn is equivalent to the optimization problem:

$$\min_{y \in Y} \left\{ \langle F(y^k) - \frac{1}{\gamma_k} y^k, y \rangle + \frac{1}{2\gamma_k} \|y\|^2 + \mu(y) \right\}, \tag{14}$$

with the unique solution y^{k+1} under the same assumptions, since the cost function in (14) is strongly convex. Moreover, in the separable case, problem (14) can be simplified. In fact, suppose that Y is a product set, i.e.,

$$Y = \prod_{i=1}^{m} Y_i, \quad \text{where } Y_i \subseteq R_+ \quad \text{for } i = 1, \dots, m.$$

and that G is diagonal, i.e., $G_i(y) = G_i(y_i)$ for $i = 1, \dots, m$. It follows that $G_i(y_i) = \mu_i'(y_i)$ and

$$\mu(y) = \sum_{i=1}^{m} \mu_i(y_i).$$

Many applied equilibrium models satisfy these assumptions; some additional examples will be given in the next sections. Then, problem (14) can be replaced with m one-dimensional optimization problems of the form: Find $y_i^{k+1} \in Y_i$ which solves the problem

$$\min_{\alpha \in Y_i} \quad \left\{ (F_i(y^k) - y_i^k/\gamma_k)\alpha + \alpha^2/(2\gamma_k) + \mu_i(\alpha) \right\} \tag{15}$$

for $i = 1, \dots, m$. Clearly, the problem (15) can be solved very easily with the help of well-known one-dimensional algorithms.

4. Application to Inverse Optimization Problems

In this section, we consider a particular case of the system (2), (3), where the bifunction Φ is additive. More precisely, let W be a open convex subset of R^n such that $W \supseteq X$, $\varphi : W \to R$ a convex function. Setting

$$\Phi(x, y) = \varphi(y) - \varphi(x) \tag{16}$$

in (2), we obtain the system

$$\varphi(x) + \langle y^*, H(x) \rangle \geq \varphi(x^*) + \langle y^*, H(x^*) \rangle \quad \forall x \in X, \tag{17}$$

$$\langle G(y^*) - H(x^*), y - y^* \rangle \geq 0 \quad \forall y \in Y. \tag{18}$$

From Proposition 2.1 it follows that the system (17), (18) represents the sufficient condition of optimality for the inverse optimization problem:

$$\min_{x\in\Omega(y^*)} \varphi(x), \tag{19}$$

where $\Omega(y^*)$ is defined in (5). However, in order to apply the approach above to this problem we need certain additional assumptions.

We recall that a function $\psi : V \to R$ is said to be *strongly convex* with modulus $\tau > 0$, if, for all $u', u'' \in V$ and for each $\alpha \in [0,1]$, we have

$$\psi(\alpha u' + (1-\alpha)u'') \leq \alpha\psi(u') + (1-\alpha)\psi(u'') - 0.5\tau\alpha(1-\alpha)\left\|u' - u''\right\|^2.$$

Next, if ψ is differentiable, then the strong convexity of ψ with modulus τ is equivalent to the strong monotonicity of its gradient ψ' with modulus τ.

In what follows in this section we suppose that $\varphi : W \to R$ is strongly convex with modulus τ and differentiable, and that **(A1)** and **(A4)** hold. Moreover, for simplicity, we replace **(A3)** with the following condition.

(A3') *$H : W \to R^m$ is a Lipschitz continuous mapping with constant L_H such that each component $H_i : W \to R$ is convex and differentiable for $i = 1, \dots, m$.*
Then the mapping X in (6) can be re-defined as follows

$$X(\tilde{y}) = \arg\min\{\varphi(x) + \langle \tilde{y}, H(x)\rangle \mid x \in X\} \tag{20}$$

which is a nonempty singleton for every $\tilde{y} \in Y$ under the assumptions above. However, the bifunction Φ in (16) cannot be strongly monotone, hence **(A5)** does not hold, although Φ is clearly monotone. For this reason, we replace the optimization problem (19) with its variational inequality format. It is well known that (19) is equivalent to the inverse variational inequality: Find $x^* \in \Omega(y^*)$ such that

$$\langle \varphi'(x^*), x - x^*\rangle \geq 0 \quad \forall x \in \Omega(y^*).$$

Similarly, we can replace the system (17), (18).

