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A REAL ANALYTICITY RESULT FOR SYMMETRIC FUNCTIONS OF THE EIGENVALUES OF A DOMAIN DEPENDENT DIRICHLET PROBLEM FOR THE LAPLACE OPERATOR

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Abstract. Let Ω be an open connected subset of $\mathbb{R}^n$ for which the imbedding of the Sobolev space $W_0^{1,2}(\Omega)$ into the space $L^2(\Omega)$ is compact. We consider the Dirichlet eigenvalue problem for the Laplace operator in the open subset $\phi(\Omega)$ of $\mathbb{R}^n$, where ϕ is a Lipschitz continuous homeomorphism of Ω onto $\phi(\Omega)$. Then we prove a result of real analytic dependence for symmetric functions of the eigenvalues upon variation of ϕ.

1. Introduction.

This paper concerns the dependence of the Dirichlet eigenvalues for the Laplace operator upon domain perturbation.

To prove our results, we need to develop some preliminary abstract results for the dependence of the eigenvalues of a compact selfadjoint operator in Hilbert space, upon perturbation of both the scalar product and the operator. With this respect, we mention the Lipschitz continuity result of Cox [1], and that of [6].

Let $(H, < \cdot, \cdot >)$ be a real Hilbert space, which we shall consider as the 'environment' space. Then we consider a variable scalar product Q on H, and we denote by H_Q the Hilbert space H endowed with the scalar product Q. We shall study the dependence of the spectrum and of the projections onto the eigenspaces of a compact selfadjoint operator T acting in H_Q upon perturbation of Q, T. For each (Q, T), we assume the eigenvalues $\mu_j[T]$ of T in H_Q to be indexed by nonzero integer numbers, and we count each eigenvalue as many times as its geometric multiplicity (see section 2.) Then we consider a finite set F of indices, and we consider the set $\mathcal{A}[F]$ of the pairs (Q, T) for which T is compact and selfadjoint in H_Q and has the j-th eigenvalue for all j in F, and for which all the eigenvalues indexed by F are distinct from the eigenvalues with indices not in F (cf. (2.5).) As it is well known, for a fixed scalar product Q, the eigenvalues of T depend with continuity upon T. For a fixed scalar product, Rellich and Nagy (cf. *e.g.*, Rellich [12, Thm. 1, p. 33]) have proved that if $\{T_\eta\}_{\eta\in\mathbb{I}}$ is a one-parameter real analytic family of compact selfadjoint operators in H, with $\mathbb{I}$ an interval of $\mathbb{R}$ containing 0, and if T_0 has a certain eigenvalue $\tilde{\mu}$ of multiplicity $m \geq 1$, then there exist m real analytic functions $\mu_1(\cdot), \ldots, \mu_m(\cdot)$ defined in a neighborhood of 0 such that $\tilde{\mu} = \mu_1(0) = \cdots = \mu_m(0)$, and such that $\mu_1(\eta), \ldots, \mu_m(\eta)$ are eigenvalues for T_η. In other words, Rellich and Nagy have shown that for a *one parameter* real analytic family of operators, an

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eigenvalue of multiplicity m 'splits' analytically into m eigenvalues of the perturbed operator. One may think of extending such analyticity result for the dependence of the eigenvalues of the operators of a one parameter family, to a result of real analytic dependence of the eigenvalues of T upon T itself, *i.e.* by thinking T as an independent variable in the Banach space $\mathcal{K}_s(H,H)$ of selfadjoint compact operators equipped with its usual operator norm. Although such idea may be enhanced by the above mentioned continuity result for the dependence of the eigenvalues of T upon T itself, easily constructed examples in finite dimension show that one cannot expect that the dependence of the eigenvalues of T upon T itself be differentiable or analytic. In this paper, we show that although the eigenvalues of T cannot be expected to depend real analytically on T, no matter whether Q is held fixed or not, the elementary symmetric functions of the eigenvalues of T indexed by the indices of F depend real analytically on $(Q,T)\in\mathcal{A}[F]$. We denote by $\mathcal{Q}\left(H^2,\mathbb{R}\right)$, the set of continuous and coercive scalar products on H. The set $\mathcal{Q}\left(H^2,\mathbb{R}\right)$ can be shown to be open in the linear space $\mathcal{B}_s\left(H^2,\mathbb{R}\right)$ of symmetric bilinear forms. Instead, the set of pairs (Q,T) for which T is a compact selfadjoint operator in H_Q is neither a linear subspace nor an open subset of $\mathcal{B}_s\left(H^2,\mathbb{R}\right)\times\mathcal{L}\left(H,H\right)$. Thus, a first difficulty is to clarify in what sense our symmetric functions of the eigenvalues of T for (Q,T) in $\mathcal{A}[F]$ depend real analytically on (Q,T). As we shall see in Theorem 2.30, one can take $(\tilde{Q},\tilde{T})$ in $\mathcal{A}[F]$ and extend locally the symmetric functions of the eigenvalues of T for (Q,T) in $\mathcal{A}[F]$ to an open neighborhood of $(\tilde{Q},\tilde{T})$ in $\mathcal{B}_s\left(H^2,\mathbb{R}\right)\times\mathcal{L}\left(H,H\right)$. To do so, we need to introduce the orthogonal projection $P_F[Q,T]$ of H_Q onto the space generated by the eigenvectors of T relative to the eigenvalues of T indexed by the indices of F, and show that $P_F[\cdot,\cdot]$ can be extended locally to an open neighborhood of $(\tilde{Q},\tilde{T})$ in $\mathcal{B}_s\left(H^2,\mathbb{R}\right)\times\mathcal{L}\left(H,H\right)$. Once the extension of $P_F[\cdot,\cdot]$ is shown to exist, we can deduce the existence of the extension for the symmetric functions by an argument of reduction to finite dimension (cf. Proof of Theorem 2.30.)

Next, we apply our abstract results to a concrete situation. We shall consider the dependence of the eigenvalues of the Laplace operator $-\Delta$ upon domain perturbation. We fix an open subset Ω of $\mathbb{R}^n$ such that the Sobolev space $W_0^{1,2}(\Omega)$ is compactly imbedded in $L^2(\Omega)$, and we consider the Dirichlet eigenvalue problem for the Laplace operator in the open subset $\phi(\Omega)$ of $\mathbb{R}^n$, where ϕ is a Lipschitz continuous homeomorphism of Ω onto $\phi(\Omega)$. Then we consider the eigenvalue problem

$$-\Delta v=\lambda v \qquad \text{in } \phi(\Omega), \tag{1.1}$$

with Dirichlet boundary conditions. Problem (1.1) has been defined on the ϕ-dependent domain $\phi(\Omega)$, and we shall transform it into a problem on Ω. To do so, we consider the Sobolev space $W_0^{1,2}(\Omega)$ obtained by taking the closure of the space $\mathcal{D}(\Omega)$ of the C^∞ functions with compact support in Ω in the Sobolev space $W^{1,2}(\Omega)$ of distributions in Ω which have weak derivatives up to the first order in $L^2(\Omega)$, endowed with its usual norm (cf. (3.1).) By the Poincaré inequality, one has an equivalent norm in $W_0^{1,2}(\Omega)$ by taking the energy norm $\left\{\int_\Omega |Du|^2\,dx\right\}^{1/2}$, for all $u\in W_0^{1,2}(\Omega)$. We denote by $w_0^{1,2}(\Omega)$ the space $W_0^{1,2}(\Omega)$ with the scalar product associated to the energy norm. The space $w_0^{1,2}(\Omega)$ will play the role of our 'environment' space H. Then we introduce the 'variable' scalar product

$$Q_\phi[u_1,u_2]=\int_\Omega Du_1(D\phi)^{-1}(D\phi)^{-t}Du_2^t|\mathrm{det}D\phi|\,dx \qquad \forall u_1,u_2\in W_0^{1,2}(\Omega)\,,$$

and we denote by $w_{0,\phi}^{1,2}(\Omega)$ the space $W_0^{1,2}(\Omega)$ endowed with the scalar product Q_ϕ. Under our assumptions on ϕ, the function u belongs to $w_0^{1,2}(\Omega)$ if and only if the function $u\circ\phi^{(-1)}$ belongs to $w_0^{1,2}(\phi(\Omega))$. Furthermore, the imbedding of $W_0^{1,2}(\phi(\Omega))$ into $L^2(\phi(\Omega))$ is compact. Thus the operator $-\Delta$ is well known to be an isomorphism of $W_0^{1,2}(\phi(\Omega))$ onto its dual space $W^{-1,2}(\phi(\Omega))$. Since $L^2(\phi(\Omega))$ in naturally included in $W^{-1,2}(\phi(\Omega))$, and $W_0^{1,2}(\phi(\Omega))$ is (compactly) imbedded in $L^2(\phi(\Omega))$, it can be shown that for all $u\in w_0^{1,2}(\Omega)$, there exists one and only one element $T_\phi u\in w_0^{1,2}(\Omega)$ such that $(T_\phi u)\circ\phi^{(-1)}\in w_0^{1,2}(\phi(\Omega))$ and

$$-\Delta\left((T_\phi u)\circ\phi^{(-1)}\right)=u\circ\phi^{(-1)} \qquad \text{in } \phi(\Omega).$$

Hence, we can consider the operator T_ϕ of $w_{0,\phi}^{1,2}(\Omega)$ to itself, and one can easily show that T_ϕ is selfadjoint in $w_{0,\phi}^{1,2}(\Omega)$. Instead, one cannot expect that T_ϕ be selfadjoint in $w_0^{1,2}(\Omega)$. Thus it becomes clear that if one wishes to preserve selfadjointness, then one has to consider different scalar products in $w_0^{1,2}(\Omega)$ for different ϕ's. We will take $w_0^{1,2}(\Omega)$ as our environment space H, and $w_{0,\phi}^{1,2}(\Omega)$ will play the role of H_Q, and we will apply the abstract results introduced above.

The paper is organized as follows. Section 2 provides the abstract results mentioned above. Section 3 contains the applications of the results of Section 2 to the Dirichlet eigenvalue problem for the Laplace operator.

