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ON A THEOREM OF ALEXANDROV

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Abstract. We give an elementary variational proof of classical theorems of Minkowski and Alexandrov on the existence and uniqueness, up to translations, of a closed convex hypersurface with a given Gaussian curvature (as a function of the exterior unit normal) or with a given surface function.

1. Introduction

1.1. **The surface function of a convex body.** Denoting by $\mathcal{M}_+(S^{n-1})$ the cone of nonnegative Radon measures defined on the sphere S^{n-1}, we recall that any convex compact set A of $\mathbb{R}^n$ with full dimension n (a convex *body* in the sequel) can be associated with a nonnegative measure $\mu_A \in \mathcal{M}_+(S^{n-1})$ defined by:

$$(1) \qquad \forall \varphi \in C^0(S^{n-1}), \quad \int_{S^{n-1}} \varphi(y)\, d\mu_A(y) = \int_{\partial A} \varphi(\nu_A(x))\; d\mathcal{H}^{n-1}(x).$$

Here $\nu_A(x)$ denotes the unit outer vector at $x \in \partial A$, we recall that ν_A is uniquely defined $\mathcal{H}^{n-1}$-a.e. on ∂A. We shall, in the sequel, say that $x \in \partial A$ is *singular*, if the unit outer vector at $x \in \partial A$ is not well defined i.e. x belongs to several distinct supporting hyperplanes of A at x. Similarly $x \in \partial A$ will be called *regular* if x is not a singular point of ∂A. Note that, by definition, μ_A simply is the image of $\mathcal{H}^{n-1} \llcorner \partial A$ by the Gauss map of ∂A.

As a consequence of the definition (1) of μ_A we have $\mu_A(S^{n-1}) = \mathcal{H}^{n-1}(\partial A)$ and

$$(2) \qquad \int_{S^{n-1}} y \; d\mu_A(y) = 0.$$

Moreover the fact that A has nonempty interior implies a certain nondegeneracy of the measure μ_A i.e. there exists no hyperplane in which μ_A is supported, more precisely:

$$(3) \qquad \int_{S^{n-1}} |(y_0, y)| d\mu_A(y) > 0, \text{ for any } y_0 \in S^{n-1}$$

For example, if A is a polyhedron, then $\mu_A = \sum \alpha_i \delta_{\nu_i}$, where ν_i is the unit outer normal vector to the face i, and α_i the corresponding surface area.

When A is a convex compact subset of $\mathbb{R}^n$ with nonempty interior, μ_A is called the *surface function* or *surface area measure* of the closed convex hypersurface ∂A. The notion of surface function, which essentially is an invention of Alexandrov around 1937, is a key tool to study curvature properties of convex bodies with a non smooth boundary.

Remark 1. Note that the surface area can be defined similarly in the degenerate case $\dim A < n$. In the following we will only consider convex bodies of full dimension, which is of course without loss of generality.

If the Gaussian curvature of the closed convex hypersurface ∂A is prescribed as a positive continuous function $K(.)$ of the unit exterior normal y then the surface function is given, for every Borel subset B of S^{n-1} by the formula:

$$\mu_A(B) = \int_B \frac{dy}{K(y)} \tag{4}$$

where dy denotes the Haar measure of S^{n-1}.

Let us remark that in the class of convex bodies, those (smooth) ones whose Gaussian curvature can be expressed as a positive continuous function of the unit exterior normal are exceptional (in the sense of Baire's category) as the striking results of Zamfirescu [15, 16] show.

1.2. **Minkowski and Alexandrov's theorems.** The classical Minkowski problem is the inverse problem of existence and uniqueness of a closed convex hypersurface with prescribed Gaussian curvature expressed as a function of the exterior normal unit vector. In 1937, Alexandrov has considered the following generalization of this inverse problem : given a measure $\mu \in \mathcal{M}_+(S^{n-1})$, can one find a convex closed hypersurface ∂A such that $\mu = \mu_A$? Alexandrov proved that conditions (2) and (3) appear as not only necessary but also as sufficient for the existence of such a convex hypersurface ∂A, moreover A is unique up to translations.

Theorem 1 (Alexandrov)**.** *Let μ be some positive measure on S^{n-1} which satisfies*

$$\int_{S^{n-1}} y d\mu(y) = 0, \quad \int_{S^{n-1}} |(y_0, y)| d\mu(y) > 0, \text{ for any } y_0 \in S^{n-1}$$

then there exists a convex body A, which is unique up to translation, for which μ is its surface function : $\mu = \mu_A$.

