Journal of Nonlinear and Convex Analysis
Volume 5, Number 1, 2004, 59–69

Yokohama Publishers
ISSN 1880-5221 ONLINE JOURNAL

A TWO POINT BOUNDARY VALUE PROBLEM FOR A CLASS OF DIFFERENTIAL INCLUSIONS

TZANKO DONCHEV AND MARC QUINCAMPOIX

Abstract. A two point boundary value problem for differential inclusions with relaxed one sided Lipschitz right-hand side in Banach spaces with uniformly convex duals are studied. We use successive approximations to obtain nonemptiness and compactness of the solution set as well as its continuous dependence from the initial conditions and the right-hand side. The relaxation theorem is also proved.

1. Introduction

This paper is about the following boundary value problem:

$$\dot{x}(t) \in F(t,x,y), \quad x(0) = x^0, \tag{1}$$

$$\dot{y}(t) \in G(t,x,y), \quad y(1) = y^0. \tag{2}$$

Here $F : I \times E_1 \times E_2 \longrightarrow CC(E_1)$, $G : I \times E_1 \times E_2 \longrightarrow CC(E_2)$, where E_1 and E_2 are Banach spaces with uniformly convex duals, $t \in I = [0,1]$ and $CC(E)$ denotes the set of all nonempty convex compact subsets of E (E is E_1 or E_2).

Such kind of systems appear for example when one obtains necessary conditions for optimal control of Pontryagin maximum principle type (see for example [14] p.219-221). Two point boundary value problems in case of single valued F and G are intensively studied. We note among others [15] and [18]. In the last one some iteration procedure is used. For more general two point boundary value problems see [7] §11, where the result of [15] is extended to differential inclusions in Banach spaces. We refer to [1], [7] and [12] where the initial point problems for differential inclusions are comprehensive studied. The boundary value problems are studied in a large number of papers. In case of R^n, we refer to [12] ch. III.2 and the references given there. In this book (see also [3]) the relaxation theorem is proved. The (existence) results (see [2], [16]), however, are obtained either in a finite dimensional case, or under some compactness type conditions ([4]). Notice also [6], where the two point boundary value problem is studied with Baire category. The problem (1) – (2), however is not included in the problems considered in the literature presented above.

Our result says that, under suitable Relaxed One-Sided Lipschitz (ROSL) condition described below the problem (1) – (2) has at least one absolutely continuous solution. Furthermore the set of all solutions is (pre)compact. We will use the successive approximations. In case of initial value problems [8, 9] this method does not work under ROSL condition and time discretization is used. In our case, however, time discretization does not work, but successive approximations do.

2000 *Mathematics Subject Classification.* 34B15, 34A60.

The solution set of (1) – (2) consists of all absolutely continuous (AC) functions (x, y) satisfying (1) for a.e. $t \in I$. Denote by $J(z) = \{l \in E^* : \ |l| = |z|; \ \langle l, z \rangle = |z|^2\}$ the duality mapping. Notice that since E_1 and E_2 have uniformly convex duals the duality mapping is single valued and uniformly continuous on the bounded sets in both spaces. It is easy to see that $J(-z) = -J(z)$. Indeed $|-J(z)| = |z|$ and $\langle -J(z), -z \rangle = |z|^2$. For A, B closed bounded define $D_H(A, B) = \max\{\max_{a\in A} \min_{b\in B} |a - b|, \max_{b\in B} \min_{a\in A} |a - b|\}$ – the Hausdorff distance and dist $(a, A) = \min_{b\in A} |a - b|$. A multimap is said to be continuous, when it is continuous with respect to the Hausdorff distance. In a Banach space $\mathcal{X}$ we will also use $[x, y]_+ = \lim_{h\to 0^+} h^{-1}\{|x + hy| - |x|\}$ (see [13] p.7). The map $[x, \cdot]_+$ is nonexpansive and $[\cdot, y]_+$ is upper semicontinuous as a real valued function. Furthermore, if the dual space $\mathcal{X}^*$ is uniformly convex then $|x|[x, y]_+ = \langle J(x), y \rangle$. By $\overline{\text{ext}}\ A$ we denote the closure of the set of all extreme points of A. Denote $||f(\cdot)||_C = \max_{t\in I} |f(t)|$ (this notation is used for real valued and for vector valued functions).

Definition 1. The multifunction F from $I \times E$ into E is said to be upper semi continuous (USC) at (t, x), when to $\varepsilon > 0$ there exists $\delta > 0$ such that $F(s, y) \subset F(t, x) + \varepsilon U$ (U is the closed unit ball in E) for every (s, y) with $|s - t| + |x - y| < \delta$. $F(\cdot, \cdot)$ is said to be almost USC when for every $\varepsilon > 0$ there exists $I_\varepsilon \subset I$ with Lebesgue measure meas $(I \backslash I_\varepsilon) < \varepsilon$ such that F is USC restricted to $I_\varepsilon \times E$. Almost continuous multimaps are defined analogously.

