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THE HYERS-ULAM STABILITY OF A WEIGHTED COMPOSITION OPERATOR ON A UNIFORM ALGEBRA

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Dedicated to Professor Tadasi Huruya on his 60th birthday (Kanreki)

Abstract. For a weighted composition operator $uC_\varphi : f \mapsto u \cdot (f \circ \varphi)$ on a uniform algebra, we give a necessary and sufficient condition for uC_φ to have the Hyers-Ulam stability, in terms of the image of the level set $\{x : |u(x)| \geq r\}$ under φ.

1. Introduction and main result

Let $\mathcal{B}$ be a Banach space and let T be a mapping from $\mathcal{B}$ into itself. We say that T has the *Hyers-Ulam stability*, if there exists a constant K with the following property:

For any $g \in T(\mathcal{B})$, $\varepsilon > 0$ and $f \in \mathcal{B}$ satisfying $\|Tf - g\| \leq \varepsilon$, we can find an $f_0 \in \mathcal{B}$ such that $Tf_0 = g$ and $\|f - f_0\| \leq K\varepsilon$.

We call K a *HUS constant* for T, and denote the infimum of all HUS constants for T by K_T. These concepts are based on the research by Hyers [3] or Ulam [13], and are introduced in the paper [7]. One of their concrete examples may be found in the papers [6, 11, 12].

Throughout this paper, X is a compact Hausdorff space and $C(X)$ denotes the Banach algebra of all continuous complex functions on X with the supremum norm. Let $\mathcal{A}$ be a uniform algebra on X, that is, a uniformly closed subalgebra of $C(X)$ which contains the constants and separates the points of X. For any $u \in \mathcal{A}$, we put $S(u) = \{x \in X : u(x) \neq 0\}$. Fix a function $u \in \mathcal{A}$ and a selfmap φ of X which is continuous on $S(u)$. Then u and φ induce an operator uC_φ defined by

$$(uC_\varphi f)(x) = u(x)\, f(\varphi(x)) \qquad (x \in X)$$

for all $f \in \mathcal{A}$. If uC_φ maps $\mathcal{A}$ into itself, then uC_φ is a bounded linear operator on $\mathcal{A}$ and is called a *weighted composition operator* on $\mathcal{A}$. The weighted composition operators on various uniform algebras are studied by Contreras and Díaz-Madrigal [1], Kamowitz [4, 5], and many other mathematicians (see also [8, 10]). The book [9] is a monograph on this subject. In [11], the authors characterize the weighted composition operators on $C(X)$ which have the Hyers-Ulam stability. In this paper, we generalize this characterization by replacing $C(X)$ by a uniform algebra $\mathcal{A}$.

For any subset E of X, we put $\ker E = \{f \in \mathcal{A} : f(x) = 0 \text{ for all } x \in E\}$ and $\overline{E}^{\mathcal{A}} = \{x \in X : f(x) = 0 \text{ for all } f \in \ker E\}$. The set $\overline{E}^{\mathcal{A}}$ is nothing but the closure of E with respect to the hull-kernel topology on X. A closed subset F of X is called a *peak set* for $\mathcal{A}$, if there exists an $f \in \mathcal{A}$ such that $f(x) = 1$ for $x \in F$ and

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$|f(x)| < 1$ for $x \in X \setminus F$. The intersection of some collection of peak sets for $\mathcal{A}$ is called a *generalized peak set* for $\mathcal{A}$. A point $x \in X$ is called a *generalized peak point* for $\mathcal{A}$, if the singleton $\{x\}$ is a generalized peak set for $\mathcal{A}$. The set of all generalized peak points for $\mathcal{A}$ is referred to as the *strong boundary* or the *Choquet boundary* of $\mathcal{A}$ and is denoted by $\mathrm{Ch}(\mathcal{A})$. While, we denote by $\Gamma(\mathcal{A})$ the Shilov boundary of $\mathcal{A}$, which means the smallest closed subset F of X such that $\|f\| = \sup\{|f(x)| : x \in F\}$ for all $f \in \mathcal{A}$. For some familiar uniform algebra $\mathcal{A}$, it occurs that $\mathrm{Ch}(\mathcal{A}) = \Gamma(\mathcal{A})$. The book [2] is a nice reference on these topics.