2. Analyticity of the symmetric functions of the eigenvalues

We first introduce some technical preliminaries and notation. Let $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Z}$ be real Banach spaces. We denote by $\mathcal{L}(\mathcal{X},\mathcal{Y})$ the Banach space of linear and continuous maps of $\mathcal{X}$ to $\mathcal{Y}$ endowed with its usual norm of the uniform convergence on the unit sphere of $\mathcal{X}$. We denote by $\mathcal{B}(\mathcal{X}\times\mathcal{Y},\mathcal{Z})$ the space of the bilinear and continuous maps of $\mathcal{X}\times\mathcal{Y}$ to $\mathcal{Z}$ endowed with the norm of the uniform convergence on the cross product of the unit sphere of $\mathcal{X}$ and of the unit sphere of $\mathcal{Y}$. We say that the space $\mathcal{X}$ is continuously imbedded in the space $\mathcal{Y}$ provided that $\mathcal{X}$ is a linear subspace of $\mathcal{Y}$, and that the inclusion map is continuous. We denote by $\mathbb{Z}$ the set of integer numbers, and by $\mathbb{N}$ the set of natural numbers including 0. The inverse function of an invertible function f is denoted $f^{(-1)}$, as opposed to the reciprocal of a complex-valued function g, or the inverse of a matrix A, which are denoted g^{-1} and A^{-1}, respectively. If $A\equiv(a_{rs})_{r,s=1,\dots,n}$ is an $n\times n$ matrix with real entries, we set $|A|\equiv\left\{\sum_{r,s=1}^n a_{rs}^2\right\}^{1/2}$, and we denote by A^t the transpose matrix of A. If A is invertible, we set $A^{-t}\equiv\left(A^{-1}\right)^t$. Let $(H,<\cdot,\cdot>)$ be a real Hilbert space. Let $\|\cdot\|$ denote the norm associated to the scalar product $<\cdot,\cdot>$ on H, and $\dim(H)$ denote the possibly infinite dimension of H. We denote by H_Q the linear space H endowed with a scalar product Q defined on H. We denote by $\|\cdot\|_Q$ the norm associated to the scalar product Q on H. We denote by I the identity operator in H. We denote by $\mathcal{K}(H,H)$ the real Banach subspace of $\mathcal{L}(H,H)$ of those elements T which are compact, *i.e.*, which map bounded subsets of H to subsets of H with compact closure. We denote by $\mathcal{K}_s(H_Q,H_Q)$ the real Banach subspace of $\mathcal{K}(H_Q,H_Q)$ of those elements T such that $Q[Tu,v]=Q[u,Tv]$ for all $u,v\in H_Q$. As it is well known, if $T\in\mathcal{K}_s(H_Q,H_Q)$, then there exists a subset $\sigma[T]$ of $\mathbb{R}$, named the spectrum, with $\sigma[T]$ finite or countable, such that $\mathbb{R}\setminus\sigma[T]$ is the set of μ such that the operator $T-\mu I$ is a linear homeomorphism of H_Q. It is also well-known that all the elements μ of $\sigma[T]\setminus\{0\}$ are eigenvalues of T, *i.e.*, the null space $\mathrm{Ker}(T-\mu I)\neq\{0\}$. Furthermore, 0 is the only possible accumulation point of $\sigma[T]$. We call multiplicity of an eigenvalue μ, the dimension of the space $\mathrm{Ker}(T-\mu I)$. We denote by $j^+[T]$ the (possibly infinite) number of elements of $\sigma[T]\cap]0,+\infty[$, each counted with its multiplicity, and we denote by $j^-[T]$ the (possibly infinite) number of elements of $\sigma[T]\cap]-\infty,0[$, each counted with its multiplicity. We also set $J^+[T]\equiv\{j\in\mathbb{Z}:\,1\leq j\leq j^+[T]\}$, $J^-[T]\equiv\{j\in\mathbb{Z}:\,-j^-[T]\leq j\leq -1\}$. Then there exists a uniquely determined function $j\mapsto\mu_j[T]$ of $J[T]\equiv J^-[T]\cup J^+[T]$ to $\mathbb{R}\setminus\{0\}$ such that $j\mapsto\mu_j[T]$ is decreasing on $J^-[T]$ and on $J^+[T]$, and such that

$$\sigma[T]\setminus\{0\}=\{\mu_j[T]:\,j\in J[T]\}\,,$$

and such that each eigenvalue is repeated as many times as its multiplicity. We shall require the imbedding of H_Q in H to be continuous, and thus that the scalar product Q be coercive on H. Thus we introduce the following Lemma concerning continuous bilinear forms on H, whose verification is straightforward.

Lemma 2.1. *Let H be a real Hilbert space. Let $\eta[\cdot]$ be the map of $\mathcal{B}\left(H^2,\mathbb{R}\right)$ to $\mathbb{R}$ defined by*

$$\eta[B]\equiv\inf\left\{\frac{B[u,u]}{\|u\|^2}\,:\,u\in H\setminus\{0\}\right\}\,,$$

for all $B\in\mathcal{B}\left(H^2,\mathbb{R}\right)$. Then we have

$$|\eta[B]|\leq\|B\|_{\mathcal{B}(H^2,\mathbb{R})},\qquad |\eta[B_1]-\eta[B_2]|\leq\|B_1-B_2\|_{\mathcal{B}(H^2,\mathbb{R})}\,,$$

for all $B,\ B_1,\ B_2\in\mathcal{B}\left(H^2,\mathbb{R}\right)$. In particular, the set $\left\{B\in\mathcal{B}\left(H^2,\mathbb{R}\right):\,\eta[B]>0\right\}$ is open in $\mathcal{B}\left(H^2,\mathbb{R}\right)$.

Since scalar products are bilinear and symmetric forms, we introduce the following notation

$$\mathcal{B}_s\left(H^2,\mathbb{R}\right)\equiv\left\{B\in\mathcal{B}\left(H^2,\mathbb{R}\right):\,B[u_1,u_2]=B[u_2,u_1]\quad\forall u_1,u_2\in H\right\}\,.$$

Clearly, $\mathcal{B}_s\left(H^2,\mathbb{R}\right)$ is a closed linear subspace of $\mathcal{B}\left(H^2,\mathbb{R}\right)$. Then the set of coercive elements of $\mathcal{B}_s\left(H^2,\mathbb{R}\right)$ is denoted

$$\mathcal{Q}\left(H^2,\mathbb{R}\right)\equiv\left\{B\in\mathcal{B}_s\left(H^2,\mathbb{R}\right):\,\eta[B]>0\right\}\,. \tag{2.2}$$

Now we observe that if Q is a scalar product on H, and if the imbedding of H_Q in H is a homeomorphism, then $Q\in\mathcal{Q}\left(H^2,\mathbb{R}\right)$, and that conversely, if $Q\in\mathcal{Q}\left(H^2,\mathbb{R}\right)$, then Q is a scalar product on H, and the identity of H_Q in H is a homeomorphism. We obviously have

$$\eta[Q]^{1/2}\|u\|\leq\|u\|_Q\leq\|Q\|^{1/2}_{\mathcal{B}(H^2,\mathbb{R})}\|u\|, \tag{2.3}$$

for all $u \in H$, and for all $Q \in \mathcal{Q}\left(H^2,\mathbb{R}\right)$. We also note that if Q belongs to $\mathcal{Q}\left(H^2,\mathbb{R}\right)$, then $\mathcal{L}\left(H_Q,H_Q\right)$ equals $\mathcal{L}\left(H,H\right)$ algebraically and topologically. Similarly, $\mathcal{K}\left(H_Q,H_Q\right)$ equals $\mathcal{K}\left(H,H\right)$ algebraically and topologically. Instead, $\mathcal{K}_s\left(H_Q,H_Q\right)$ may vary with $Q \in \mathcal{Q}\left(H^2,\mathbb{R}\right)$, although the topology of H_Q does not.

We now set

$$\mathcal{M} \equiv \left\{(Q,T) \in \mathcal{B}_s\left(H^2,\mathbb{R}\right) \times \mathcal{K}\left(H,H\right) :\, Q\left[Tu,v\right] = Q\left[u,Tv\right]\ \forall u,v \in H\right\} .$$

Clearly, $\mathcal{M}$ is a closed subset of $\mathcal{B}_s\left(H^2,\mathbb{R}\right) \times \mathcal{K}\left(H,H\right)$. Furthermore, the set

$$\begin{aligned}\mathcal{O} \equiv{}& \mathcal{M} \cap \left(\mathcal{Q}\left(H^2,\mathbb{R}\right) \times \mathcal{K}\left(H,H\right)\right)\\ &= \left\{(Q,T) \in \mathcal{Q}\left(H^2,\mathbb{R}\right) \times \mathcal{K}\left(H,H\right) :\, T \in \mathcal{K}_s\left(H_Q,H_Q\right)\right\}\end{aligned}$$

is obviously open in $\mathcal{M}$. For a more detailed analysis of the set $\mathcal{O}$, we refer to [8]. Unless otherwise specified, we think of $\mathcal{O}$ as endowed of a product norm of $\mathcal{B}_s\left(H^2,\mathbb{R}\right) \times \mathcal{K}\left(H,H\right)$.

By exploiting the variational formulas for the eigenvalues of a compact selfadjoint operator, one can prove the following (cf. [6, §2].)

Theorem 2.4. *Let H be a real Hilbert space. Let $j \in \mathbb{Z} \setminus \{0\}$, then the set*

$$\mathcal{A}_j \equiv \{(Q,T) \in \mathcal{O} :\, j \in J[T]\}$$

is open in $\mathcal{M}$. The function $\mu_j[\cdot]$ of $\mathcal{A}_j$ to $\mathbb{R}$ which takes $(Q,T) \in \mathcal{A}_j$ to $\mu_j[T]$ is continuous.

Now we consider a certain finite subset F of $\mathbb{Z} \setminus \{0\}$, and the set of pairs (Q,T) for which $F \subseteq J[T]$ and for which the eigenvalues $\mu_j[T]$ with $j \in F$ do not equal any of the eigenvalues $\mu_l[T]$ of T with $l \in J[T] \setminus F$. Thus we introduce the following notation.