The multimap F is said to be lower semi continuous (LSC) at (t, x) when for every $f \in F(t, x)$ and every $(t_i, x_i) \to (t, x)$ there exist $f_i \in F(t_i, x_i)$ such that $f_i \to f$. Almost LSC maps are defined analogously.

Given $M > 0$, define $\Gamma^M = \{(t, x) \in I \times E : \ |x| \leq Mt\}$. The single valued map $f : I \times E \to E$ is said to be Γ^M continuous, when $(t_n - t, x_n - x) \in \Gamma^M$ and $(t_n, x_n) \to (t, x)$ implies $f(t_n, x_n) \to f(t, x)$. Almost Γ^M continuous maps are defined analogously.

We refer to [7] for all concepts used in the paper but not discussed in details.

We need the following assumptions:

A1. The multimaps F and G are bounded on the bounded sets and almost USC.
A2. There exist constants A, B, C, D (with B, C not negative) such that $\forall (x_1, y_1), (x_2, y_2) \in E_1 \times E_2$ and a.a. $t \in I$ the following two inequalities hold:
 i) $\sigma(J(x_1 - x_2), F(t, x_1, y_1)) - \sigma(J(x_1 - x_2), F(t, x_2, y_2)) \leq A|x_1 - x_2|^2 + B|x_1 - x_2||y_1 - y_2|$.
 ii) $\sigma(J(y_1 - y_2), G(t, x_1, y_1)) - \sigma(J(y_1 - y_2), G(t, x_2, y_2)) \geq -C|x_1 - x_2||y_1 - y_2| - D|y_1 - y_2|^2$.

Here $\sigma(l, A) = \sup_{a\in A} \langle l, a \rangle$ is the support function of the set A.

The paper consists of two sections and an introduction. In the next section the main results are presented. In the last one the main results are proved.

2. Main Theorem

In this section we formulate our main result Theorem 1 and its corollary, the relaxation theorem (Theorem 2). Firstly we will change variable in (2). We let $\tau = 1 - t$. Obviously in this case $\dot{y}(t) = -\dot{y}(\tau)$. The system (1) – (2) becomes:

$$\dot{x}(t) \in F(t, x, y), \quad x(0) = x^0, \tag{3}$$

$$\dot{y}(\tau) \in H(\tau, x, y), \quad y(0) = y^0. \tag{4}$$

Where we have denoted $H(\tau, x, y) = -G(1 - \tau, x, y)$. The following lemma is then true.

Lemma 1. $H(\cdot,\cdot,\cdot)$ *satisfies the following condition:*

$$\begin{aligned}\sigma(J(y_1 - y_2), H(\tau, x_1, y_1)) - \sigma(J(y_1 - y_2), H(\tau, x_2, y_2))& \\ \le C|x_1 - x_2||y_1 - y_2| + D|y_1 - y_2|^2&\end{aligned}$$

$\forall (x_1, y_1), (x_2, y_2) \in E_1 \times E_2$ *and a.a.* $t \in I$.

The proof is trivial and is omitted.

Given nonnegative constant B and C we define the real-valued functions $r(\cdot,\cdot)$ and $s(\cdot,\cdot)$ as follows:

$$r(A, D) = \begin{cases} \dfrac{BC}{A}\Big[\dfrac{e^{A+D} - 1}{A + D} - \dfrac{e^D - 1}{D}\Big] & \text{for } A > 0, D > 0 \\ \dfrac{BC}{A}\Big[\dfrac{1 - e^D}{-D} - \dfrac{1 - e^{D-A}}{A - D}\Big] & \text{for } A > 0, D < 0 \\ \dfrac{BC}{-A}\Big[\dfrac{e^D - 1}{D} - \dfrac{1 - e^{A-D}}{D - A}\Big] & \text{for } A < 0, D > 0, A + D \neq 0 \\ \dfrac{BC}{D^2}\Big[e^D - 1 - D\Big] & \text{for } A < 0, D = -A \\ \dfrac{BC}{-A}\Big[\dfrac{1 - e^D}{-D} - \dfrac{1 - e^{A+D}}{-A - D}\Big] & \text{for } A < 0, D < 0 \\ \dfrac{BC}{D^2}\Big[De^D - e^D + 1\Big] & \text{for } A = 0, D \neq 0 \\ \dfrac{BC}{A^2}\Big[e^A - 1 - A\Big] & \text{for } A \neq 0, D = 0 \\ \dfrac{BC}{2} & \text{for } A = D = 0 \end{cases}$$

$$s(A,D)=\begin{cases}\frac{BC}{D}\Big[\frac{e^{A+D}-1}{A+D}-\frac{e^A-1}{A}\Big] & \text{for } A>0, D>0\\ \frac{BC}{D}\Big[\frac{1-e^{A}}{-A}-\frac{1-e^{A-D}}{D-A}\Big] & \text{for } A<0, D>0\\ \frac{BC}{-D}\Big[\frac{e^{A}-1}{A}-\frac{1-e^{D-A}}{A-D}\Big] & \text{for } A>0, D<0, A+D\neq 0\\ \frac{BC}{A^2}\Big[e^A-1-A\Big] & \text{for } A>0, D=-A\\ \frac{BC}{-D}\Big[\frac{1-e^{A}}{-A}-\frac{1-e^{A+D}}{-A-D}\Big] & \text{for } A<0, D<0\\ \frac{BC}{A^2}\Big[Ae^A-e^A+1\Big] & \text{for } A\neq 0, D=0\\ \frac{BC}{D^2}\Big[e^D-1-D\Big] & \text{for } A=0, D\neq 0\\ \frac{BC}{2} & \text{for } A=D=0\end{cases}$$