The main result in this paper is the following:

Theorem. *Let $\mathcal{A}$ be a uniform algebra on X with $\mathrm{Ch}(\mathcal{A}) = \Gamma(\mathcal{A})$, and let uC_φ be a weighted composition operator on $\mathcal{A}$. Suppose that $\overline{\varphi(S(u))}^{\mathcal{A}}$ is a generalized peak set for $\mathcal{A}$. Then uC_φ has the Hyers-Ulam stability if and only if there exists a constant $r > 0$ such that*

$$\varphi(\{x \in X : |u(x)| \geq r\}) \supset \overline{\varphi(S(u))}^{\mathcal{A}} \cap \Gamma(\mathcal{A}). \tag{1}$$

Moreover, if R is the supremum of all r such that (1) *holds, then $K_{uC_\varphi} = 1/R$.*

2. Proof of Theorem

We begin with the general theory of the Hyers-Ulam stability. Let T be a bounded linear operator on a Banach space $\mathcal{B}$. We write $\mathcal{N}(T)$ for the kernel of T. By $\tilde{T}$, we denote the induced one-to-one operator from the quotient space $\mathcal{B}/\mathcal{N}(T)$ into $\mathcal{B}$ defined by

$$\tilde{T}(f + \mathcal{N}(T)) = Tf \qquad (f \in \mathcal{B}).$$

In [11], the authors investigate the relation of the Hyers-Ulam stability of T and the inverse operator $\tilde{T}^{-1}$ from $T(\mathcal{B})$ onto $\mathcal{B}/\mathcal{N}(T)$:

Lemma 1 ([11, Theorem 2])**.** *For a bounded linear operator T on a Banach space, the following statements are equivalent:*

- *T has the Hyers-Ulam stability.*
- *T has closed range.*
- *$\tilde{T}^{-1}$ is bounded.*

Moreover, we have $K_T = \|\tilde{T}^{-1}\|$.

For the proof of Theorem, we will need the following lemma:

Lemma 2. *Let $\mathcal{A}$ be a uniform algebra on X and let uC_φ be a weighted composition operator on $\mathcal{A}$. Suppose that $\overline{\varphi(S(u))}^{\mathcal{A}}$ is a generalized peak set for $\mathcal{A}$. Then we have*

$$\|f + \mathcal{N}(uC_\varphi)\| = \sup\left\{ |f(y)| : y \in \overline{\varphi(S(u))}^{\mathcal{A}} \cap \Gamma(\mathcal{A})\right\},$$

for all $f \in \mathcal{A}$.

Proof. Pick $f \in \mathcal{A}$ and put $\alpha = \sup\left\{ |f(y)| : y \in \overline{\varphi(S(u))}^{\mathcal{A}} \cap \Gamma(\mathcal{A})\right\}$. Choose $h \in \mathcal{N}(uC_\varphi)$ arbitrarily. Then $h(y) = 0$ for $y \in \varphi(S(u))$ and so for $y \in \overline{\varphi(S(u))}^{\mathcal{A}}$. Hence we have

$$\alpha = \sup\left\{ |f(y) + h(y)| : y \in \overline{\varphi(S(u))}^{\mathcal{A}} \cap \Gamma(\mathcal{A})\right\} \leq \|f + h\|.$$

Since $h \in \mathcal{N}(uC_\varphi)$ was arbitrary, it follows that $\alpha \le \|f + \mathcal{N}(uC_\varphi)\|$.

To verify $\|f + \mathcal{N}(uC_\varphi)\| \le \alpha$, choose $\varepsilon > 0$ arbitrarily. Put $F = \{y \in \Gamma(\mathcal{A}) : |f(y)| \ge \alpha + \varepsilon\}$. Then F is a closed set in X which does not intersect $\overline{\varphi(S(u))}^{\mathcal{A}}$. Since $\overline{\varphi(S(u))}^{\mathcal{A}}$ is a generalized peak set for $\mathcal{A}$, we can find a $g \in \mathcal{A}$ such that $\|g\| \le 1$, $g(x) = 1$ for $x \in \overline{\varphi(S(u))}^{\mathcal{A}}$ and $|g(x)| < 1$ for $x \in F$. Moreover, for some sufficiently large integer n, we have $|g(x)|^n < \varepsilon/(\|f\| + 1)$ for $x \in F$. Put $h = f(1 - g^n)$. It is clear that $h \in \mathcal{A}$ and $h(y) = 0$ for $y \in \varphi(S(u))$. This implies $h \in \mathcal{N}(uC_\varphi)$. While, we have

$$|f(x) - h(x)| = |f(x)|\,|g(x)|^n \le \begin{cases} \|f\| \cdot \frac{\varepsilon}{\|f\|+1} < \varepsilon & \text{if } x \in F \\ (\alpha+\varepsilon)\|g\|^n \le \alpha + \varepsilon & \text{if } x \in \Gamma(\mathcal{A}) \setminus F \end{cases}.$$

Hence

$$\|f + \mathcal{N}(uC_\varphi)\| \le \|f - h\| = \sup\{|f(x) - h(x)| : x \in \Gamma(\mathcal{A})\} \le \alpha + \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, we get $\|f + \mathcal{N}(uC_\varphi)\| \le \alpha$, concluding the proof. □