(2.5)

$$\mathcal{A}[F] \equiv \{(Q,T) \in \mathcal{O} :\, j \in J[T]\ \forall j \in F, \mu_l[T] \notin \{\mu_j[T] :\, j \in F\}\ \forall l \in J[T] \setminus F\} .$$

By Theorem 2.4, the functions $\mu_j[\cdot]$ are continuous on $\mathcal{A}[F]$, for all $j \in F$, and $\mathcal{A}[F]$ is open in $\mathcal{M}$.

For each finite subset F of $\mathbb{Z} \setminus \{0\}$, and $(Q,T) \in \mathcal{A}[F]$, we define the orthogonal projection $P_F[Q,T]$ of H_Q onto the subspace $E[T,F]$ of H_Q generated by the set

$$\{u \in H_Q :\, Tu = \mu u,\ \text{for some}\ \mu \in \{\mu_j[T] :\, j \in F\}\} .$$

Then we have the following obvious statement.

Proposition 2.6. *Let H be a real Hilbert space. Let F be a finite subset of $\mathbb{Z} \setminus \{0\}$. Let $(Q,T) \in \mathcal{A}[F]$. Then $E[T,F]$ has dimension equal to the number $|F|$ of elements of F, and $P_F[Q,T]$ satisfies the following system*

$$\left\{\begin{array}{ll}(I - P_F[Q,T]) \circ T \circ P_F[Q,T] = 0 & \text{in } \mathcal{L}(H,H),\\ Q\left[a - P_F[Q,T](a), P_F[Q,T](b)\right] = 0 & \forall a,b \in H.\end{array}\right. \tag{2.7}$$

In particular, $E[T,F]$ is an invariant subspace of H for T.

Our goal is now to analyze the regularity of the dependence of $P_F[Q,T]$ upon the pair (Q,T). To do so, we need to introduce a result of Kato, and thus some preliminaries.

Let $\hat{H}$ be the complexified space of H. The complexified operator $\hat{T}$ of T is the complex linear operator in $\hat{H}$ defined by $\hat{T}[u+iv] \equiv T[u] + iT[v]$ for all $u, v \in H$. The complexified scalar product $\hat{Q}$ of Q on $\hat{H}$ is defined by

$$\hat{Q}[u_1 + iv_1, u_2 + iv_2] \equiv Q[u_1, u_2] + Q[v_1, v_2] + i(Q[v_1, u_2] - Q[u_1, v_2]) \,,$$

for all u_1, u_2, v_1, $v_2 \in H$. As usual, we set $\mathrm{Re}(u+iv) \equiv u$, $\mathrm{Im}(u+iv) \equiv v$, for all u, $v \in H$. We denote by $\hat{I}$ the identity operator in the complexified space $\hat{H}$, and by $\mathcal{L}_{\mathbb{C}}\left(\hat{H}, \hat{H}\right)$ the space of continuous complex-linear maps in $\hat{H}$. If $S \in \mathcal{L}_{\mathbb{C}}\left(\hat{H}, \hat{H}\right)$, then the resolvent $\rho_{\mathbb{C}}[S]$ of S denotes the set of $\mu \in \mathbb{C}$ such that $\left(S - \mu\hat{I}\right)$ is a linear homeomorphism of $\hat{H}$. Then $\sigma_{\mathbb{C}}[S]$ denotes the spectrum $\mathbb{C} \setminus \rho_{\mathbb{C}}[S]$ of S. Then we have the following classical result (see Kato [4, III, §6, and pp. 276, 277].)

Theorem 2.8. *Let H be a real Hilbert space, $Q \in \mathcal{Q}\left(H^2, \mathbb{R}\right)$. Let $T \in \mathcal{L}(H, H)$. Let $\tilde{\mu}$ be an isolated point of $\sigma_{\mathbb{C}}[T]$. Let $r > 0$ be smaller than the distance of $\tilde{\mu}$ from $\sigma_{\mathbb{C}}[\hat{T}] \setminus \{\tilde{\mu}\}$. Let*

$$\gamma_r(\theta) \equiv \tilde{\mu} + re^{i\theta} \qquad \forall \theta \in [0, 2\pi] \,.$$

Then the operator

$$P[T, \tilde{\mu}] \equiv -\frac{1}{2\pi i} \int_{\gamma_r} \left(\hat{T} - \xi\hat{I}\right)^{(-1)} d\xi \,, \tag{2.9}$$

is a projection in $\hat{H}$, i.e., $\hat{H} = \left(\hat{I} - P[T, \tilde{\mu}]\right)\left[\hat{H}\right] \oplus P[T, \tilde{\mu}]\left[\hat{H}\right]$, although such direct sum is not necessarily orthogonal. The operator $P[T, \tilde{\mu}]$ is continuous. The operator $\hat{T}$ maps the image of $P[T, \tilde{\mu}]$ to itself and the image of $\hat{I} - P[T, \tilde{\mu}]$ to itself. The spectrum of the restriction of $\hat{T}$ to the image of $P[T, \tilde{\mu}]$ coincides with $\{\tilde{\mu}\}$, and the spectrum of the restriction of $\hat{T}$ to the image of $\hat{I} - P[T, \tilde{\mu}]$ coincides with $\sigma_{\mathbb{C}}\left[\hat{T}\right] \setminus \{\tilde{\mu}\}$. If we further assume that T is normal, i.e. $T^*T = TT^*$ where T^* is the adjoint of T in H_Q, then the projection $P[T, \tilde{\mu}]$ is orthogonal with respect to the scalar product $\hat{Q}$ in $\hat{H}$. In particular, if T is selfadjoint, i.e. if $T = T^*$ in H_Q, then $\tilde{\mu}$ is real, and the restriction of $\mathrm{Re}P[T, \tilde{\mu}]$ *to H_Q is the orthogonal projection of H_Q onto the kernel* $\mathrm{Ker}\,(T - \tilde{\mu}I)$*, and T maps the image of* $\mathrm{Re}P[T, \tilde{\mu}]$ *to itself, and the image of $I - \mathrm{Re}P[T, \tilde{\mu}]$ to itself.*

By exploiting the above Theorem of Kato [4], one can prove the following (cf. [6, §2], Kato [4, Thm. 3.16, p. 212].)

Theorem 2.10. *Let H be a real Hilbert space. Let F be a finite subset of $\mathbb{Z} \setminus \{0\}$. Then the map P_F of $\mathcal{A}[F]$ to $\mathcal{L}(H, H)$ which takes $(Q, T) \in \mathcal{A}[F]$ to $P_F[Q, T]$ is continuous.*

We are interested in a result of real analytic dependence of $P_F[Q, T]$ upon (Q, T). However, $\mathcal{A}[F]$ is an open subset of $\mathcal{M}$, but $\mathcal{M}$ does not have a linear structure. Then we will be looking for real analytic extensions of $P_F[\cdot, \cdot]$ to open subsets of some Banach space. As we shall see shortly, a natural Banach space to consider is $\mathcal{B}_s\left(H^2, \mathbb{R}\right) \times \mathcal{L}(H, H)$. Then one could exploit the right hand side of (2.9) to define an extension of $P_F[\cdot, \cdot]$. However, if T is not selfadjoint (or at least normal), the right hand side of (2.9) is not an orthogonal projection and does not necessarily satisfy system (2.7). By exploiting a different avenue, we can prove the existence of an extension of $P_F[\cdot,\cdot]$, which satisfies a weaker version of (2.7) (see (2.12) below.) Thus, we introduce the following.

Theorem 2.11. *Let H be a real Hilbert space. Let $(\tilde{Q},\tilde{T}) \in \mathcal{Q}\left(H^2,\mathbb{R}\right) \times \mathcal{L}\left(H,H\right)$. Let $\tilde{\mu} \in \mathbb{R}\setminus\{0\}$. Let $\tilde{E} \equiv \mathrm{Ker}\left(\tilde{T}-\tilde{\mu}I\right) \neq \{0\}$. Let $\tilde{P}$ be the orthogonal projection in $H_{\tilde{Q}}$ onto $\tilde{E}$. If $\tilde{T}-\tilde{\mu}I$ is a linear homeomorphism of the orthogonal space $\tilde{E}^{\perp,\tilde{Q}}$ of $\tilde{E}$ in $H_{\tilde{Q}}$ onto $\tilde{E}^{\perp,\tilde{Q}}$, then there exist an open neighborhood $\tilde{\mathcal{Q}}$ of $\tilde{Q}$ in $\mathcal{Q}\left(H^2,\mathbb{R}\right)$, and an open neighborhood $\tilde{\mathcal{U}}$ of $\tilde{T}$ in $\mathcal{L}\left(H,H\right)$, and an open neighborhood $\tilde{\mathcal{V}}$ of $\tilde{P}$ in $\mathcal{L}\left(H,H\right)$, and a real analytic map $P^\sharp[\cdot,\cdot]$ of $\tilde{\mathcal{Q}}\times\tilde{\mathcal{U}}$ to $\tilde{\mathcal{V}}$ such that the graph of $P^\sharp[\cdot,\cdot]$ coincides with the set of triples (Q,T,P) of $\tilde{\mathcal{Q}}\times\tilde{\mathcal{U}}\times\tilde{\mathcal{V}}$ such that the following system holds*

$$
\begin{cases} (I-P)\circ T\circ P_{|\tilde{E}} = 0 & \text{in } \mathcal{L}(\tilde{E},H), \\ Q\left[a-P(a),P(b)\right]=0 & \forall (a,b)\in \tilde{E}^{\perp,\tilde{Q}}\times H. \end{cases} \tag{2.12}
$$