The main result in the paper is the following:

Theorem 1. *Let F, G be convex compact valued multimaps, satisfying A1, A2. If*

$$l(A,D) := \max\{r(A,D), s(A,D)\} = q < 1, \tag{5}$$

then the system (1) has a solution. Furthermore the solution set is $C(I, E_1 \times E_2)$ compact and depends continuously on the initial condition and on the right-hand side.

As a corollary we will prove the relaxation theorem, which extends the corresponding results of [3] and [12].

Theorem 2. *Let $F(\cdot,\cdot,\cdot)$ and $G(\cdot,\cdot,\cdot)$ be almost continuous. Under the conditions of Theorem 1 the solution set of*

$$\dot{x}(t) \in \overline{\text{ext}}\ F(t,x,y), \quad x(0) = x^0, \tag{6}$$

$$\dot{y}(t) \in \overline{\text{ext}}\ G(t,x,y), \quad y(1) = y^0. \tag{7}$$

is nonempty and dense in the solution set of (1)–(2).

Remark 1. The condition **A2** is a kind of so called Relaxed One Sided Lipschitz (ROSL) condition. This condition is introduced from the first author under different name. We recall that the multifunction $R : I \times E \to CC(E)$ is said to be ROSL when there exists a constant L such that: $\sigma(J(x-y), F(t,x)) - \sigma(J(x-y), F(t,y)) \leq y|x-y|^2$ for every $x, y \in E$. With the help of this condition in [8, 9] the differential inclusion

$$\dot{x}(t) \in R(t, x(t),\ x(0) = x_0,\ t \in [0,1] \tag{8}$$

is comprehensively studied. We refer to [10, 11, 14] where many examples of differential inclusions, where this condition is useful are given.

The condition **A2** i) is for example fulfilled when $F(t,x,\cdot)$ is Lipschitz with a constant B and $F(t,\cdot,y)$ is ROSL with a constant A. When we study the differential inclusion (8)for $t \in [-1,0]$ then the opposite inequality $\geq$ in the definition of ROSL must be used.

Of course the conditions **A1, A2** are not sufficient for the existence of solutions. We employed $l(A, D) = q < 1$. It is easy to see that $\lim_{A \to -\infty} l(A, D) = \lim_{D \to -\infty} l(A, D) = 0$. If $BC = 0$ then the system (1)–(2) has a solution.

In the next section we present the proofs of Theorem 1 and Theorem 2.

Now we present two examples of systems where Theorem 1 and Theorem 2 are useful.

Example 1. This example is a modification of the well known counter example of Plis (see [17] for instance).

Consider the following system:

$$\begin{aligned} \dot{x}(t) &\in \{-1 - |y|, 1 + |y|\}, \quad x(0) = 0 \\ \dot{y}(t) &= \sqrt[3]{y} + |x|, \quad y(1) = 0. \end{aligned}$$

It is easy to see that **A2** holds with $A = D = 0$ and $B = C = 1$. Therefore all the assumptions of Theorem 2 hold.

Consider the initial valued problem:

$$\begin{aligned} \dot{x}(t) &\in \{-1 - |y|, 1 + |y|\}, \quad x(0) = 0 \\ \dot{y}(t) &= \sqrt[3]{y} + |x|, \quad y(0) = 0. \end{aligned} \tag{9}$$

It is easy to see that $y(t) \geq \left(\frac{2t}{3}\right)^{3/2}$ for every solution of (9). However, $y(t) \equiv 0$ is a solution of the convexified problem.

This example can be modified in case of Hilbert space.

Example 2. Let $E_1 \equiv E_2 \equiv l_2$ (the space of all sequences $\{k_n\}_{n=1}^{\infty}$ with $\sum_{n=1}^{\infty} |k_n|^2 < \infty$). Let $\vec{e} \in l_2$ be with positive coordinates.

Consider the following system:

$$\begin{aligned} \dot{x}(t) &\in \{-\vec{e} - \|y\|, \vec{e} + \|y\|\}, \quad x(0) = 0 \\ \dot{y}(t) &= f(y) + \|x\|, \quad y(1) = 0. \end{aligned} \tag{10}$$

here

$$f(y) = \begin{cases} 0 & \text{for } y = 0, \\ \dfrac{y}{\sqrt[3]{\|y\|^2}} & \text{for } y \neq 0. \end{cases}$$

It is easy to see that **A2** holds with $A = D = 0$ and $B = C = 1$. Therefore all the assumptions of Theorems 1 and 2 hold. If we replace the second equation of (10) by

$$\begin{aligned} &\cdots \\ &\dot{y}_{i+1}(t) = \frac{\sqrt[3]{y_{i+1}}}{i+1} + |x_i| + \frac{y_i}{i+1}, \; y_{i+1}(1) = 0 \\ &\cdots \end{aligned}$$

then Theorems 1 and 2 still hold. However, for the corresponding initial value problem ($y_{i+1}(1)$ replaced by $y_{i+1}(0)$) Theorem 2 does not hold.