In view of (4), this result appears as a generalization of an earlier theorem proved by Minkowski in 1903 [9]:

Theorem 2 (Minkowski)**.** *Let K be a positive continuous function on S^{n-1} satisfying:*

$$\int_{S^{n-1}} \frac{y}{K(y)} d\mu(y) = 0$$

then there exists a convex body A, which is unique up to translation, for which $K(y)$ is the Gaussian curvature at any point $x \in \partial A$ with exterior unit normal y.

Theorems 1 and 2 are well-known : see the original articles of Minkowski [9], Alexandrov [1], and for a more modern presentation, for instance, Bakelman's book [2]. The usual strategy for proving theorem 1 is to prove the result for convex polyhedra (for convex polyhedra, i.e. when μ is a weighted sum of Dirac masses, the theorem was proved by Minkowski himself, in the case $n = 3$, in 1897, see [10]) and then to use an approximation argument to prove Alexandrov's theorem in its generality.

1.3. **Variational characterizations: how and why?** As a matter of fact, the classical proof of theorem 1 for polyhedra and Minkowski's original approach rely on an optimization problem, for which the Euler-Lagrange equation exactly yields the result. The purpose of the present article is firstly to answer the following natural question: given μ which satisfies the assumptions of theorem 1, is there a simple variational characterization of convex bodies A for which $\mu = \mu_A$? Secondly, can one give an elementary and direct (i.e. not using approximation by polyhedra) proof of theorem 1 using some extremum problem which generalizes that introduced by Minkowski?

Some motivations for obtaining and emphasizing a variational characterization are: investigation of numerical methods for solving Minkowski-Alexandrov inverse problem as well as the study of some shape optimization problems subject to a convexity constraint (see [4], [5]).

As far as I know, the known variational characterization of solutions of Minkowski-Alexandrov problem actually uses theorem 1 and *Minkowski's first inequality for convex bodies*. This characterization reads as : if A is a convex solution of $\mu_A = \mu$ then, up to an homothety, A minimizes the functional:

$$L \mapsto \int_{S^{n-1}} \sup_{x \in L} (x \cdot y)\, d\mu(y)$$

under the condition $\mathrm{vol}(L) = 1$ (see for instance the recent article of Gardner [8], where connections with the *Wulff* equilibrium shape of crystals are mentioned). The idea of the present work is to directly study a variational problem (not the one mentioned above, but a kind of dual one) and to deduce from an elementary study of this problem theorem 1 as well as a variational characterization.

In the following section, we introduce a variational problem related to Alexandrov's problem. More precisely, we shall show that this problem admits a unique (up to some normalizations) solution and that the Euler-Lagrange equation of this problem is equivalent to solving $\mu = \mu_A$ with A convex.

2. A variational problem related to Alexandrov-Minkowski's problem

Given a nonnegative Borel measure μ which satisfies (2) and (3), our goal is to find some *Alexandrov body* associated with μ i.e. a convex body A such that $\mu = \mu_A$.

To that end, we consider the following maximization problem:

$$\sup_{\varphi \in \mathcal{S}} J(\varphi) \tag{5}$$

where we define:

$$\mathcal{S} := \{\varphi \in C^0(S^{n-1}, \mathbb{R}) : \varphi \geq 0, \int_{S^{n-1}} \varphi d\mu = 1\} \tag{6}$$

$$J(\varphi) := \mathrm{vol}(A(\varphi)) \tag{7}$$

and:

$$A(\varphi) := \{x \in \mathbb{R}^n : x \cdot y \le \varphi(y), \text{ for all } y \in S^{n-1}\} \tag{8}$$

Let us first remark that the condition $\varphi \ge 0$ is equivalent to $0 \in A(\varphi)$ which is some normalization condition on $A(\varphi)$. Secondly, note that $J^{\frac{1}{n}}$ is concave (hence J is quasiconcave) on $\mathcal{S}$, indeed for $(\varphi, \psi) \in \mathcal{S}^2$ and $t \in [0,1]$, one obviously has $tA(\varphi) + (1-t)A(\psi) \subset A(t\varphi + (1-t)\psi)$, hence Brunn-Minkowski inequality yields:

$$J^{1/n}(t\varphi + (1-t)\psi) \ge tJ^{1/n}(\varphi) + (1-t)J^{1/n}(\psi)$$

and the inequality is strict unless $A(\varphi)$ and $A(\psi)$ are dilates or translates to each other. Concerning Brunn-Minkowski inequality and its numerous applications, we refer the reader to the books of Bakelman [2], Schneider [14] and to the recent inspiring article of Gardner [8].