Proposition 4.1. *System (17), (18) is equivalent to the following system of MVIs:*

$$\langle \varphi'(x^*), x - x^*\rangle + \langle y^*, H(x) - H(x^*)\rangle \geq 0 \quad \forall x \in X, \tag{21}$$

$$\langle G(y^*) - H(x^*), y - y^*\rangle \geq 0 \quad \forall y \in Y. \tag{22}$$

Proof. It suffices to show that (17) is equivalent to (21). The necessary and sufficient condition of optimality for the optimization problem (17) is the following:

$$\langle \varphi'(x^*), x - x^*\rangle + \sum_{i=1}^{m} y_i^* \langle H_i'(x^*), x - x^*\rangle \geq 0 \quad \forall x \in X$$

However, this variational inequality is equivalent to (21); see e.g. [7, Proposition 1.3]. □

It is clear that system (21), (22) coincides with (2), (3) where

$$\Phi(x, y) = \langle \varphi'(x), y - x\rangle,$$

but the bifunction Φ is now strongly monotone with modulus τ. Therefore, we can apply the splitting algorithm (10), (12) for solving system (21), (22). In accordance with Theorem 3.1, it generates a sequence $\{(x^k, y^k)\}$ which converges to its solution, and hence, to a solution of the system (17), (18) because of Proposition 4.1. In comparison with the general case, the computation of a value of the mapping X (or F) requires a solution of the convex minimization problem in (20), which is nothing but minimization of the Lagrange function $\varphi(x) + \langle y, H(x)\rangle$ in the primal variables.

We have used the differentiability of φ and H only to simplify the exposition. The results of this section remain true in the case where φ and H are not differentiable since we can utilize the same scheme of the splitting algorithm.

5. Application to Network Equilibrium Problems

There are a great number of various formulations of network equilibrium problems; see e.g. [13] and references therein. In this section, we consider a multicommodity formulation, which is a modification of the known ones; see e.g. [14, Section 3.6.2].

The model is determined on a transportation network given by a set of nodes N and a set of arcs A. We denote by D the subset of destination nodes, $D \subseteq N$. The variable x^l_a denotes the flow on arc a with destination $l \in D$, so that we have

$$x^l = (x^l_a)_{a\in A} \quad \text{and} \quad x = (x^l)_{l\in D}.$$

Similarly, the variable t^l_i denotes the minimal cost to reach destination l from node i, so that we have

$$t^l = (t^l_i)_{i\in N} \quad \text{and} \quad t = (t^l)_{l\in D}.$$

For each pair $(l, i) \in D \times N$, d^l_i denotes the flow demand, i.e. the minimal demand for transportation from node i to node l, which may depend on t. For each $i \in N$, A^+_i and A^-_i denote the sets of outgoing and incoming arcs at i. Next, for each arc a, c_a denotes the flow cost on this arc, which is dependent of the flow vector

$$f = \sum_{l\in D} x^l.$$

The pair $(\tilde{x}, \tilde{t})$ is said to be in equilibrium if the following conditions hold:

$$\begin{array}{lll} & \tilde{x}^l_a \geq 0, & \tilde{t}^l_i \geq 0; \\ & \tilde{t}^l_j - \tilde{t}^l_i + c_a(\tilde{f}) \geq 0, & \sum\limits_{a\in A^+_i} \tilde{x}^l_a - \sum\limits_{a\in A^-_i} \tilde{x}^l_a - d^l_i(\tilde{t}) \geq 0; \\ (23) & & \\ & \tilde{x}^l_a \left[\tilde{t}^l_j - \tilde{t}^l_i + c_a(\tilde{f})\right] = 0, & \tilde{t}^l_i \left[\sum\limits_{a\in A^+_i} \tilde{x}^l_a - \sum\limits_{a\in A^-_i} \tilde{x}^l_a - d^l_i(\tilde{t})\right] = 0; \end{array}$$

for all $a = (i, j) \in A, l \in D$ and for all $i \in N, l \in D$. The first pair of relations represents the non-negativity of flows and costs. The second pair of inequalities in (23) means that the difference of minimal costs at any two nodes cannot exceed the flow cost on the corresponding arc and that the travel demand cannot exceed the difference between outgoing and incoming flows. The third pair of relations in (23) means that the positive flow on each arc (respectively, the positive minimal cost at each node) yields the equality in the previous series of conditions. Hence, if it is necessary to provide the flow balance at each node:

$$\sum_{a\in A_i^+} \tilde{x}_a^l - \sum_{a\in A_i^-} \tilde{x}_a^l = d_i^l(\tilde{t}),$$

one has to simply guarantee the positivity of the transportation costs. It is clear that conditions (23) determine a nonlinear complementarity problem, hence they may be rewritten in the form of a system of EPs: Find $(\tilde{x},\tilde{t}) \geq 0$ such that