3. Proof of the main results

In this section Theorem 1 and Theorem 2 are proved. The proofs will be given in case of positive A, D, because if some of them is nonpositive one has to do only obvious modifications.

Proof of Theorem 1. We let $x_0(\cdot) \equiv x_0$ (the initial function). If we replace $x(\cdot)$ in (4) by $x_0(\cdot)$, then we are able to obtain a solution $y^0(\cdot)$ of (4) thanks to theorem 1 of [8]. If we replace y in (3) by $y^0(\cdot)$, then due to Theorem 1 of [8] there exists a solution $x_1(\cdot)$ of (3). The rest of the proof will be done in three steps.

Step 1. Construction of the initial approximate solutions.

Consider the differential inclusion:

$$\dot{y}(\tau) \in H(\tau, x_1(\tau), y), \quad y(0) = y^0.$$

We will find a solution $y^1(\cdot)$ of the last system satisfying the inequality:

$$\frac{d}{d\tau}|y^1(\tau) - y^0(\tau)| \le C|x_1(\tau) - x_0(\tau)| + D|y^1(\tau) - y^0(\tau)|.$$

To this end define the multifunction:

$$G_1(\tau, u) = \{v \in H(\tau, x_1(\tau), u) : \langle J(y^0(\tau) - u), \dot{y}^0(\tau) - v\rangle \le$$
$$C|x_1(\tau) - x_0(\tau)||y^0(\tau) - u| + |y^0(\tau) - u|^2\}.$$

Due to **A2 ii)** the set valued map $G_1(\cdot,\cdot)$ has nonempty values. Furthermore it is not difficult to show that $G_1(\cdot,\cdot)$ is almost USC with convex compact values. Since $G_1(\tau, u) \subset H(\tau, x_1(\tau), u)$ one has that the differential inclusion:

$$\dot{z}(\tau) \in G_1(\tau, z(\tau)), \; z(0) = y^0$$

admits a solution $y^1(\cdot)$ thanks to theorem 1 of [8]. This solution satisfies

$$\langle J(y^0(\tau) - y^1(\tau)), \dot{y}^0(\tau) - \dot{y}^1(\tau)\rangle \le$$
$$C|x_1(\tau) - x_0(\tau)||y^0(\tau) - y^1(\tau))| + D|y^0(\tau) - y^1(\tau))|^2. \tag{11}$$

The function $|y^0(\tau) - y^1(\tau))|$ is absolutely continuous (AC) and hence a.e. differentiable. Due to (11) we have $\frac{1}{2}\frac{d}{dt}|y^0(\tau) - y^1(\tau))|^2 \le C|x_1(\tau) - x_0(\tau)||y^0(\tau) - y^1(\tau))| + D|y^0(\tau) - y^1(\tau))|^2$. If $|y^0(\tau) - y^1(\tau))| > 0$ then we can divide the inequality by it. If $|y^0(\tau) - y^1(\tau))| = 0$ on a set with positive (Lebesgue) measure say ν then the points of density of this set have measure also ν and on every such a point if the derivative of $|y^0(\tau) - y^1(\tau))|$ exists, then it is 0. Therefore denoting $|y^0(\tau) - y^1(\tau))| = r_1(\tau)$ and $|x_1(\tau) - x_0(\tau)| = s_1(\tau)$ one has that

$$\dot{r}_1(\tau) \le Cs_1(\tau) + Dr_1(\tau), \; r_1(0) = 0.$$

Now we are looking for $x_2(\cdot)$ satisfying

$$\frac{d}{dt}|x_1(t) - x_2(t)| \le A|x_1(t) - x_2(t)| + B|y^1(t) - y^0(t)|.$$

The construction is the same as above, i.e. consider the multimap:

$$F_2(t, z) = \{w \in F(t, z, y^1(t)) : \langle J(x_1(t) - z), \dot{x}_1(t) - w\rangle \le$$
$$A|x_1(t) - z|^2 + B|y^0(t) - y^1(t)||x_1(t) - z|\}$$

one has that the multimap $F_2(\cdot,\cdot)$ is almost USC with nonempty convex compact values. Hence the differential inclusion:

$$\dot{z}(t) \in F_2(t, z(t)),\ z(0) = x_0$$

has a solution $x_2(\cdot)$ which satisfies the required inequality above. Denote $|x_1(t) - x_2(t)| = s_2(t)$. We have $\dot{s}_2(t) \le As_2(t) + Br_1(t)$, $s_2(0) = 0$.

Step 2. The existence of solutions and continuous dependence.