Finally, given $\varphi \in \mathcal{S}$ define σ the support function of $A(\varphi)$:

$$\sigma(y) := \sup_{x \in A(\varphi)} x \cdot y, \text{ for all } y \in S^{n-1}$$

obviously one has $0 \le \sigma \le \varphi$ and, by Hahn-Banach's theorem, $A(\varphi) = A(\sigma)$. Let us define then

$$\widetilde{\varphi} := \frac{\sigma}{\int_{S^{n-1}} \sigma d\mu} \in \mathcal{S} \tag{9}$$

we have $J(\widetilde{\varphi}) \ge J(\varphi)$ and the inequality is strict unless $\varphi = \sigma = \widetilde{\varphi}$ on the support of μ. Therefore, if the supremum of (5) is attained at $\varphi \in \mathcal{S}$ one has $\varphi = \widetilde{\varphi}$ on the support of μ.

2.1. **Existence.** Given $e \in S^{n-1}$, we denote in the sequel by e^{++} the open half space $e^{++} := \{x \in \mathbb{R}^n : x \cdot e > 0\}$.

Lemma 1. *There exists $R > 0$ such that $A(\varphi) \subset B(0,R)$ for all $\varphi \in \mathcal{S}$.*

Proof. Assume on the contrary that there exists a sequence $(\varphi_k)_k \in \mathcal{S}^N$ and $x_k \in A_k := A(\varphi_k)$ such that $\lim_k |x_k| = +\infty$. Up to a subsequence, we may assume that $x_k/|x_k|$ converges to some $e \in S^{n-1}$:

$$x_k = |x_k|(e + \varepsilon_k), \text{ with } \lim_k \varepsilon_k = 0.$$

Define for $\delta \in (0,1)$:

$$S_\delta := \{y \in S^{n-1} : e \cdot y \ge \delta\}.$$

If $y \in S_\delta$, then we have

$$\varphi_k(y) \ge x_k \cdot y = |x_k|(e + \varepsilon_k) \cdot y \ge |x_k|(\delta - |\varepsilon_k|)$$

so that:

$$\int_{S^{n-1}} \varphi_k d\mu \ge |x_k|(\delta - |\varepsilon_k|)\mu(S_\delta) \tag{10}$$

Note now that since $S_\delta \uparrow S^{n-1} \cap e^{++}$ as $\delta \to 0^+$:

$$\lim_{\delta \to 0^+} \mu(S_\delta) = \mu(S^{n-1} \cap e^{++})$$

and $\mu(S^{n-1} \cap e^{++}) > 0$ since otherwise one would have $e \cdot y \leq 0$ μ-a.e., which together with (2) would imply $e \cdot y = 0$ μ-a.e., contradicting (3).

Let us now take $\delta \in (0,1)$ such that $\mu(S_\delta) > 0$ in (10), we get:

$$\lim_k \int_{S^{n-1}} \varphi_k d\mu = +\infty$$

which contradicts $\int_{S^{n-1}} \varphi_k d\mu = 1$ for every k. □

Proposition 1. *The maximization problem (5) admits at least one solution φ. If φ and ψ are two solutions of (5), then $A(\varphi)$ and $A(\psi)$ are translates to each other: there exists $x_0 \in -A(\varphi) \cap A(\psi)$ such that $A(\psi) = x_0 + A(\varphi)$ and $\psi(y) = \varphi(y) + x_0 \cdot y$ for all y in the support of μ.*

Remark 2. Note that if φ and ψ are two solutions of (5), then defining $\widetilde{\varphi}$ and $\widetilde{\psi}$ by (9), there exists $x_0 \in -A(\varphi) \cap A(\psi)$ such that $\widetilde{\psi}(y) = \widetilde{\varphi}(y) + x_0 \cdot y$ for all $y \in S^{n-1}$.