Proof. To prove the Theorem, we shall now recast system (2.12) into an abstract equation and apply the Implicit Function Theorem for real analytic operators. To do so, we introduce the map Λ of $\mathcal{Q}\left(H^2,\mathbb{R}\right)\times\mathcal{L}\left(H,H\right)\times\mathcal{L}\left(H,H\right)$ to $\mathcal{L}\left(\tilde{E},H\right)\times\mathcal{B}\left(\tilde{E}^{\perp,\tilde{Q}}\times H,\mathbb{R}\right)$ by setting

$$
\Lambda[Q,T,P] \equiv \left((I-P)\circ T\circ P_{|\tilde{E}}, Q\left[I-P,P\right]\right),
$$

for all $(Q,T,P)\in\mathcal{Q}\left(H^2,\mathbb{R}\right)\times\mathcal{L}\left(H,H\right)\times\mathcal{L}\left(H,H\right)$. Furthermore, the differential of Λ with respect to the variable P at $(\tilde{Q},\tilde{T},\tilde{P})$ is delivered by the formula

$$
\begin{aligned} & d_P\Lambda\left[\tilde{Q},\tilde{T},\tilde{P}\right](\dot{P}) \\ & \qquad = \left(-\dot{P}\circ\tilde{T} + \left(I-\tilde{P}\right)\circ\tilde{T}\circ\dot{P}, \tilde{Q}\left[I-\tilde{P},\dot{P}\right]-\tilde{Q}\left[\dot{P},\tilde{P}\right]\right), \end{aligned}
$$

for all $\dot{P}\in\mathcal{L}\left(H,H\right)$. We now prove that $d_P\Lambda\left[\tilde{Q},\tilde{T},\tilde{P}\right]$ is a bijection. It suffices to show that for each $(M,S)\in\mathcal{L}\left(\tilde{E},H\right)\times\mathcal{B}\left(\tilde{E}^{\perp,\tilde{Q}}\times H,\mathbb{R}\right)$, there exists a unique $\dot{P}\in\mathcal{L}\left(H,H\right)$ such that

$$
\begin{cases} -\dot{P}\circ\tilde{T}+\left(I-\tilde{P}\right)\circ\tilde{T}\circ\dot{P}=M & \text{in } \mathcal{L}(\tilde{E},H), \\ \tilde{Q}\left[a_1,\dot{P}[b_1+b_2]\right]-\tilde{Q}\left[\dot{P}[a_1],b_2\right] & \\ \qquad = S[a_1,b_1+b_2] & \forall a_1,b_1\in\tilde{E}^{\perp,\tilde{Q}}, b_2\in\tilde{E}\,. \end{cases} \tag{2.13}
$$

Obviously, $\tilde{T}=\tilde{\mu}I$ on $\tilde{E}$. Consequently, $\left(I-\tilde{P}\right)\circ\tilde{T}\circ\tilde{P}\circ\dot{P}=0$. Furthermore, $\tilde{T}-\tilde{\mu}I$ maps $\tilde{E}^{\perp,\tilde{Q}}$ to itself. In particular, $\tilde{T}$ maps $\tilde{E}^{\perp,\tilde{Q}}$ to itself, and

$$
\left(I-\tilde{P}\right)\circ\tilde{T}\circ\left(I-\tilde{P}\right)\circ\dot{P}=\tilde{T}\circ\left(I-\tilde{P}\right)\circ\dot{P} \qquad \forall\dot{P}\in\mathcal{L}\left(H,H\right).
$$

Thus by setting

$$\dot{P}_1 \equiv \left(I-\tilde{P}\right)\circ \dot{P}_{|\tilde{E}},\ \dot{P}_2 \equiv \tilde{P}\circ \dot{P}_{|\tilde{E}},\ \dot{P}_3 \equiv \left(I-\tilde{P}\right)\circ \dot{P}_{|\tilde{E}^{\perp,\tilde{Q}}},\ \dot{P}_4 \equiv \tilde{P}\circ \dot{P}_{|\tilde{E}^{\perp,\tilde{Q}}}\,,$$

the first equation of (2.13) can be rewritten as

$$-\tilde{\mu}\dot{P}_2 - \tilde{\mu}\dot{P}_1 + \tilde{T}\circ \dot{P}_1 = \tilde{P}\circ M + \left(I-\tilde{P}\right)\circ M\,,$$

or equivalently as

$$\begin{cases} -\tilde{\mu}\dot{P}_2 = \tilde{P}\circ M & \text{in } \mathcal{L}(\tilde{E},H), \\ \left(\tilde{T}-\tilde{\mu}I\right)\circ \dot{P}_1 = \left(I-\tilde{P}\right)\circ M & \text{in } \mathcal{L}(\tilde{E},H). \end{cases} \tag{2.14}$$

In particular, it follows that $\dot{P}_1$ and $\dot{P}_2$ are uniquely determined by M. Indeed,

$$\dot{P}_1 = \left[\left(\tilde{T}-\tilde{\mu}I\right)_{|\tilde{E}^{\perp,\tilde{Q}}}\right]^{(-1)} \circ \left(I-\tilde{P}\right)\circ M, \qquad \dot{P}_2 = -\tilde{\mu}^{-1}\tilde{P}\circ M\,. \tag{2.15}$$

Now we note that

$$\tilde{Q}\left[a_1, \dot{P}_4[b_1]\right] = \tilde{Q}\left[a_1, \dot{P}_2[b_2]\right] = \tilde{Q}\left[\dot{P}_3[a_1], b_2\right] = 0,$$

for all $a_1, b_1 \in \tilde{E}^{\perp,\tilde{Q}}$, $b_2 \in \tilde{E}$. Then it follows that the second equation of (2.13) can be written as follows

$$\tilde{Q}\left[a_1, \dot{P}_3[b_1] + \dot{P}_1[b_2]\right] - \tilde{Q}\left[\dot{P}_4[a_1], b_2\right] = S[a_1, b_1 + b_2]\,,$$

for all $a_1, b_1 \in \tilde{E}^{\perp,\tilde{Q}}$, $b_2 \in \tilde{E}$. Then by setting $b_2 = 0$, it follows that the second equation of (2.13) is equivalent to the following system

$$\begin{cases} \tilde{Q}\left[a_1, \dot{P}_3[b_1]\right] = S[a_1, b_1] \\ \tilde{Q}\left[a_1, \dot{P}_1[b_2]\right] - \tilde{Q}\left[\dot{P}_4[a_1], b_2\right] = S[a_1, b_2]\,, \end{cases} \tag{2.16}$$

for all $a_1, b_1 \in \tilde{E}^{\perp,\tilde{Q}}$, $b_2 \in \tilde{E}$. By the Riesz-Frechét Representation Theorem for the dual of the Hilbert Space $\left(\tilde{E}^{\perp,\tilde{Q}}, \tilde{Q}\right)$, the first equation of (2.16) determines uniquely $\dot{P}_3$. By (2.15), we know that $\dot{P}_1$ is a uniquely determined linear and continuous map of $\tilde{E}$ to $\tilde{E}^{\perp,\tilde{Q}}$. Thus the bilinear form $\tilde{Q}\left[\cdot, \dot{P}_1[\cdot]\right] - S[\cdot,\cdot]$ is continuous on $\tilde{E}^{\perp,\tilde{Q}} \times \tilde{E}$. Thus, by applying the Riesz-Frechét Representation Theorem to the dual space of $\left(\tilde{E}, \tilde{Q}\right)$, it follows that the second equation of (2.16) determines uniquely $\dot{P}_4$. Thus we have proved that $\dot{P}_1, \ldots, \dot{P}_4$ are uniquely determined by M and S. Accordingly, $d_P\Lambda$ is a bijection. Then the statement follows by applying the Implicit Function Theorem in its formulation for real analytic operators (cf. *e.g.*, Prodi and Ambrosetti [11, Thm. 11.6]) to equation $\Lambda[Q,T,P] = 0$ at the point $(\tilde{Q}, \tilde{T}, \tilde{P})$. □

Remark 2.17. Concerning the statement of Theorem 2.11, we note that we have not assumed that T be normal or selfadjoint for T close to $\tilde{T}$, an assumption which would guarantee that the projector delivered by the real part of the integral representation (2.9) be orthogonal, and map H onto the eigenspace relative to $\tilde{\mu}$ for

$T = \tilde{T}$. Accordingly, we could not deduce the result of Theorem 2.11 by exploiting the classical result of Kato, i.e., Theorem 2.8. On the other hand, the operator $P^\sharp[Q,T]$ for (Q,T) in $\tilde{\mathcal{Q}} \times \tilde{\mathcal{U}}$ does not have an explicit representation as the real part of that in (2.9) of Kato.

Theorem 2.18. *Let H be a real Hilbert space. Let F be a finite nonempty subset of $\mathbb{Z} \setminus \{0\}$. Let $(\tilde{Q}, \tilde{T}) \in \mathcal{A}[F]$. Then there exist an open neighborhood $\tilde{\mathcal{W}}$ of $(\tilde{Q}, \tilde{T})$ in $\mathcal{Q}\left(H^2, \mathbb{R}\right) \times \mathcal{L}(H,H)$, and a real analytic operator $P_F^\sharp$ of $\tilde{\mathcal{W}}$ to $\mathcal{L}(H,H)$ such that $P_F^\sharp[Q,T] = P_F[Q,T]$ for all $(Q,T) \in \tilde{\mathcal{W}} \cap \mathcal{A}[F]$.*

Proof. We first prove the statement in the specific case in which there exists $\tilde{\mu} \in \mathbb{R}$ such that $\tilde{\mu} = \mu_j[\tilde{T}]$ for all $j \in F$. Under such assumption, $\tilde{\mu}$ has multiplicity $|F|$, and $\tilde{E} = E[\tilde{T}, F]$ coincides with the eigenspace of $\tilde{\mu}$. By well known properties of compact selfadjoint operators in Hilbert space, $\tilde{T} - \tilde{\mu} I$ is a linear homeomorphism of $\tilde{E}^{\perp, \tilde{Q}}$ onto itself. Then there exist neighborhoods $\tilde{\mathcal{Q}}$, $\tilde{\mathcal{U}}$, $\tilde{\mathcal{V}}$, and a real analytic map $P_F^\sharp[\cdot,\cdot]$ as in Theorem 2.11. Clearly $P_F^\sharp[\tilde{Q}, \tilde{P}] = P_F[\tilde{Q}, \tilde{P}]$. By continutity of $P_F[\cdot,\cdot]$ on $\mathcal{A}[F]$ (cf. Theorem 2.10), and possibly by shrinking $\tilde{\mathcal{Q}} \times \tilde{\mathcal{U}}$, we can assume that $P_F[Q,T] \in \tilde{\mathcal{V}}$ for all $(Q,T) \in \left(\tilde{\mathcal{Q}} \times \tilde{\mathcal{U}}\right) \cap \mathcal{A}[F]$. Then by Proposition 2.6, and by Theorem 2.11, we must have $P_F[Q,T] = P_F^\sharp[Q,T]$ for all $(Q,T) \in \left(\tilde{\mathcal{Q}} \times \tilde{\mathcal{U}}\right) \cap \mathcal{A}[F]$. We now consider the case in which the eigenvalues $\mu_j[\tilde{T}]$ are not necessarily coincident. Then we can assume that there exist a finite subset F^* of $\mathbb{N}$, and a family $(F_l)_{l \in F^*}$ such that $F = \cup_{l \in F^*} F_l$, and such that $\mu_j[\tilde{T}]$ assumes a common value $\tilde{\mu}_l$ for all $j \in F_l$, and that $\tilde{\mu}_{l_1} \neq \tilde{\mu}_{l_2}$ if $l_1, l_2 \in F^*$, $l_1 \neq l_2$. Clearly,

$$\left\{\mu_j[\tilde{T}] : j \in F\right\} = \{\tilde{\mu}_l : l \in F^*\} .$$

Now let $\mathcal{W}^*$ be a neighborhood of $(\tilde{Q}, \tilde{T})$ in $\mathcal{Q}\left(H^2, \mathbb{R}\right) \times \mathcal{L}(H,H)$ such that

$$\mathcal{W}^* \cap \mathcal{A}[F] \subseteq \cap_{l \in F^*} \mathcal{A}[F_l] .$$