We go on by induction. Let the sequences $\{x_i(\cdot)\}_{i=1}^n$ and $\{y^i(\cdot)\}_{i=1}^n$ be already defined. Recalling that $\tau = 1 - t$ and arguing as above we find functions $x_{n+1}(\cdot)$ and $y^{n+1}(\cdot)$ satisfying

$$\dot{x}_{n+1}(t) \in F(t, x_{n+1}(t), y^n(t)),\ x_{n+1}(0) = x_0$$
$$\dot{y}^{n+1}(\tau) \in F(\tau, x_{n+1}(\tau), y^{n+1}(\tau)),\ y^{n+1}(0) = y^0$$

to be such that:

$$\frac{d}{dt}|x_n(t) - x_{n+1}(t)| \le A|x_n(t) - x_{n+1}(t)| + B|y^{n-1}(t) - y^n(t)|$$
$$\frac{d}{d\tau}|y^n(\tau) - y^{n+1}(\tau)| \le C|x_n(\tau) - x_{n+1}(\tau)| + D|y^{n+1}(\tau) - y^n(\tau)|.$$

Consequently $|x_n(t) - x_{n+1}(t)| \le s_{n+1}(t)$ and $|y^n(\tau) - y^{n+1}(\tau)| \le r_{n+1}(\tau)$, where

$$\dot{s}_{n+1}(t) \le As_{n+1}(t) + Br_n(t),\ s_{n+1}(0) = 0 \tag{12}$$
$$\dot{r}_{n+1}(\tau) \le Cs_{n+1}(\tau) + Dr_{n+1}(\tau),\ r_{n+1}(0) = 0 \tag{13}$$

Now we will estimate $||s_{n+1}(\cdot)||_C$ and $\|r_{n+1}(\cdot)\|_C$. From (13) we obtain:

$$r_{n+1}(\tau) \le e^{D\tau}\int_0^\tau e^{-D\alpha}Cs_{n+1}(\alpha)\,d\alpha = Ce^{D\tau}\int_0^\tau e^{-D\alpha}s_{n+1}(\alpha)\,d\alpha$$
$$\text{and } s_{n+1}(t) \le Be^{At}\int_0^t e^{-A\beta}r_n(\beta)\,d\beta.$$

If we let $\bar{r}_n = ||r_n(\cdot)||_C$, then $s_{n+1}(t) \le \bar{r}_n Be^{At}\int_0^t e^{-A\tau}\,d\tau = \bar{r}_n Be^{At}\dfrac{1 - e^{-At}}{A}$. Consequently

$$\begin{aligned} r_{n+1}(\tau) &\le \bar{r}_n\frac{BC}{A}e^{D\tau}\int_0^\tau e^{-Dt}[e^{A(1-t)} - 1]\,dt \\ &= \bar{r}_n\frac{BC}{A}e^{D\tau}\Big[\int_0^\tau e^{A-(A+D)t}\,dt - \int_0^\tau e^{-Dt}\,dt\Big] \\ &= \bar{r}_n\frac{BC}{A}e^{D\tau}\Big(e^A\frac{1 - e^{-(A+D)\tau}}{A+D} - \frac{1 - e^{-D\tau}}{D}\Big). \end{aligned}$$

After some technical and straightforward computations we get:

$$||r_{n+1}(\cdot)||_C < \bar{r}_n\frac{BC}{A}\Big[\frac{e^{A+D} - 1}{A+D} - \frac{e^D - 1}{D}\Big],$$

i.e. $\bar{r}_{n+1} < \bar{r}_n r(A, D) \le q\bar{r}_n$. Analogously, using (12), we get

$$||s_{n+1}(\cdot)||_C < \bar{s}_n\frac{BC}{D}\Big[\frac{e^{A+D} - 1}{A+D} - \frac{e^A - 1}{A}\Big],$$

i.e. $\bar{s}_{n+1} < \bar{s}_n s(A,D) \le q\bar{s}_n$. Therefore $\displaystyle\sum_{n=1}^{\infty} \bar{r}_n + \bar{s}_n < \infty$. Thus there exist limits $\lim\limits_{n\to\infty} x_n(t) = x(t)$ and $\lim\limits_{n\to\infty} y^n(t) = y(t)$ with respect to $C(I,E_1)$ $(C(I,E_2))$ topology. Since F, G are bounded on the bounded sets one has (passing to subsequence if necessary) that $\dot{y}^k(\cdot) \rightharpoonup \dot{y}(\cdot)$ and $\dot{x}_k(\cdot) \rightharpoonup \dot{x}(\cdot)$ in L_1–weak topology. By virtue of Mazur's lemma taking into account that F and G are almost USC one has that $(x(\cdot), y(\cdot))$ is a solution of (3)–(4). That is the system (1)–(2) has a solution.