Proof. Let us first prove the existence part. Let $(\varphi_k)_k \in \mathcal{S}^N$ be a maximizing sequence of (5) and define $A_k := A(\varphi_k)$. Thanks to Lemma 1, we may assume, up to some subsequence that A_k converges to some compact set A for Hausdorff distance and also in the sense of Kuratowski (see for instance [13]):

$$A = \overline{\lim_k} A_k = \underline{\lim}_k A_k \tag{11}$$

Obviously we have $0 \in A$, A is convex and $\lim_k \mathrm{vol}(A_k) = \mathrm{vol}(A) = \sup_{\varphi \in \mathcal{S}} J(\varphi)$.

Let σ be the support function of A so that $A = A(\sigma)$. Let $x \in A$ and $y \in S^{n-1}$; using (11), there exists a sequence $x_k \in A_k$ converging to x so that $\underline{\lim}_k \varphi_k(y) \geq \underline{\lim}_k x_k \cdot y = x \cdot y$ hence $\sigma \leq \underline{\lim}_k \varphi_k$ so that $\int_{S^{n-1}} \sigma d\mu \leq 1$. If we define:

$$\widetilde{\varphi} := \frac{\sigma}{\int_{S^{n-1}} \sigma d\mu} \in \mathcal{S}$$

we have

$$\sup_{\varphi \in \mathcal{S}} J(\varphi) \geq J(\widetilde{\varphi}) = \mathrm{vol}(A) \left(\int_{S^{n-1}} \sigma d\mu \right)^{-n} \geq \sup_{\varphi \in \mathcal{S}} J(\varphi)$$

hence we have $\int_{S^{n-1}} \sigma d\mu = 1$, $\sigma \in \mathcal{S}$ and σ is a solution of (5).

Assume now that φ and ψ are two solutions of (5) and define $\widetilde{\varphi}$, $\widetilde{\psi}$ by (9). As already noted, $\widetilde{\varphi}$ and $\widetilde{\psi}$ also are solutions of (5) and $\varphi = \widetilde{\varphi}$, $\psi = \widetilde{\psi}$ on the support of μ. Now Brunn-Minkowski inequality implies that for all $t \in (0,1)$:

$$J(t\varphi + (1-t)\psi) \geq \sup_{\varphi \in \mathcal{S}} J(\varphi) = \mathrm{vol}(A(\varphi)) = \mathrm{vol}(A(\psi))$$

and the inequality is strict unless $A(\varphi)$ and $A(\psi)$ are translates to each other. Hence $A(\psi) = x_0 + A(\varphi)$ with $x_0 \in -A(\varphi) \cap A(\psi)$ since $0 \in A(\varphi) \cap A(\psi)$. This finally implies $\widetilde{\psi}(y) = \widetilde{\varphi}(y) + x_0 \cdot y$ for all $y \in S^{n-1}$, which ends the proof.

□

2.2. **Euler-Lagrange Equation.** We first need a preliminary result:

Lemma 2. *Let $g \in C^0(S^{n-1}, \mathbb{R}_+^*)$, let σ_g be the support function of $A(g)$:*

$$\sigma_g(\nu) := \sup_{x \in A(g)} x \cdot \nu, \text{ for all } \nu \in S^{n-1}$$

then for every regular point x of $\partial A(g)$, with outer unit normal $\nu(x)$ one has $\sigma_g(\nu(x)) = g(\nu(x))$. Let $\mu_g := \mu_{A(g)}$ be the surface function of $A(g)$ then we have:

$$\mu_g(\{\nu \in S^{n-1} : \sigma_g(\nu) \neq g(\nu)\}) = 0.$$

Proof. Let $x \in \partial A(g)$ and let $\nu \in S^{n-1}$ be such that $x \cdot \nu = \sigma_g(\nu)$. Assume that $\sigma_g(\nu) \neq g(\nu)$ hence $\sigma_g(\nu) < g(\nu)$.

First, we claim that there exists $\nu' \in S^{n-1}$ such that $x \cdot \nu' = g(\nu')$, otherwise, by compactness of S^{n-1} we would have:

$$\sup_{\nu' \in S^{n-1}} x \cdot \nu' - g(\nu') < 0$$

which would imply that x lies in the interior of $A(g)$; since it is not the case, there exists then $\nu' \in S^{n-1}$ such that $g(\nu') = \sigma_g(\nu') = x \cdot \nu'$, in particular $\nu' \neq \nu$.

This proves that x belongs to two different supporting hyperplanes of $A(g)$, with respective normal vectors ν and ν', x is therefore a singular point of $\partial A(g)$.