We obviously have

$$E[T,F] = \oplus_{l \in F^*} E[T, F_l]$$

for all $(Q,T) \in \mathcal{W}^* \cap \mathcal{A}[F]$. Now, if l_1, $l_2 \in F^*$, $l_1 \neq l_2$, then we have $\tilde{\mu}_{l_1} \neq \tilde{\mu}_{l_2}$, and accordingly $E[T, F_{l_1}]$ and $E[T, F_{l_2}]$ are mutually orthogonal in H_Q, and

$$P_F[Q,T] = \sum_{l \in F^*} P_{F_l}[Q,T] \qquad \forall (Q,T) \in \mathcal{W}^* \cap \mathcal{A}[F] . \tag{2.19}$$

(cf. *e.g.*, Taylor and Lay [14, Thm. 12.8, Ch. IV].) Since $\mu_j[\tilde{T}] = \tilde{\mu}_l$ for all $j \in F_l$, then the previous part of the proof ensures that for each $l \in F^*$ there exists a neighborhood $\mathcal{W}_l$ of $(\tilde{Q}, \tilde{T})$ in $\mathcal{Q}\left(H^2, \mathbb{R}\right) \times \mathcal{L}(H,H)$, and a real analytic operator $P_l^\sharp[\cdot,\cdot]$ of $\mathcal{W}_l$ to $\mathcal{L}(H,H)$ such that

$$P_l^\sharp[Q,T] = P_{F_l}[Q,T] \qquad \forall (Q,T) \in \mathcal{W}_l \cap \mathcal{A}[F_l] .$$

Now we set $\tilde{\mathcal{W}} \equiv \mathcal{W}^* \cap (\cap_{l \in F^*} \mathcal{W}_l)$, and

$$P_F^\sharp[Q,T] \equiv \sum_{l \in F^*} P_l^\sharp[Q,T] \qquad \forall (Q,T) \in \tilde{\mathcal{W}} .$$

By (2.19), $P_F^\sharp$ and $\tilde{\mathcal{W}}$ satisfy the properties required in the statement. □

Next we show that one can locally choose an orthonormal basis of $E[T,F]$ depending real analytically on (Q,T).

Proposition 2.20. *Let H be a real Hilbert space. Let F be a finite subset of $\mathbb{Z}\setminus\{0\}$, $(\tilde{Q},\tilde{T}) \in \mathcal{A}[F]$. Let $\{\tilde{u}_j : j \in F\}$ be an orthonormal basis of $E[\tilde{T},F]$ in $H_{\tilde{Q}}$. Then there exist an open neighborhood $\mathcal{W}^o$ of $(\tilde{Q},\tilde{T})$ in $\mathcal{Q}\left(H^2,\mathbb{R}\right)\times\mathcal{L}\left(H,H\right)$ contained in the neighborhood $\tilde{\mathcal{W}}$ of Theorem 2.18, and $|F|$ real analytic operators $u_j[\cdot,\cdot]$ $\forall j \in F$ of $\mathcal{W}^o$ to H such that*

- (i) *$\{u_j[Q,T] : j \in F\}$ is an orthonormal set in H_Q, for all $(Q,T) \in \mathcal{W}^o$.*
- (ii) *$\{u_j[Q,T] : j \in F\}$ is an orthonormal basis of the image of $P_F^\sharp[Q,T]$, which coincides with $E[T,F]$, in H_Q, for all $(Q,T) \in \mathcal{W}^o \cap \mathcal{A}[F]$.*
- (iii) *$u_j[\tilde{Q},\tilde{T}] = \tilde{u}_j$ for all $j \in F$.*

Proof. Obviously, $P_F[\tilde{Q},\tilde{T}]$ restricts the identity on $E[\tilde{T},F]$. A simple contradiction argument shows that possibly by shrinking the neighborhood $\tilde{\mathcal{W}}$ of Theorem 2.18, the restriction of $P^\sharp[Q,T]$ to $E[\tilde{T},F]$ is injective for all $(Q,T) \in \tilde{\mathcal{W}}$. Thus the set $\left\{P_F^\sharp[Q,T](\tilde{u}_j) : j \in F\right\}$ is a linearly independent set contained in the image of $P_F^\sharp[Q,T]$. In particular, if $(Q,T) \in \tilde{\mathcal{W}}\cap\mathcal{A}[F]$, then the image of $P_F^\sharp[Q,T] = P_F[Q,T]$ has dimension $|F|$, and $\left\{P_F^\sharp[Q,T](\tilde{u}_j) : j \in F\right\}$ is a basis for the image of $P_F^\sharp[Q,T]$. Thus it suffices to define $\{u_j[Q,T] : j \in F\}$ to be the orthonormal system obtained by applying the Gram-Schmidt procedure to $\left\{P_F^\sharp[Q,T](\tilde{u}_j) : j \in F\right\}$ in the Hilbert space H_Q (cf. *e.g.*, Taylor and Lay [14, Thm. 6.5, Ch. II].) It is immediate to recognize that $u_j[Q,T]$ depends real analytically on $(Q,T) \in \tilde{\mathcal{W}}$, for all $j \in F$. □

Then we have the following technical statement that enables us to reduce our eigenvalue problem to finite dimension.

Proposition 2.21. *Let H be a real Hilbert space. Let F be a finite subset of $\mathbb{Z}\setminus\{0\}$. Let $(\tilde{Q},\tilde{T}) \in \mathcal{A}[F]$. Let $\left\{\tilde{u}_1,\ldots,\tilde{u}_{|F|}\right\}$ be an orthonormal basis of $E[\tilde{T},F]$ in $H_{\tilde{Q}}$, and $\{u_j[Q,T] : j = 1,\ldots,|F|\}$ be as in Proposition 2.20. Let $\mathcal{S}$ be the map of $\mathcal{W}^o$ to the set $M_{|F|}(\mathbb{R})$ of $|F|\times|F|$-matrices with real entries, defined by*

$$\mathcal{S}[Q,T] \equiv (\mathcal{S}_{hk}[Q,T])_{h,k=1,\ldots,|F|} \equiv \left(Q\left[T\left[u_k[Q,T]\right],u_h[Q,T]\right]\right)_{h,k=1,\ldots,|F|}\,, \tag{2.22}$$

for all $(Q,T) \in \mathcal{W}^o$. Then $\mathcal{S}[\cdot,\cdot]$ is real analytic, and $\mathcal{S}[Q,T]$ is symmetric for all $(Q,T) \in \mathcal{W}^o \cap \mathcal{A}[F]$. Furthermore, if $(Q,T) \in \mathcal{W}^o \cap \mathcal{A}[F]$, then the numbers $\mu_j[T]$ for $j \in F$ are the eigenvalues of $\mathcal{S}[Q,T]$ counted with their multiplicity. Finally, if we further assume that $\mu_j[\tilde{T}]$ assumes a common value $\tilde{\mu}$ for all $j \in F$, then the differential of $\mathcal{S}[\cdot,\cdot]$ at $(\tilde{Q},\tilde{T})$ is delivered by the formula

$$d\mathcal{S}[\tilde{Q},\tilde{T}](\dot{Q},\dot{T}) = \left(\tilde{Q}\left[\dot{T}\left[\tilde{u}_k\right],\tilde{u}_h\right]\right)_{h,k=1,\ldots,|F|} \qquad \forall(\dot{Q},\dot{T}) \in \mathcal{B}_s\left(H^2,\mathbb{R}\right)\times\mathcal{L}\left(H,H\right). \tag{2.23}$$

Proof. The analyticity of $\mathcal{S}$ follows immediately by Proposition 2.20. If $(Q,T) \in \mathcal{W}^o \cap \mathcal{A}[F]$, then $\mathcal{S}[Q,T]$ is clearly the matrix of $T_{|E[T,F]}$ with respect to the basis $\{u_j[Q,T] : j = 1,\dots,|F|\}$, and accordingly its eigenvalues counted with their multiplicity are the numbers $\mu_j[T]$ for $j \in F$. The symmetry of $\mathcal{S}[Q,T]$ follows by the selfadjointness of T in H_Q for $(Q,T) \in \mathcal{W}^o \cap \mathcal{A}[F]$. We now prove the formula for the differential of $\mathcal{S}$ at $(\tilde{Q},\tilde{T})$. By assumption, we have

$$Q\left[u_k[Q,T],u_h[Q,T]\right] = \delta_{hk} \qquad \forall (Q,T) \in \mathcal{W}^o, \tag{2.24}$$

for all $h,k = 1,\dots,|F|$, where δ_{hk} denotes the Kronecker symbol defined by $\delta_{hk} = 1$ if $h = k$, $\delta_{hk} = 0$ if $h \neq k$. By differentiating equation (2.24) with respect to (Q,T) at the point $(\tilde{Q},\tilde{T})$, we obtain

$$\begin{aligned}\dot{Q}\left[u_k[\tilde{Q},\tilde{T}],u_h[\tilde{Q},\tilde{T}]\right] + \tilde{Q}\left[du_k[\tilde{Q},\tilde{T}](\dot{Q},\dot{T}),u_h[\tilde{Q},\tilde{T}]\right]& \\ +\tilde{Q}\left[u_k[\tilde{Q},\tilde{T}],du_h[\tilde{Q},\tilde{T}](\dot{Q},\dot{T})\right] &= 0\,,\end{aligned} \tag{2.25}$$

for all $(\dot{Q},\dot{T}) \in \mathcal{B}_s\left(H^2,\mathbb{R}\right) \times \mathcal{L}\left(H,H\right)$, $h,k = 1,\dots,|F|$. Furthermore,

$$\tilde{T}\left[u_k[\tilde{Q},\tilde{T}]\right] = \tilde{\mu} u_k[\tilde{Q},\tilde{T}]\,, \tag{2.26}$$

for all $k \in \{1,\dots,|F|\}$. Then by differentiating $\mathcal{S}_{hk}$ at $(\tilde{Q},\tilde{T})$, and by exploiting (2.25) and (2.26), we obtain (2.23). □

We now present a variant of the Rellich-Nagy Theorem, which we employ later. The original result of Rellich and Nagy holds for fixed scalar products. Our variant holds for variable scalar products.