It remains to show that the solution set depends continuously on the initial conditions and on the right-hand sides. To this end we will show that for every solution $(x(\cdot), y(\cdot))$ of (1) there exists a solution $(x_1(\cdot), y_1(\cdot))$ of

$$\dot{x}(t) \in F(t,x,y), \quad x(0) = x^1,$$
$$\dot{y}(\tau) \in H(\tau,x,y), \quad y(0) = y^1.$$

which is sufficiently close to $(x(\cdot), y(\cdot))$, when (x^1, y^1) is near enough to (x^0, y^0). We will follow the arguments above. Define

$$F_1(t,u,v) = \{w \in F(t,u,v) : \ \big\langle J(x(t)-u), \dot{x}(t) - v\big\rangle \le A|x(t)-u|^2 + B|x(t)-u||y(t)-v|\}.$$

It is standard to prove, proceeding as above, that F_1 is almost USC with nonempty convex compact values. Let $x_1(\cdot)$ be a solution of

$$\dot{x}(t) \in F_1(t,x,y), \quad x(0) = x^1,$$

Thus $|x(t) - x_1(t)| \le s_1(t)$, where $\dot{s}_1(t) \le As_1(t)$, $s(0) = |x^1 - x^0| = s_0$. Hence $s_1(t) \le e^{At}s_0$. Analogously define

$$H_1(\tau,u,v) = \{z \in F(\tau,u,v) : \ \big\langle J(y(\tau)-u), \dot{y}(\tau)-w\big\rangle \le D|y(\tau)-v|^2 + D|x(\tau)-u||y(\tau)-v|\}.$$

The multifunction H_1 is almost USC with convex compact values. Let $y_1(\cdot)$ be a solution of

$$\dot{y}(\tau) \in H_1(\tau,x,y), \quad y(0) = y^1.$$

Consequently, $|y(\tau) - y_1(\tau)| \le r_1(\tau)$, where

$$\dot{r}_1(\tau) \le Cs_1(\tau) + Dr_1(\tau), \ r_1(0) = |y^1 - y^0| = r_0,$$

recall that $\tau = 1 - t$. Therefore

$$\begin{aligned} r_1(\tau) &\le e^{D\tau}r_0 + e^{D\tau}\int_0^\tau Cs_1(t)e^{-Dt}\,dt \le e^{D\tau}r_0 + Ce^{D\tau}\int_0^\tau e^{-Dt}e^{A(1-t)}s_0\,dt \\ &\le \frac{Cs_0}{A+D}e^A e^{D\tau}(1 - e^{-(D+A)\tau}) + e^{D\tau}r_0 \le \frac{Cs_0}{A+D}(e^{D+A} - 1) + e^D r_0. \end{aligned}$$

Due to the assumptions of the theorem and using the construction above one can obtain successively the sequences $\{x_n(\cdot)\}_{n=1}^\infty$ and $\{y_n(\cdot)\}_{n=1}^\infty$ such that denoting $\bar{s}_{n+1} = |x_{n+1}(t) - x_n(t)|_C$ and $\bar{r}_{n+1} = |y_{n+1}(t) - y_n(t)|_C$ we will have $\bar{s}_{n+1} < q\bar{s}_n$ and $\bar{r}_{n+1} < q\bar{r}_n$. Consequently there exists a solution $(x_\infty(\cdot), y_\infty(\cdot))$ of the system (3) - (4) such that $|x_\infty(\cdot) - x(\cdot)|_C \le s_0\displaystyle\sum_{n=0}^\infty q^n = \frac{s_0}{1-q}$ and $|y_\infty(\cdot) - y(\cdot)|_C \le$

$r_0 \sum_{n=0}^{\infty} q^n = \frac{r_0}{1-q}$. Therefore the solution set of 3) - 4) depends continuously on the initial conditions.

Let now $D_H(F(t,x,y), F_1(t,x,y)) \le \varepsilon$ and $D_H(H(\tau,x,y), H_1(\tau,x,y)) \le \varepsilon$ for every (t,x,y). Let also F_1, H_1 satisfy the assumptions **A1** and **A2**. Arguing as above for any solution $(x(\cdot), y(\cdot))$ of (1) we will find a solution $(u(\cdot), v(\cdot))$ of (1) with F, H replaced by F_1, H_1, such that $|u(t) - x(t)| \le s(t)$, $|v(\tau) - y(\tau)| \le r(\tau)$ such that $\dot{s}(t) \le As(t) + Br(t) + \varepsilon, \quad s(0) = 0$ and $\dot{r}(\tau) \le As(\tau) + Br(\tau) + \varepsilon, \quad r(0) = 0$. We have to repeat the calculus above to complete the proof.

Step 3. The compactness of the solution set.