By definition of μ_g we have:

$$\mu_g(\{\nu \in S^{n-1} : \sigma_g(\nu) \neq g(\nu)\}) \leq \mathcal{H}^{n-1}(\{x \in \partial A(g) : x \text{ is singular}\})$$

since the rightmost member of this inequality is 0, we are done. □

Proposition 2. *Let $\varphi \in C^0(S^{n-1}, \mathbb{R}_+^*)$ and $f \in C^0(S^{n-1}, \mathbb{R})$, one has:*

$$\lim_{t \to 0^+} \frac{1}{t}\left[J(\varphi + tf) - J(\varphi)\right] = \int_{\partial A(\varphi)} f(\nu(x)) d\mathcal{H}^{n-1}(x) \tag{12}$$

where $\nu(x)$ denotes the exterior normal unit vector to $\partial A(\varphi)$ at $x \in \partial A(\varphi)$.

Proof. Define $A := A(\varphi)$ and $A_t := A(\varphi + tf)$. For $t > 0$ small enough $\varphi + tf > 0$ and $A_t \subset B_t$ with:

$$B_t := \{x \in \mathbb{R}^n : x.\nu(z) \leq \varphi(\nu(z)) + tf(\nu(z)), \text{ for every regular point } z \text{ of } \partial A\}$$

Using Lemma 2, if $z \in \partial A$ is regular, we have $z.\nu(z) = \varphi(\nu(z))$, so that:

$$B_t = \{x \in \mathbb{R}^n : (x - z).\nu(z) \leq tf(\nu(z)), \text{ for every regular point } z \text{ of } \partial A\}$$

we get then:

$$\text{vol}(A_t) \leq \text{vol}(B_t) = \text{vol}(A) + t \int_{\partial A(\varphi)} f(\nu(x)) d\mathcal{H}^{n-1}(x) + o(t) \tag{13}$$

so that:

$$\overline{\lim}_{t \to 0^+} \frac{1}{t}\left[J(\varphi + tf) - J(\varphi)\right] \leq \int_{\partial A} f(\nu(x)) d\mathcal{H}^{n-1}(x) \tag{14}$$

Similarly, we have

$$A \subset C_t := \{x \in \mathbb{R}^n : x.\nu(z) \leq \varphi(\nu(z)), \text{ for every regular point } z \text{ of } \partial A_t\}$$

Using Lemma 2, for every regular point z of ∂A_t, one has:

$$z \cdot \nu(z) = \varphi(\nu(z)) + tf(\nu(z))$$

which yields:

$$C_t = \{x \in \mathbb{R}^n : (x - z).\nu(z) \leq -tf(\nu(z)),\ \text{for every regular point } z \text{ of } \partial A_t\}$$

hence, we obtain:

$$\text{vol}(A) \leq \text{vol}(C_t) = \text{vol}(A_t) - t \int_{\partial A_t} f(\nu(x)) d\mathcal{H}^{n-1}(x) + o(t) \tag{15}$$

so that

$$\underline{\lim}_{t \to 0^+} \frac{1}{t} \left[J(\varphi + tf) - J(\varphi) \right] \geq \underline{\lim}_{t \to 0^+} \int_{\partial A_t} f(\nu(x)) d\mathcal{H}^{n-1}(x) \tag{16}$$

To achieve the proof, using (14) and (16) it is enough to prove that:

$$\underline{\lim}_{t \to 0^+} \int_{\partial A_t} f(\nu(x)) d\mathcal{H}^{n-1}(x) = \int_{\partial A} f(\nu(x)) d\mathcal{H}^{n-1}(x) \tag{17}$$

Denoting by μ_0 the surface function of A and by μ_t the surface function of A_t, to prove (17), it is enough to prove that μ_t converges weakly $*$ to μ_0 in $\mathcal{M}(S^{n-1})$ (where $\mathcal{M}(S^{n-1})$ denotes the space of Radon measures on S^{n-1}). The weak $*$ convergence of μ_t to μ_0 is a well-known result see for instance [2] and [14]. For the sake of completness however, we prefer to prove (17) directly, closely following arguments of Buttazzo and Guasoni (see [4]) based on a theorem of Reshetnyak (see [11]).

Denoting by χ_C the indicatrix function of a measurable set $C \subset \mathbb{R}^n$, it is straightforward to see that χ_{A_t} converges to χ_A in L^1 as t goes to 0. Define the vector measures of finite variation:

$$\tau_t := D\chi_{A_t},\ \text{for } t > 0, \text{ and } \tau_0 := D\chi_A$$

where $D\chi_C$ stands for the derivative in the sense of Schwartz's distributions of χ_C.