Proof of Theorem. Suppose that there exists an $r > 0$ which satisfies (1). Then we use Lemma 2 and compute as follows:

$$\begin{aligned} \|f + \mathcal{N}(uC_\varphi)\| &= \sup\{\,|f(y)| : y \in \overline{\varphi(S(u))}^{\mathcal{A}} \cap \Gamma(\mathcal{A})\} \\ &\le \sup\{\,|f(y)| : y \in \varphi(\{x \in X : |u(x)| \ge r\})\,\} \\ &= \sup\{\,|f(\varphi(x))| : |u(x)| \ge r\,\} \\ &= \sup\left\{\frac{1}{|u(x)|}|(uC_\varphi f)(x)| : |u(x)| \ge r\right\} \\ &\le \frac{1}{r}\,\sup\{\,|(uC_\varphi f)(x)| : |u(x)| \ge r\} \\ &\le \frac{1}{r}\,\|uC_\varphi f\|, \end{aligned}$$

for all $f \in \mathcal{A}$. Hence $\widetilde{uC_\varphi}^{-1}$ is bounded and $\|\widetilde{uC_\varphi}^{-1}\| \le 1/r$. According to Lemma 1, uC_φ has the Hyers-Ulam stability, and the "if" part is proved. Moreover, if r is taken all over the numbers satisfying (1), we get

$$\|\widetilde{uC_\varphi}^{-1}\| \le \frac{1}{R}. \tag{2}$$

Conversely, suppose that uC_φ has the Hyers-Ulam stability. By Lemma 1, $\widetilde{uC_\varphi}^{-1}$ is bounded. Choose $r > 0$ so that $\|\widetilde{uC_\varphi}^{-1}\| < 1/r$. Let us assume that

$$\varphi(\{x \in X : |u(x)| \ge r\}) \not\supset \overline{\varphi(S(u))}^{\mathcal{A}} \cap \Gamma(A).$$

Then there is a point $y_0 \in \overline{\varphi(S(u))}^{\mathcal{A}} \cap \Gamma(A)$ with $y_0 \notin \varphi(\{x \in X : |u(x)| \ge r\})$. Here $\varphi(\{x \in X : |u(x)| \ge r\})$ is a closed set, because $\{x \in X : |u(x)| \ge r\}$ is a compact subset of $S(u)$ and φ is continuous on $S(u)$. While y_0 is a generalized peak point for $\mathcal{A}$, because $y_0 \in \Gamma(\mathcal{A}) = \mathrm{Ch}(\mathcal{A})$. Hence we find a $g \in \mathcal{A}$ such that $\|g\| \le 1$, $g(y_0) = 1$ and $|g(y)| < 1$ for $y \in \varphi(\{x \in X : |u(x)| \ge r\})$. Moreover, for some large integer n, $|g(y)|^n < r/(\|u\|+1)$ for $y \in \varphi(\{x \in X : |u(x)| \geq r\})$. If we put $h = g^n$, then $h \in \mathcal{A}$ and

$$|(uC_\varphi h)(x)| = |u(x)|\,|g(\varphi(x))|^n \leq \begin{cases} \|u\| \cdot \frac{r}{\|u\|+1} < r & \text{if } |u(x)| \geq r \\ r\,\|g\|^n \leq r & \text{if } |u(x)| < r \end{cases},$$

namely, $\|uC_\varphi h\| \leq r$. Hence we use Lemma 2 to see that

$$\begin{aligned} 1 &= |g(y_0)|^n = |h(y_0)| \\ &\leq \sup\left\{\, |h(y)| : y \in \overline{\varphi(S(u))}^{\mathcal{A}} \cap \Gamma(\mathcal{A}) \right\} \\ &= \|h + \mathcal{N}(uC_\varphi)\| = \|u\widetilde{C}_\varphi^{-1}(uC_\varphi h)\| \\ &\leq \|u\widetilde{C}_\varphi^{-1}\|\,\|uC_\varphi h\| < \frac{1}{r} \cdot r = 1, \end{aligned}$$

which is a contradiction. Thus we obtain (1). The "only if" part is proved. Since (1) holds for any $r > 0$ with $\|u\widetilde{C}_\varphi^{-1}\| < 1/r$, it follows that $1/R \leq \|u\widetilde{C}_\varphi^{-1}\|$. Together with (2), we obtain $\|u\widetilde{C}_\varphi^{-1}\| = 1/R$, which implies $K_{uC_\varphi} = 1/R$ by Lemma 1. □

3. Applications

We first consider the case $\mathcal{A} = C(X)$. In this case, for each subset E of X, $\overline{E}^{C(X)}$ coincides with the usual closure of E in X. Also, every closed set in X is a generalized peak set for $C(X)$. Hence $\mathrm{Ch}(C(X)) = \Gamma(C(X)) = X$. These facts are easy consequences of Urysohn's lemma.