Let $u(\cdot)$ and $v(\cdot)$ be two different initial functions. Denote by $R(u)$ and $R(v)$ the solution set of (4), when $x(\cdot)$ is replaced by $u(\cdot)$ or $v(\cdot)$ respectively. Using the same fashion as in the previous steps one can show that $D_H(R(u), R(v)) \le M$ (with respect to the Hausdorff distance in $C(I, E_2)$), where $m = \max_{\tau \in [0,1]} r(\tau)$ and $\dot{r}(\tau) = C|u(\tau) - v(\tau)| + Dr(\tau)$, $r(0) = 0$. Denote by $S(R(u))$ and $S(R(v))$ the solution set of (3), where $y(\cdot)$ is replaced by $R(u)$ or $R(v)$ respectively. Analogously $D_H(S(R(u)), S(R(v)) \le M$, where $M = \max_{t\in[0,1]} s(t)$ and $\dot{s}(t) = As(t) + Br(t)$, $s(1) = 0$. Denote by X_n the solution set of (3)–(4) obtained in the n-th stage in Step 2. It is easy to see that $D_H(X_n, X_{n+1}) \le \max\{r(A,D), s(A,D)\}$. Consequently there exists a $C(I, E_1 \times E_2)$ compact set $X = \lim_{n\to\infty} X_n$ (with respect to the Hausdorff distance in $C(I, E_1 \times E_2)$). Obviously X is a subset of the solution set of (1)–(2). Let $(x(\cdot), y(\cdot))$ be a solution of (1)–(2). One can consider $x_n(t) \equiv x(t)$ and $y^n(\tau) \equiv y(\tau)$. Consequently dist $\Big((x(\cdot), y(\cdot)), X_n\Big) \le q^n \max\{|x(\cdot) - x_0(\cdot)|, |y(\cdot) - y^0(\cdot)|\}$. Therefore $(x, y) \in X$. □

Remark 2. Using the result of [9] one can prove similar result in case of arbitrary Banach spaces when, however, $F(\cdot,\cdot,\cdot)$ and $G(\cdot,\cdot,\cdot)$ are almost continuous. Here $\big\langle J(x(t)), \dot{x}(t)\big\rangle$ has to be replaced by $[x(t), \dot{x}(t)]_+$, where $[x,y]_+ = \lim_{h \mapsto 0^+} h^{-1}\{|x + hy| - |x|\}$ (cf. [13] p. 7). Notice also that even for (single valued) differential equations in first order the successive approximations do not necessarily converge (see [13] §2 for instance).

Furthermore using more careful computations one can obtain better inequality for the constants A, B, C, D. In fact If $x_0(t) = x_0$ one has that $|x_1(t) - x_0(t)| \le Mt$. Using technical, but straightforward calculation one can prove for example:

If $A = D = 1$ then the problem (1) - (2) has a solution for $BC \le 1$. However, in general case we have difficulties to obtain the exact estimations. So we hope that the reader will be able to find better inequalities in the general case.

It is very important to notice that our estimations **are valid only on the interval** $[0,1]$. If we consider (1) – (2) on an interval $[0,T]$ one has to assume that $T^2 BC \le \max\{r(A,D), s(A,D)\} = q < 1$.

Proof of Theorem 2. Due to corollary 6.2 of [7] the maps $\overline{\text{ext}}\ F(\cdot,\cdot,\cdot)$ and $G(\cdot,\cdot,\cdot)$ are almost LSC. Furthermore the solution set of (1) – (2) consists of M–Lipschitz functions (with appropriate constant M). From theorem 2 of [5] we know that there exist Γ^{2M+1} continuous selections $f(t,x,y) \in \overline{\text{ext}}\ F(t,x,y)$ and $g(t,x,y) \in \overline{\text{ext}}\ G(t,x,y)$.

Let $(x^n(\cdot), y^n(\cdot)) \in X_n$ be arbitrary. it is easy to see that $\{(x^n(\cdot), y^n(\cdot))\}_{n=1}^{\infty}$ is $C(I, E_1 \times E_2)$ precompact set, i.e. passing to subsequences $(x^n(\cdot), y^n(\cdot)) \to (x(\cdot), y(\cdot)) \in (X, Y)$. If $\dot{x}^n = f(t, x^n, y^n)$ and $\dot{y}^n = g(t, x^{n-1}, y^n)$, then $\dot{x} = f(t, x, y)$ and $\dot{y} = g(t, x, y)$ (cf. [7] ch. 6). Therefore the solution set of (6)–(7) is nonempty. Furthermore for every almost LSC $\tilde{F}(t, x, y) \subset \overline{\text{ext}}\ F(t, x, y)$ and $\tilde{G}(t, x, y) \subset G(t, x, , y)$ one has that the system

$$\dot{x}(t) \in \tilde{F}(t, x, y), \quad x(0) = x^0, \tag{14}$$

$$\dot{y}(t) \in \tilde{G}(t, x, y), \quad y(1) = y^0. \tag{15}$$

has a solution. Let $(x(\cdot), y(\cdot)) \in X$ (in accordance of the proof of Theorem 1 X is the solution set of (1)– (2)). Fix $\varepsilon > 0$ and define

$$\begin{aligned} &\tilde{F}_\varepsilon(t, u, v) \\ &= cl(\{w \in \overline{\text{ext}}\ F(t, x, y) : \ [x(t) - u, \dot{x}(t) - w]_+ < A|x(t) - u| + B|y(t) - v| + \varepsilon\}) \end{aligned}$$

(where cl means the closure), and

$$\tilde{G}_\varepsilon(t, u, v) = cl(\{z \in \overline{\text{ext}}\ G(t, x, y) : \ [y(t) - v, \dot{y}(t) - z]_+ < C|x(t) - u| + D|y(t) - v| + \varepsilon\}.$$