Using Lemma 3.1 of [4], the sequence τ_t is compact with respect to the convergence in variation, this obviously implies that τ_t converges to τ_0 with respect to the convergence in variation. Consider now the functional:

$$\tau \in \mathcal{M} \mapsto F(\tau) := \int_{\mathbb{R}^n} f\left(-\frac{d\tau}{d|\tau|}\right) d|\tau|$$

where $\mathcal{M}$ denotes the class of measures of finite variation with values in $\mathbb{R}^n$, $|\tau|$ denotes the variation measure associated to $\tau \in \mathcal{M}$ and $\frac{d\tau}{d|\tau|}$ denotes the Radon-Nikodym derivative of τ with respect to $|\tau|$. A theorem of Reshetnyak [11] states that F is continuous with respect to the convergence in variation, this yields:

$$\lim_{t \to 0^+} F(\tau_t) = F(\tau_0) \tag{18}$$

Since:

$$|\tau_t| = \mathcal{H}^{n-1} \llcorner \partial A_t \text{ and } \frac{d\tau_t}{d|\tau_t|} = -\nu_{\partial A_t},\ \text{for } t \geq 0$$

we have:

$$F(\tau_t) = \int_{\partial A_t} f(\nu(x)) d\mathcal{H}^{n-1}(x),\ \text{for } t > 0,\ \text{and } F(\tau_0) = \int_{\partial A} f(\nu(x)) d\mathcal{H}^{n-1}(x)$$

with (18), this finally establishes (17).

□

As a consequence, the Euler-Lagrange equation of (5) turns out to be equivalent (up to some normalizations) to finding an Alexandrov body associated with μ:

Theorem 3. *Let $\varphi \in \mathcal{S}$, then φ is a solution of (5) if and only if there exists $\lambda > 0$ such that $\mu_{A(\varphi)} = \lambda\mu$ i.e. $A(\varphi)$ is an Alexandrov body associated with $\lambda\mu$.*

Proof. Let φ be a solution of (5) as already noticed we may assume that $\varphi = \widetilde{\varphi}$. If 0 belongs to the interior of $A(\varphi)$ then $\varphi > 0$ and if $0 \in \partial A(\varphi)$ changing $A(\varphi)$ into $A(\varphi) + \varepsilon\nu(0)$ and $\varphi(y)$ into $\varphi(y) + \varepsilon\nu(0) \cdot y$ where $\nu(0)$ is the unit exterior vector to some supporting hyperplane at 0 to $A(\varphi)$ and $\varepsilon > 0$ is such that 0 is interior to $A(\varphi + \varepsilon\nu(0))$, we may assume in any case after this translation that $\varphi > 0$.

Let $f \in C^0(S^{n-1}, \mathbb{R})$ be such that $\int_{S^{n-1}} f d\mu = 0$, for small enough t we have $\varphi + tf \in \mathcal{S}$ so that, with proposition 2

$$\lim_{t \to 0^+} \frac{1}{t} \left[J(\varphi + tf) - J(\varphi) \right] = \int_{\partial A(\varphi)} f(\nu(x)) d\mathcal{H}^{n-1}(x) \leq 0$$

changing f into $-f$ we obtain $\int_{S^{n-1}} f d\mu_{A(\varphi)} = 0$ so that μ and $\mu_{A(\varphi)}$ are proportional : $\mu_{A(\varphi)} = \lambda\mu$ the fact that $\lambda > 0$ follows from $\int_{S^{n-1}} d\mu > 0$ and $\int_{S^{n-1}} d\mu_{A(\varphi)} > 0$.

Conversely assume that there exists $\lambda > 0$ such that $\mu_{A(\varphi)} = \lambda\mu$, without loss of generality we may assume again that $\varphi = \widetilde{\varphi} > 0$. Let $\psi \in \mathcal{S}$, since $J^{1/n}$ is concave and using proposition 2 we have

$$J^{1/n}(\psi) \leq J^{1/n}(\varphi) + \frac{1}{nJ^{\frac{n-1}{n}}(\varphi)} \int_{\partial A(\varphi)} (\psi - \varphi)(\nu(x)) d\mathcal{H}^{n-1}(x)$$

since:

$$\int_{\partial A(\varphi)} (\psi - \varphi)(\nu(x)) d\mathcal{H}^{n-1}(x) = \lambda \left(\int_{S^{n-1}} \psi d\mu - \int_{S^{n-1}} \varphi d\mu \right) = 0$$

we obtain that φ is a solution of (5).