$$\sum_{a\in A} c_a(\tilde{f})(f_a - \tilde{f}_a) + \sum_{a=(i,j)\in A}\sum_{l\in D}(\tilde{t}_j^l - \tilde{t}_i^l)(x_a^l - \tilde{x}_a^l) \geq 0 \quad \forall x \geq 0, \tag{24}$$

$$\sum_{i\in N}\sum_{l\in D}\left[\sum_{a\in A_i^+} \tilde{x}_a^l - \sum_{a\in A_i^-} \tilde{x}_a^l - d_i^l(\tilde{t})\right](t_i^l - \tilde{t}_i^l) \geq 0 \quad \forall t \geq 0; \tag{25}$$

where $\tilde{f} = \sum\limits_{l\in D} \tilde{x}^l$. Clearly, system (24), (25) is a particular case of system (2), (3), hence we can make use of algorithm (10), (12) to find its solution. The choice of primal and dual variables in system (24), (25) is rather arbitrary and will depend on the properties of the mappings c and d to satisfy convergence assumptions. Besides, if either c or d is integrable, then the system (24), (25) becomes a particular case of system (17), (18), i.e. it can be treated as an inverse optimization problem.

For instance, we consider the case where the mapping $-d$ is strongly monotone with modulus τ and continuous. Then we can take t as the primal variable and x as the dual variable, and set

$$\Phi(t,s) = \sum_{i\in N}\sum_{l\in D} d_i^l(t)(t_i^l - s_i^l).$$

Then we have $H_{l,a}(t) = t_i^l - t_j^l$ with $a = (i,j)$ and all the assumptions **(A1)**–**(A5)** hold and we can specialize algorithm (10) as follows.

Algorithm 1. Choose an arbitrary point $x^{(0)} \geq 0$. At the kth iteration, $k = 0,1,\dots$, we have a point $x^{(k)} \geq 0$, set $h_i^{l,(k)} = \sum\limits_{a\in A_i^+} x_a^{l,(k)} - \sum\limits_{a\in A_i^-} x_a^{l,(k)}$ and find the unique solution $t^{(k)} = (t_i^{l,(k)})_{l\in D, i\in N}$ of the problem

$$\sum_{i\in N}\sum_{l\in D}\left[h_i^{l,(k)} - d_i^l(t^{(k)})\right](t_i^l - t_i^{l,(k)}) \geq 0 \quad \forall t \geq 0.$$

Next, we set $p_a^{l,(k)} = t_j^{l,(k)} - t_i^{l,(k)}$ for each arc $a = (i,j) \in A$ and find $x^{(k+1)} \geq 0$ as the unique solution of the auxiliary problem:

$$\begin{aligned}\sum_{a\in A} c_a\left(\sum_{l\in D} x^{l,(k+1)}\right)\left[\sum_{l\in D} x_a^l - \sum_{l\in D} x_a^{l,(k+1)}\right] + \sum_{a\in A}\sum_{l\in D} p_a^{l,(k)}(x_a^l - x_a^{l,(k+1)}) \\ + \frac{1}{\gamma_k}\sum_{l\in D}\sum_{a\in A}(x_a^{l,(k+1)} - x_a^{l,(k)})(x_a^l - x_a^{l,(k+1)}) \geq 0 \quad \forall x \geq 0,\end{aligned} \tag{26}$$

where γ_k is chosen in conformity with (12).

It is clear that (26) is nothing but the splitting algorithm iteration. Therefore, from Theorem 3.1 it follows that the iteration sequence $\{(t^{(k)}, x^{(k)})\}$ will converge to a solution of system (24), (25).

6. Application to Economic Equilibrium Problems

In this section, we consider a class of economic equilibrium models, which can be viewed as an extension and a modification of the known Cassel-Wald models; see [15, 16], and adjust the splitting algorithm (10), (12) for finding its solution.

The model describes an economic system which deals in l commodities, n technologies of production, and m pure factors of production. In what follows, c_k denotes the price of the kth commodity, b_i denotes the total inventory of the ith factor, and a_{ij} denotes the inventory of the ith factor which is required for the unit level of activity of the jth technology, so that we set $c = (c_1, \dots, c_l)^T$, $b = (b_1, \dots, b_m)^T$, $A = (a_{ij})_{m \times n}$. Next, x_j denotes the value of activity of the ith technology, z_k denotes the output of the kth commodity, so that $x = (x_1, \dots, x_n)^T$ and $z = (z_1, \dots, z_l)^T$. We suppose that the output mapping $T : R^n_+ \to R^l_+$ gives the relation between x and z, i.e.

$$z = T(x). \tag{27}$$

The vectors c and b are not fixed in general, but $c = c(z)$ and $b = b(y)$, where $y = (y_1, \dots, y_m)^T$ is the vector of shadow prices of factors.