Theorem 2.27. *Let H be a real Hilbert space. Let F be a finite subset of $\mathbb{Z}\setminus\{0\}$. Let $(\tilde{Q},\tilde{T}) \in \mathcal{A}[F]$ be such that $\mu_j[\tilde{T}]$ assumes a common value $\tilde{\mu}$ for all $j \in F$. Let $\mathcal{W}^0$ be as in Proposition 2.21. Let $\mathbb{I}$ be an open interval of the real line containing* 0. *Let $\{(Q(\epsilon),T(\epsilon))\}_{\epsilon\in\mathbb{I}}$ be a real analytic family in $\mathcal{W}^o \cap \mathcal{A}[F]$, with $(Q(0),T(0)) = (\tilde{Q},\tilde{T})$. Then, possibly shrinking $\mathbb{I}$, there exists a family $\{\zeta_j(\cdot)\}_{j\in F}$ of real analytic functions of $\mathbb{I}$ to $\mathbb{R}$ such that for each $\epsilon \in \mathbb{I}$ there exists a bijection σ of F to itself with $\zeta_{\sigma(j)}(\epsilon) = \mu_j[T(\epsilon)]$ for all $j \in F$.*

Proof. Let $\mathcal{S}\left[Q(\epsilon),T(\epsilon)\right]$ be the matrix of (2.22) for $(Q,T) = (Q(\epsilon),T(\epsilon))$. By assumption, and by Proposition 2.21, the entries of $\mathcal{S}\left[Q(\epsilon),T(\epsilon)\right]$ depend real analytically upon $\epsilon \in \mathbb{I}$. Since $\mathcal{S}\left[Q(\epsilon),T(\epsilon)\right]$ is the matrix associated to the linear operator $T(\epsilon)$ in the space $E[T(\epsilon),F]$, which is generated by the eigenvectors of $T(\epsilon)$ relative to the eigenvalues $\{\mu_j[T(\epsilon)] : j \in F\}$, it follows that the numbers $\mu_j[T(\epsilon)]$ for $j \in F$ coincide with the eigenvalues of the matrix $\mathcal{S}\left[Q(\epsilon),T(\epsilon)\right]$ counted with their multiplicity, for all $\epsilon \in \mathbb{I}$. Thus the result follows by the classical Rellich-Nagy Theorem (cf. *e.g.*, Rellich [12, Thm. 1, p. 33]) applied to the real analytic family of symmetric matrices $\left(\mathcal{S}\left[Q(\epsilon),T(\epsilon)\right]\right)_{\epsilon\in\mathbb{I}}$. □

By the above variant of the Rellich-Nagy Theorem, we can prove the following Corollary, where we also compute the derivatives of the branches of the eigenvalues which split by a multiple eigenvalue.

Corollary 2.28. *Let the assumptions of Theorem 2.27 hold. Let* I *denote the identity matrix in* $M_{|F|}(\mathbb{R})$*. Let* $\mathcal{S}\left[Q(\epsilon),T(\epsilon)\right]$ *be the symmetric matrix of (2.22) computed at* $(Q(\epsilon),T(\epsilon))$*, for all* $\epsilon\in\mathbb{I}$*. Let* $\left\{\tilde u_1,\dots,\tilde u_{|F|}\right\}$ *be an orthonormal basis for* $E[\tilde T,F]$ *with respect to the scalar product* $\tilde Q$ *on* H*. Then the following statements hold.*

(i) *Let* $j_0\in F$*. Then there exist* $j_1,\ j_2\in F$*,* $\delta>0$ *such that* $\mu_{j_0}[T(\epsilon)]=\zeta_{j_1}(\epsilon)$ *for* $0\le\epsilon<\delta$*,* $\mu_{j_0}[T(\epsilon)]=\zeta_{j_2}(\epsilon)$ *for* $-\delta<\epsilon\le 0$*. In particular, the function* $\epsilon\mapsto\mu_{j_0}[T(\epsilon)]$ *has right and left derivatives at* 0*, and the set of all such right and left derivatives as* j_0 *ranges in* F *coincides with the set* $\left\{\zeta_j'(0)\,:\,j\in F\right\}$*.*

(ii) *If* $\mu\in\mathbb{R}$ *and* $\epsilon\in\mathbb{I}$*, then*

$$\det\left(\mu\mathrm{I}-\mathcal{S}[Q(\epsilon),T(\epsilon)]\right)=\Pi_{j\in F}\left(\mu-\mu_j[T(\epsilon)]\right)=\Pi_{j\in F}\left(\mu-\zeta_j(\epsilon)\right)\,. \tag{2.29}$$

(iii) *Let* $\dot{\mathcal{S}}_0$ *be the matrix* $\left(\tilde Q\left[\frac{dT(\epsilon)}{d\epsilon}_{|\epsilon=0}\left[\tilde u_k\right],\tilde u_h\right]\right)_{h,k=1,\dots,|F|}$*. Then* $\dot{\mathcal{S}}_0$ *is symmetric and*

$$\det\left(\nu\mathrm{I}-\dot{\mathcal{S}}_0\right)=\Pi_{j\in F}(\nu-\zeta_j'(0))$$

for all $\nu\in\mathbb{R}$*.*

Proof. We first prove statement (*i*). Since the functions $\{\zeta_j(\cdot)\}_{j\in F}$ are real analytic, there exists $\delta>0$ such that in the set $]-\delta,\delta[\setminus\{0\}$, any two of such functions are either identical, or equal for no value of ϵ. Now let $\epsilon^\sharp\in]0,\delta[$ be fixed. By assumption, and by Theorem 2.27, there exists j_1 such that $\mu_{j_0}[T(\epsilon^\sharp)]=\zeta_{j_1}(\epsilon^\sharp)$. Then by Theorem 2.27, and by the continuity of the function $\epsilon\mapsto\mu_{j_0}[T(\epsilon)]$, and by the connectivity of $]0,\delta[$, it easily follows that $\mu_{j_0}[T(\epsilon)]=\zeta_{j_1}(\epsilon)$, for all $\epsilon\in]0,\delta[$. Similarly, one can argue for $\epsilon\in]-\delta,0[$. Statement (*ii*) is an immediate consequence of Theorems 2.21 and 2.27. We now prove statement (*iii*). We note that $\zeta_j(\epsilon)=\zeta_j(0)+\epsilon\zeta_{j,1}(\epsilon)$, $\mathcal{S}[Q(\epsilon),T(\epsilon)]=\tilde\mu\mathrm{I}+\epsilon\mathcal{S}_1(\epsilon)$ for $\zeta_{j,1}(\cdot)$, $\mathcal{S}_1(\cdot)$ real analytic functions of $\mathbb{I}$ to $\mathbb{R}$, $M_{|F|}(\mathbb{R})$ respectively, with $\zeta_{j,1}(0)=\zeta_j'(0)$, $\mathcal{S}_1(0)=\dot{\mathcal{S}}_0$ (see (2.23).) Then we fix $\nu\in\mathbb{R}$, and we replace μ, ζ_j, $\mathcal{S}[Q(\epsilon),T(\epsilon)]$ in the first and last term of (2.29) with $\tilde\mu+\nu\epsilon$, $\zeta_j(0)+\epsilon\zeta_{j,1}(\epsilon)$, $\tilde\mu\mathrm{I}+\epsilon\mathcal{S}_1(\epsilon)$, and deduce the validity of (*iii*). □

Then we have the following.

Theorem 2.30. *Let* H *be a real Hilbert space,* F *a finite nonempty subset of* $\mathbb{Z}\setminus\{0\}$*. Let*

$$\begin{aligned}
M_{F,1}[T] &= \sum_{j_1\in F}\mu_{j_1}[T]\\
\dots &= \dots\\
M_{F,s}[T] &= \sum_{j_1,\dots,j_s\in F,\ j_1<\dots<j_s}\mu_{j_1}[T]\cdots\mu_{j_s}[T]\qquad\forall s\in\{1,\dots,|F|\}\,,\\
\dots &= \dots\\
M_{F,|F|}[T] &= \Pi_{j\in F}\,\mu_j[T]\,,
\end{aligned}$$

for all $(Q,T)\in\mathcal{A}[F]$*, be the elementary symmetric functions of the eigenvalues* $\mu_j[T]$ *indexed by* $j\in F$*. Let* $(\tilde Q,\tilde T)\in\mathcal{A}[F]$*. Then there exist an open neighborhood*

$\tilde{\mathcal{W}}$ of $(\tilde{Q},\tilde{T})$ in $\mathcal{Q}\left(H^2,\mathbb{R}\right)\times\mathcal{L}(H,H)$, and real analytic functions $M^\sharp_{F,s}[\cdot,\cdot]$, $s=1,\dots,|F|$ of $\tilde{\mathcal{W}}$ to $\mathbb{R}$ such that

$$M^\sharp_{F,s}[Q,T]=M_{F,s}[T] \tag{2.31}$$

for all $(Q,T)\in\tilde{\mathcal{W}}\cap\mathcal{A}[F]$, and for all $s=1,\dots,|F|$. If we further assume that there exists $\tilde{\mu}\in\mathbb{R}$ such that $\tilde{\mu}=\mu_j[\tilde{T}]$ for all $j\in F$, and if $\{\tilde{u}_1,\dots,\tilde{u}_{|F|}\}$ is an orthonormal basis for $E[\tilde{T},F]$ in $H_{\tilde{Q}}$, then the partial differential of $M^\sharp_{F,s}$ with respect to the variable T at $(\tilde{Q},\tilde{T})$ satisfies the equality

$$d_T M^\sharp_{F,s}[\tilde{Q},\tilde{T}](\dot{T})=\left(\begin{array}{c}|F|-1\\ s-1\end{array}\right)\tilde{\mu}^{s-1}\sum_{l=1}^{|F|}\tilde{Q}\left[\dot{T}[\tilde{u}_l],\tilde{u}_l\right], \tag{2.32}$$

for all $\dot{T}\in\mathcal{K}_s\left(H_{\tilde{Q}},H_{\tilde{Q}}\right)$, and for all $s=1,\dots,|F|$.