Using standard arguments (cf. [9]) one can show that $\tilde{F}_\varepsilon$ and $\tilde{G}_\varepsilon$ are almost LSC with nonempty compact values. Therefore the system:

$$\begin{aligned} \dot{x}(t) &\in \tilde{F}_\varepsilon(t, x, y), \quad x(0) = x^0, \\ \dot{y}(t) &\in \tilde{G}_\varepsilon(t, x, y), \quad y(1) = y^0 \end{aligned}$$

has a solution $(\tilde{x}(\cdot), \tilde{y}(\cdot))$. Due to the definition of $\tilde{F}_\varepsilon$ and $\tilde{G}_\varepsilon$ we have $|x(t) - \tilde{x}(t)| \le r(t)$ and $|y(t) - \tilde{y}(t)| \le s(t)$, where

$$\begin{aligned} \dot{r}(t) &= Ar + Bs + \varepsilon, \quad r(0) = 0, \\ \dot{s}(t) &= Cr + Ds + \varepsilon, \quad s(1) = 0. \end{aligned}$$

It is easy to see that $\lim\limits_{\varepsilon \to 0} \Big[\|r(\cdot)\|_C + ||s(\cdot)||_C\Big] = 0$. The proof is therefore complete. □

Acknowledgement. The work of the first author was mainly carried out during a stay at the University of Bretagne Occidentale. He would like to thank the Laboratoire de Mathématiques for its hospitality.

The work of the first author on this project has been also funded in part by the National Academy of Sciences under the Collaboration in Basic Science and Engineering Program, supported by Contract No. INT-0002341 from the National Science Foundation.

References

[1] J.P. Aubin and A. Cellina, *Differential Inclusions*, Springer Verlag, Berlin, 1984.

[2] D. Averna and S.A. Marano, Existence of solutions for operator inclusions: a unified approach, *Rend. Sem. Mat. Univ. Padova* **102** (1999), 285-303.

[3] E. Avgerinos and N. Papageorgiou, Existence and relaxation theorems for nonlinear multivalued boundary value problem, *Appl. Math. Opt.* **39** (1999), 257-279.

[4] M. Benchora and S. Ntouyas, On three and four point boundary value problems for second order differential inclusions, *Math. Notes (Miscolć)* **2** (2001), 93-101.

[5] A. Bressan and G. Colombo, Selections and representations of multifunctions in paracompact spaces, *Studia Math.* **102** (1992), 209-216.

[6] F. De Blasi and G. Pianigiani, Solution sets of boundary value problems for nonconvex differential inclusions, *Topol. Methods Nonlinear Anal.* **1** (1993), 303-313.

[7] K. Deimling, *Multivalued Differential Equations*, De Gruyter, Berlin 1992.

[8] T. Donchev, Semicontinuous differential inclusions, *Rend. Sem. Mat. Univ. di Padova* **101** (1999), 147-160.

[9] T. Donchev, Qualitative properties of a class differential inclusions, *Glasnik Matematički* **31(51)** (1996), 269-271.

[10] T. Donchev and E. Farkhi, Stability and Euler approximations of one sided Lipschitz convex differential inclusions, *SIAM J. Control Optim.* **36** (1998), 780-796.

[11] T. Donchev and E. Farkhi, Euler approximation of discontinuous one-sided Lipschitz convex differential inclusions, *Calculus of Variations and Differential Equations* A. Ioffe, S. Reich and I. Shafrir (editors), *Chapman & Hall/CRC* Boca Raton, New York, 1999, 101-118.

[12] S. Hu and N. Papageorgiou, *Handbook of Multivalued Analysis*, vol. II *Applications*, Kluwer, Dordrecht, 2000.

[13] V. Lakshimantham and S. Leela, *Nonlinear Differential Equations in Abstract Spaces*, Pergamon, Oxford, 1981.

[14] F. Lempio and V. Veliov, Discrete approximations of differential inclusions, *Bayreuth. Math. Schr.* **54** (1998), 149-232.

[15] H. Mönch, Boundary value problems for nonlinear ordinary differential equations of second order in Banach spaces, *Nonlinear Analysis* **4** (1980), 985-999.

[16] O. Naselli Ricceri and B. Ricceri, An existence theorem for inclusions of the type $\Psi(u)(t) \in F(t, \Phi(u)(t))$ and application to a multivalued boundary value problem, *Appl. Anal.* **38** (1990), 259-270.

[17] A. Plis, Trajectories and quasitrajectories of an orientor field, *Bull. Acad. Polon. Sci., Ser. Math.* **11** (1963) 369-370.

[18] A. Samoilenko and N. Ronto, Modification of the numerical analytic method of successive approximations for boundary value problem of differential equation, *Ukr. Math. Journal* **34** (1982), 796-802. (in Rusian)

Manuscript received May 20, 2003
revised January 26, 2004

Tzanko Donchev
Department of Mathematics, University of Architecture and Civil Engineering, 1 "Hr. Smirnenski" str., 1046 Sofia, Bulgaria
E-mail address: `tdd51us@yahoo.com`

Marc Quincampoix
Laboratoire de Mathématiques, Université de Bretagne Occidentale; 6 av. Victor Le Gorgeu B.P. 809; 29285 Brest Cedex, France
E-mail address: `Marc.Quincampoix@univ-brest.fr`