Let uC_φ be a weighted composition operator on $C(X)$. Then the above facts show that $\overline{\varphi(S(u))}^{C(X)}$ is always a generalized peak set for $C(X)$, and that the set $\overline{\varphi(S(u))}^{C(X)} \cap \Gamma(C(X))$ is the usual closure of $\varphi(S(u))$ in X. If $r > 0$, then $\varphi(\{x \in X : |u(x)| \geq r\})$ is closed in X, and we easily see that

$$\begin{aligned} &\varphi(\{x \in X : |u(x)| \geq r\}) \supset \overline{\varphi(S(u))}^{C(X)} \cap \Gamma(C(X)) \\ &\qquad \iff \varphi(\{x \in X : |u(x)| \geq r\}) = \varphi(S(u)). \end{aligned}$$

Thus we reach the $C(X)$-version of Theorem:

Corollary 1 ([11, Theorem 3]). *Let uC_φ be a weighted composition operator on $C(X)$. Then uC_φ has the Hyers-Ulam stability if and only if there exists a positive constant r such that*

$$\varphi(\{x \in X : |u(x)| \geq r\}) = \varphi(S(u)). \tag{3}$$

Moreover, if R is the supremum of all r such that (3) *holds, then $K_{uC_\varphi} = 1/R$.*

Next, consider the case that $\mathcal{A}$ is the disc algebra. Let $\mathbb{D}$ be the open unit disc in the complex plane and let $\overline{\mathbb{D}}$ be its closure. By $A(\mathbb{D})$, we denote the disc algebra, that is, the uniform algebra of all continuous complex functions on $\overline{\mathbb{D}}$ which are analytic on $\mathbb{D}$. It is known that both $\mathrm{Ch}(A(\mathbb{D}))$ and $\Gamma(A(\mathbb{D}))$ are the unit circle $\mathbb{T}$.

Let $u \in A(\mathbb{D})$ and put $S(u) = \{z \in \overline{\mathbb{D}} : u(z) \neq 0\}$. Let φ be a selfmap of $\overline{\mathbb{D}}$ which is continuous on $S(u)$ and analytic on $S(u) \cap \mathbb{D}$. Then we can define a weighted composition operator uC_φ on $A(\mathbb{D})$ as follows:

$$(uC_\varphi f)(z) = u(z)\, f(\varphi(z)) \qquad (z \in \overline{\mathbb{D}})$$

for all $f \in A(\mathbb{D})$. In case u is the zero function on $\overline{\mathbb{D}}$, or in case φ is constant on $S(u)$, then the induced operator uC_φ has closed range (see [8, Proposition 1]) and so uC_φ has the Hyers-Ulam stability by Lemma 1. Since it is convenient to exclude these two trivial cases, we assume that u is nonzero and φ is nonconstant on $S(u)$. Under this assumption, the induced operator uC_φ is said to be *nontrivial*. By using the open mapping theorem and the uniqueness theorem for analytic functions, we can easily see that $\varphi(S(u)\cap\mathbb{D})$ is a nonempty open set and that $\overline{\varphi(S(u))}^{A(\mathbb{D})} = \overline{\mathbb{D}}$, which is clearly a generalized peak set for $A(\mathbb{D})$. Also, we get $\overline{\varphi(S(u))}^{A(\mathbb{D})} \cap \Gamma(A(\mathbb{D})) = \mathbb{T}$. Noting that $\varphi(\mathbb{D})$ does not intersect $\mathbb{T}$ by the maximum modulus principle, we see the following equivalence:

$$\varphi(\{z \in \overline{\mathbb{D}} : |u(z)| \geq r\}) \supset \overline{\varphi(S(u))}^{A(\mathbb{D})} \cap \Gamma(A(\mathbb{D}))$$
$$\iff \varphi(\{z \in \mathbb{T} : |u(z)| \geq r\}) \supset \mathbb{T},$$

where $r > 0$. These observation yields the $A(\mathbb{D})$-version of Theorem:

Corollary 2. *Let uC_φ be a nontrivial weighted composition operator on $A(\mathbb{D})$. Then uC_φ has the Hyers-Ulam stability if and only if there exists a positive constant r such that*

$$\varphi(\{z \in \mathbb{T} : |u(x)| \geq r\}) \supset \mathbb{T}. \tag{4}$$

Moreover, if R is the supremum of all r such that (4) *holds, then $K_{uC_\varphi} = 1/R$.*

Combining this corollary with Lemma 1, we see the following fact: *For any nontrivial weighted composition operator uC_φ on $A(\mathbb{D})$, uC_φ has closed range if and only if there is an $r > 0$ satisfying* (4). This was proved in [8].

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