□

Remark 3. Let us remark that λ in the previous result (a Lagrange multiplier associated with the constraint $\int_{S^{n-1}} \varphi d\mu = 1$) is unique i.e. does not depend on the maximizer φ. Indeed let $\varphi_i = \widetilde{\varphi}_i$, $i = 1, 2$ be two maximizers of (5), so that $\mu_{A(\varphi_i)} = \lambda_i\mu$, $i = 1, 2$. Note that:

$$\mathrm{vol}(A(\varphi_i)) = \frac{1}{n} \int_{\partial A(\varphi_i)} x.\nu(x) d\mathcal{H}^{n-1}(x) = \frac{1}{n} \int_{\partial A(\varphi_i)} \varphi_i(\nu(x)) d\mathcal{H}^{n-1}(x)$$

$$= \frac{\lambda_i}{n} \int_{S^{n-1}} \varphi_i d\mu = \frac{\lambda_i}{n}$$

since $\mathrm{vol}(A(\varphi_1)) = \mathrm{vol}(A(\varphi_2))$ we obtain:

$$\lambda_1 = \lambda_2 = n \sup_{\varphi \in \mathcal{S}} J(\varphi).$$

3. Link with Alexandrov's theorem

Let us shortly prove that theorem 3 actually implies Alexandrov's theorem (and Minkowski's as well in the special case where μ has a continuous positive density with respect to the Haar measure of S^{n-1}). Indeed, if φ is a solution of (5) then

$$A := \left(\frac{\mu(S^{n-1})}{\mathcal{H}^{n-1}(\partial A(\varphi))}\right)^{\frac{1}{n-1}} A(\varphi)$$

obviously is an Alexandrov body associated with μ.

Theorem 3 also implies uniqueness up to translations of Alexandrov's bodies. Assume there are two such convex bodies A_i, $\mu_{A_i} = \mu$ for $i = 1, 2$. Translating those sets if necessary assume $0 \in A_1 \cap A_2$, define σ_i as the support function of A_i and

$$\varphi_i := \frac{\sigma_i}{\int_{S^{n-1}} \sigma_i d\mu}, \text{ for } i = 1, 2$$

since φ_1 and φ_2 satisfy the requirement of theorem 3, they are both solutions of (5), we then have:

$$\frac{\mathrm{vol}(A_1)}{\left(\int_{S^{n-1}} \sigma_1 d\mu\right)^n} = \frac{\mathrm{vol}(A_2)}{\left(\int_{S^{n-1}} \sigma_2 d\mu\right)^n} \tag{19}$$

and $A_1/(\int_{S^{n-1}} \sigma_1 d\mu)$ and $A_2/(\int_{S^{n-1}} \sigma_2 d\mu)$ are translates to each other. Note now that

$$\mathrm{vol}(A_1) = \frac{1}{n}\int_{\partial A_1} x.\nu(x) d\mathcal{H}^{n-1}(x) = \frac{1}{n}\int_{\partial A_1} \sigma_1(\nu(x)) d\mathcal{H}^{n-1}(x) = \frac{1}{n}\int_{S^{n-1}} \sigma_1 d\mu$$

using a similar formula for A_2 and (19) we get $\int_{S^{n-1}} \sigma_1 d\mu = \int_{S^{n-1}} \sigma_2 d\mu$ so that A_1 and A_2 are translates to each other.

This actually proves Alexandrov's theorem.

Remark 4. We conclude this paper by two remarks. It is easy to check that the variational characterization provided by theorem 3 is equivalent to the *dual* one mentioned in section 1.3. More precisely, $\varphi \in \mathcal{S}$ is a solution of (5) if and only if $A(\varphi)$ minimizes the functional:

$$L \mapsto \int_{S^{n-1}} \sup_{x \in L} (x \cdot y)\, d\mu(y)$$

under the condition $\mathrm{vol}(L) = v_0$ where $v_0 := \sup_{\varphi \in \mathcal{S}} J(\varphi)$. Eventhough (5) and the previous problem are equivalent, it is clear that (5), as a very simple concave programming problem, is much easier to study.

Finally, we would like to mention a formal analogy with the variational approach used in this paper to prove Alexandrov's theorem and the method used by Gangbo [7] to prove Brenier's polar factorization theorem [3].

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