Proof. Let $\mathcal{W}^o$, $\{u_j[Q,T]:\, j=1,\dots,|F|\}$ be as in Proposition 2.21. Then the matrix $\mathcal{S}[Q,T]$ of Proposition 2.21 is the matrix of T in the image $E[T,F]$ of $P^\sharp_F[Q,T]$ with respect to the basis $\{u_j[Q,T]:\, j=1,\dots,|F|\}$ for all $(Q,T)\in\mathcal{W}^o\cap\mathcal{A}[F]$. The polynomial

$$\det\left(\mu\mathrm{I}-\mathcal{S}[Q,T]\right) \tag{2.33}$$

has real coefficients and degree $|F|$. We define $M^\sharp_{F,s}[Q,T]$ to be the coefficient of $\mu^{|F|-s}$ in the polynomial in the right hand side of (2.33) multiplied by $(-1)^s$, for all $s=1,\dots,|F|$. By Proposition 2.21, the functions $M^\sharp_{F,s}[\cdot,\cdot]$ are real analytic on $\mathcal{W}^o$. As is well known, $M^\sharp_{F,s}[Q,T]$ is the s-th elementary symmetric function of the (possibly complex) zeros of the characteristic polynomial in (2.33). If $(Q,T)\in\mathcal{W}^o\cap\mathcal{A}[F]$, then the image of $P^\sharp_F[Q,T]$, which coincides with the image of $P_F[Q,T]$, is the space generated by the eigenvectors of T relative to the eigenvalues $\{\mu_j[T]:\, j\in F\}$, and thus the zeros of the characteristic polynomial in (2.33) coincide with the numbers $\mu_j[T]$ for $j\in F$. Accordingly, (2.31) holds. We now prove (2.32). Let $\dot{T}\in\mathcal{K}_s\left(H_{\tilde{Q}},H_{\tilde{Q}}\right)$. Since $\mathcal{W}^o$ is open, and $\mathcal{A}[F]$ is open in $\mathcal{M}$, then there exists $\epsilon_0>0$ such that $\left(\tilde{Q},T(\epsilon)\equiv\tilde{T}+\epsilon\dot{T}\right)\in\mathcal{W}^o\cap\mathcal{A}[F]$ for all $\epsilon\in]-\epsilon_0,\epsilon_0[$. By Corollary 2.28, it follows that the functions $\mu_j[T(\cdot)]$ of $[0,\epsilon_0[$ to $\mathbb{R}$ are differentiable at 0. Furthermore, the set $\left\{\frac{d}{d\epsilon}_{|\epsilon=0}\mu_j[T(\epsilon)]:\, j\in F\right\}$ coincides with the set of the eigenvalues of the matrix $\dot{\mathcal{S}}_0$ of Corollary 2.28 (*iii*). Thus we have that

$$\begin{aligned}
d_T M^\sharp_{F,s}[\tilde{Q},\tilde{T}](\dot{T})&=\frac{d}{d\epsilon}_{|\epsilon=0^+}M^\sharp_{F,s}[\tilde{Q},T(\epsilon)]\\
&=\sum_{j_1,\dots,j_s\in F,\ j_1<\cdots<j_s}\left\{\left(\Pi_{l=1,\dots,s,\,l\neq 1}\,\mu_{j_l}[\tilde{T}]\right)\frac{d}{d\epsilon}_{|\epsilon=0^+}\mu_{j_1}[T(\epsilon)]+\dots\right.\\
&\qquad\left.\cdots+\left(\Pi_{l=1,\dots,s,\,l\neq s}\,\mu_{j_l}[\tilde{T}]\right)\frac{d}{d\epsilon}_{|\epsilon=0^+}\mu_{j_s}[T(\epsilon)]\right\}
\end{aligned}$$

$$= \tilde{\mu}^{s-1} \sum_{j_1,\dots,j_s \in F,\ j_1<\dots<j_s} \left[\frac{d}{d\epsilon}_{|\epsilon=0^+} \mu_{j_1}[T(\epsilon)] + \cdots + \frac{d}{d\epsilon}_{|\epsilon=0^+} \mu_{j_s}[T(\epsilon)] \right]$$

$$= \begin{pmatrix} |F|-1 \\ s-1 \end{pmatrix} \tilde{\mu}^{s-1} \sum_{j\in F} \frac{d}{d\epsilon}_{|\epsilon=0^+} \mu_j[T(\epsilon)] \,.$$

Then (2.32) follows by Corollary 2.28. □

In particular, the previous Theorem implies that if $\mathcal{U}$ is an open set of some Banach space $\mathcal{X}$, $\mathcal{G}$ some real analytic map of $\mathcal{U}$ to $\mathcal{A}[F]$, then the functions $M_{F,s}\,[\mathcal{G}(\cdot)]$ for $s = 1, \dots, |F|$ are real analytic in $\mathcal{U}$. We shall illustrate such a situation in the next section.

We now note that if $(\tilde{Q}, \tilde{T}) \in \mathcal{A}[F]$, and if $\mu_j[\tilde{T}]$ has multiplicity higher than 1, then we cannot expect that $\mu_j[\cdot]$ be real analytic, or even only one time differentiable around $\tilde{T}$, or even that $\mu_j[\cdot]$ be the restriction of a differentiable function of (Q,T) in a neighborhood of $(\tilde{Q}, \tilde{T})$ in $\mathcal{B}_s\left(H^2, \mathbb{R}\right) \times \mathcal{L}\left(H, H\right)$. What we can say however, is that $\mu_j[\cdot]$ depends real analytically on T, in a sense which we clarify below, provided that we restrict μ_j to the set of T's such that $\mu_j[T]$ have a common value, for all $j \in F$. We do so by means of the following immediate consequence of Theorem 2.30, where we set $\mathrm{sgn}(t) = 1$ for $t > 0$, $\mathrm{sgn}(t) = -1$ for $t < 0$.

Theorem 2.34. *Let the same assumptions of Theorem 2.30 hold. Let*

(2.35)
$$\Theta[F] \equiv \{(Q,T) \in \mathcal{A}[F] : \mu_j[T] \text{ have a common value } \mu_F[T] \text{ for all } j \in F\} \,.$$

Let $(\tilde{Q}, \tilde{T}) \in \Theta[F]$. *Let* $\tilde{\mathcal{W}}_1$ *be an open neighborhood of* $(\tilde{Q}, \tilde{T})$ *contained in the neighborhood* $\tilde{\mathcal{W}}$ *of Theorem 2.30, and such that* $\mu_j[T]$ *have the same sign of* $\mu_j[\tilde{T}]$ *for all* $(Q,T) \in \tilde{\mathcal{W}}_1$. *Let* $M^\sharp_{F,s}$ *for* $s = 1, \dots, |F|$ *be as in Theorem 2.30. Then the real valued functions*

$$\left(\mathrm{sgn}(\mu_F[\tilde{T}])\right)^{1+1} \left(\begin{pmatrix} |F| \\ 1 \end{pmatrix}^{-1} M^\sharp_{F,1}[\cdot,\cdot] \right)^{\frac{1}{1}}, \dots$$

$$\dots, \left(\mathrm{sgn}(\mu_F[\tilde{T}])\right)^{|F|+1} \left(\begin{pmatrix} |F| \\ |F| \end{pmatrix}^{-1} M^\sharp_{F,|F|}[\cdot,\cdot] \right)^{\frac{1}{|F|}}$$

are real analytic extensions to $\tilde{\mathcal{W}}_1$ *of the function of* $\Theta[F] \cap \tilde{\mathcal{W}}_1$ *to* $\mathbb{R}$ *which takes* (Q,T) *to* $\mu_F[T]$.

3. Applications to the Dirichlet eigenvalue problem for the Laplace operator

In this section, we consider the dependence of the Dirichlet eigenvalues of the Laplace operator upon perturbation of the domain of definition.

Let Ω be an open subset of $\mathbb{R}^n$. Throughout this section, we shall consider only case $n \geq 2$. We denote by $L^2(\Omega)$ the space of square summable real valued measurable functions defined on Ω, and by $W^{1,2}_0(\Omega)$ the Sobolev space obtained by taking the closure of the space $\mathcal{D}(\Omega)$ of the C^∞ functions with compact support in

Ω in the Sobolev space $W^{1,2}(\Omega)$ of distributions in Ω which have weak derivatives up to the first order in $L^2(\Omega)$, endowed with the norm defined by

$$\|u\|_{W^{1,2}(\Omega)} \equiv \left\{ \|u\|^2_{L^2(\Omega)} + \sum_{l=1}^{n} \|u_{x_l}\|^2_{L^2(\Omega)} \right\}^{1/2}, \tag{3.1}$$

for all $u \in W^{1,2}(\Omega)$. Now, we are interested in open connected subset Ω of $\mathbb{R}^n$ for which the Poincaré constant $c[\Omega]$ is finite, *i.e.*, for which

$$c[\Omega] \equiv \sup{}^{1/2} \left\{ \frac{\int_\Omega |u|^2\, dx}{\int_\Omega |Du|^2\, dx} : u \in W^{1,2}_0(\Omega) \setminus \{0\} \right\} < \infty\,. \tag{3.2}$$

Then we have the following well-known result.