The pair $(x^*, y^*) \in R^n_+ \times R^m_+$ is said to be in equilibrium if it solves the following system of problems

$$\langle c(z^*), z^* - z\rangle + \langle y^*, Ax - Ax^*\rangle \geq 0 \quad \forall x \geq 0; \tag{28}$$

$$\langle b(y^*) - Ax^*, y - y^*\rangle \geq 0 \quad \forall y \geq 0; \tag{29}$$

where $z^* = T(x^*)$. Clearly, this system is a particular case of (2), (3), where

$$\Phi(x, x') = \langle c(T(x)), T(x) - T(x')\rangle, H(x) = Ax, G(y) = b(y), \tag{30}$$

$X = R^n_+$, and $Y = R^m_+$. Moreover, system (28), (29) can be viewed as an extension of the model (24), (25). At the same time, if T is affine and c is the gradient map of a concave function, then it is easy to see that the system (28), (29) becomes a particular case of the system (21), (22) (or equivalently, (17), (18)). Note that problems similar to (28) were investigated by Yao; see [17]. In the case where $n = l$ and T is the identity map, the system (28), (29) coincides with the system of complementarity problems:

$$\begin{array}{ll} x^* \geq 0, & y^* \geq 0; \\ A^T y^* - c(x^*) \geq 0, & b(y^*) - Ax^* \geq 0; \\ \langle x^*, A^T y^* - c(x^*)\rangle = 0, & \langle y^*, b(y^*) - Ax^*\rangle = 0. \end{array} \tag{31}$$

The well known Cassel-Wald model corresponds to the case where b is fixed. If both the vectors c and b are fixed, then system (31) becomes equivalent to the following pair of linear programming problems:

$$\begin{array}{ccc} \max \to \langle c, x\rangle & \text{and} & \min \to \langle b, y\rangle \\ \text{subject to} & & \text{subject to} \\ Ax \leq b, x \geq 0; & & A^T y \geq c, y \geq 0. \end{array}$$

Next, recall that a mapping $Q : R^l_+ \to R^l$ is said to be *strongly monotone with respect to a mapping* $T : R^n_+ \to R^l_+$ with modulus $\tau > 0$, if, for all $x', x'' \in R^n_+$, we have

$$\langle Q(T(x')) - Q(T(x'')), T(x') - T(x'') \rangle \geq \tau \left\| x' - x'' \right\|^2.$$

For instance, this condition is satisfied with $\tau = \tau'(\tau'')^2$ if $l = n$, Q and T are strongly monotone with moduli τ' and τ'', respectively.

In order to apply algorithm (10), (12) to the system (28), (29), we have to verify conditions **(A1)–(A5)**. If we suppose that $b : R^m_+ \to R^m$ is a monotone continuous mapping, $T : R^n_+ \to R^l_+$ is a continuous mapping with convex components $T_i : R^n_+ \to R_+$, and $c : R^l_+ \to R^l_+$ is a continuous mapping such that $-c$ is strongly monotone with respect to T with constant τ, then, taking into account (30), we see that **(A1)–(A5)** are satisfied. The splitting algorithm (10), (12) can be then specialized to system (28), (29) as follows.

Algorithm 2. Choose an arbitrary point $y^0 \in R^m_+$. At the kth iteration, $k = 0, 1, \dots$, we have a point $y^k \in R^m_+$ and find $x^k \in R^n_+$ as the unique solution to the auxiliary problem:

$$\langle c(T(x^k)), T(x^k) - T(x) \rangle + \langle y^k, Ax - Ax^k \rangle \geq 0 \quad \forall x \in R^n_+.$$

Afterwards, we find $y^{k+1} \in R^m_+$ as the unique solution to the auxiliary problem:

$$\langle b(y^{k+1}) + \frac{1}{\gamma_k}(y^{k+1} - y^k) - Ax^k, y - y^{k+1} \rangle \geq 0 \quad \forall y \in R^m_+,$$

where γ_k is given in (12).

From Theorem 3.1 it follows that Algorithm 2 generates a sequence $\{(x^k, y^k)\}$ which converges to a solution of the system (28), (29) under the assumptions above.

We have chosen the three examples above only to illustrate the diversity of possible applications of the described approach related to the system of EPs of form (2), (3). Clearly, this approach can be applied to various different problems which are formulated as a system of equilibrium type problems.

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