Proposition 3.3. *Let Ω be an open connected subset of $\mathbb{R}^n$ such that (3.2) holds. Then the bilinear map $< \cdot, \cdot >$ of $\left(W^{1,2}_0(\Omega)\right)^2$ to $\mathbb{R}$ defined by*

$$< u_1, u_2 > \equiv \int_\Omega Du_1 Du_2^t\, dx \qquad \forall u_1, u_2 \in W^{1,2}_0(\Omega), \tag{3.4}$$

is also a scalar product on $W^{1,2}_0(\Omega)$, which induces a norm equivalent to that of (3.1). We shall denote by $w^{1,2}_0(\Omega)$ the Hilbert space $W^{1,2}_0(\Omega)$ endowed with the scalar product of (3.4). The strong dual $w^{-1,2}(\Omega) \equiv \left(w^{1,2}_0(\Omega)\right)'$ of $w^{1,2}_0(\Omega)$ coincides with the strong dual $W^{-1,2}(\Omega) \equiv \left(W^{1,2}_0(\Omega)\right)'$ of $W^{1,2}_0(\Omega)$ both algebraically and topologically. We shall always consider $w^{-1,2}(\Omega)$ as endowed with the norm

$$\|F\|_{w^{-1,2}(\Omega)} \equiv \sup_{0 \neq u \in w^{1,2}_0(\Omega)} \frac{|F(u)|}{\|u\|_{w^{1,2}_0(\Omega)}}\,,$$

for all $F \in w^{-1,2}(\Omega)$, where

$$\|u\|_{w^{1,2}_0(\Omega)} \equiv \left\{ \int_\Omega |Du|^2 dx \right\}^{1/2} \qquad \forall u \in w^{1,2}_0(\Omega)\,,$$

defines the 'energy' norm associated to the scalar product in (3.4).

Then we have the following well-known result.

Theorem 3.5. *Let Ω be a nonempty open connected subset of $\mathbb{R}^n$ such that*

$$W^{1,2}_0(\Omega) \text{ is compactly imbedded in } L^2(\Omega). \tag{3.6}$$

Then the following statements hold.

(i) *The Poincaré constant $c[\Omega]$ is finite, and the operator $-\Delta$ is a linear homeomorphism of $W^{1,2}_0(\Omega)$ onto its strong dual $W^{-1,2}(\Omega) \equiv \left(W^{1,2}_0(\Omega)\right)'$.*

(ii) *The eigenvalue problem*

$$-\Delta u = \lambda u \tag{3.7}$$

for $\lambda \in \mathbb{R}$, $u \in W^{1,2}_0(\Omega)$ has an increasing sequence of eigenvalues, which we write as

$$0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_j \leq \ldots$$

For each $j \in \mathbb{N} \setminus \{0\}$, the eigenspace $\left\{u \in W_0^{1,2}(\Omega) : -\Delta u = \lambda_j u\right\}$ has a finite dimension, which we call the multiplicity of λ_j. Also, we shall write each eigenvalue in the above sequence as many times as its multiplicity.

By the argument of Evans [2, Proof of Thm. 1, p. 275], assumption (3.6) implies that Ω has a finite Poincaré constant. Then statement (i) follows by the Lax-Milgram Theorem. For a proof of statement (ii), we refer to Nečas [10, ch. 1, §5].

Now we shall consider perturbations of Ω in the form of homeomorphic images $\phi(\Omega)$ of Ω by some homeomorphism ϕ of Ω onto $\phi(\Omega)$ such that $\phi(\Omega)$ still satisfies condition (3.6). Then it makes sense to consider the Dirichlet eigenvalues $\{\lambda_j[\phi]\}_{j\in\mathbb{N}\setminus\{0\}}$ of $-\Delta$ in the perturbed domain $\phi(\Omega)$. We are interested in the dependence of $\lambda_j[\phi]$ upon ϕ. As a first step, we exploit a standard procedure to convert equation (3.7) into an eigenvalue equation in $w_0^{1,2}(\Omega)$ for a compact selfadjoint operator. Thus we introduce the following known Lemma, which follows immediately by Theorem 3.5.

Lemma 3.8. *Let Ω be an open connected subset of $\mathbb{R}^n$ satisfying (3.6). Let $\mathcal{I}$ be the imbedding of $W_0^{1,2}(\Omega)$ into $L^2(\Omega)$. Let $\mathcal{J}$ be the canonical inclusion of $L^2(\Omega)$ into $W^{-1,2}(\Omega)$. Then equation (3.7) for $u \in W_0^{1,2}(\Omega)$, $\lambda > 0$ is equivalent to equation*

$$u = -\lambda\Delta^{(-1)} \circ \mathcal{J} \circ \mathcal{I}[u] \tag{3.9}$$

for $u \in W_0^{1,2}(\Omega)$, $\lambda > 0$. Both equation (3.7) and equation (3.9) have solutions $u \neq 0$ only for $\lambda > 0$.

Thus we will now consider equation (3.9) on $\phi(\Omega)$ for a suitable homeomorphism ϕ. Accordingly, we must impose conditions on ϕ so as to guarantee that $\phi(\Omega)$ still satisfies condition (3.6), and that we can change the variables in equation (3.9) in order to transform (3.9) into a problem in Ω. To do so, we now introduce the following class of functions ϕ's.

Definition 3.10. Let Ω be an open subset of $\mathbb{R}^n$. Then we set

$$\begin{aligned} L^{1,\infty}(\Omega) &\equiv \left\{ f \in L^1_{\mathrm{loc}}(\Omega) : \frac{\partial f}{\partial x_l} \in L^\infty(\Omega)\ \forall l = 1,\dots,n \right\}, \\ \Phi(\Omega) &\equiv \Big\{ \phi \in \left(L^{1,\infty}(\Omega)\right)^n : \text{the continuous representative of} \\ &\qquad \phi \text{ is injective, } \operatorname*{ess\,inf}_{\Omega} |\det D\phi| > 0 \Big\}, \end{aligned}$$

where $L^1_{\mathrm{loc}}(\Omega)$ denotes the space of (equivalence classes of) locally summable measurable functions in Ω, and $L^\infty(\Omega)$ denotes the space of (equivalence classes of) essentially bounded measurable functions.

If $\mathbb{D}$ is a subset of $\mathbb{R}^n$, then we set

$$\mathrm{Lip}(\mathbb{D}) \equiv \left\{ f \in \mathbb{R}^{\mathbb{D}} : |f|_1 \equiv \sup\left\{ \frac{|f(x)-f(y)|}{|x-y|} : x,y \in \mathbb{D},\ x \neq y \right\} < \infty \right\}.$$

It is well known that $(\mathrm{Lip}(\mathbb{D}), |\cdot|_1)$ is a complete semi-normed space. However, we prefer to deal with a normed space, rather than with a semi-normed space. Then we will state our results for an arbitrary normed space $\mathcal{X}$, continuously imbedded in $(\mathrm{Lip}(\mathbb{D}), |\cdot|_1)$.

Now we introduce the following variant of [9, Lem. 4.22, Cor. 4.24].

Lemma 3.11. *Let $\mathbb{D}$ be a subset of $\mathbb{R}^n$, $n \geq 1$. Let*

$$l_{\mathbb{D}}[\phi] \equiv \inf\left\{\frac{|\phi(x)-\phi(y)|}{|x-y|} : x, y \in \mathbb{D}, x \neq y\right\},$$

for all $\phi \in (\mathrm{Lip}(\mathbb{D}))^n$. Then

$$|l_{\mathbb{D}}[\phi_1] - l_{\mathbb{D}}[\phi_2]| \leq |\phi_1 - \phi_2|_1 \tag{3.12}$$

for all ϕ_1, $\phi_2 \in (\mathrm{Lip}(\mathbb{D}))^n$. In particular, the set

$$\mathcal{A}_{\mathbb{D}} \equiv \{\phi \in (\mathrm{Lip}(\mathbb{D}))^n : l_{\mathbb{D}}[\phi] > 0\}$$

is open in $((\mathrm{Lip}(\mathbb{D}))^n, |\cdot|_1)$. Furthermore, ϕ is differentiable at almost all points of the interior $\overset{o}{\mathbb{D}}$ of $\mathbb{D}$, and

$$l_{\mathbb{D}}[\phi] \leq |\det D\phi(x)|^{1/n}, \tag{3.13}$$

for almost all $x \in \overset{o}{\mathbb{D}}$.

Proof. Inequality (3.12) is an immediate consequence of the inequality

$$\left|\frac{|\phi_1(x)-\phi_1(y)|}{|x-y|} - \frac{|\phi_2(x)-\phi_2(y)|}{|x-y|}\right| \leq |\phi_1 - \phi_2|_1,$$

for all ϕ_1, $\phi_2 \in (\mathrm{Lip}(\mathbb{D}))^n$. For a proof of (3.13), we refer to [9, Lem. 4.22]. □

Remark 3.14. By Lemma 3.11, it follows immediately that if Ω is an open subset of $\mathbb{R}^n$, then $\mathcal{A}_\Omega$ is contained in the class $\Phi(\Omega)$ introduced in Definition 3.10.

As in [6], we introduce the following.

Definition 3.15. Let Ω be an open subset of $\mathbb{R}^n$. Let $\phi \in \Phi(\Omega)$. Then we define the following two operators.

(i) Let $\mathcal{J}_\phi$ be the operator of $L^2(\Omega)$ to $W^{-1,2}(\Omega)$ defined by

$$\mathcal{J}_\phi[u][w] \equiv \int_\Omega uw|\det D\phi|\, dx \qquad \forall w \in W_0^{1,2}(\Omega). \tag{3.16}$$

(ii) Let Δ_ϕ be the operator of $W_0^{1,2}(\Omega)$ to $W^{-1,2}(\Omega)$ which takes $u \in W_0^{1,2}(\Omega)$ to the element $\Delta_\phi[u]$ of $W^{-1,2}(\Omega)$ defined by

$$\Delta_\phi[u][w] = -\int_\Omega Du(D\phi)^{-1}(D\phi)^{-t}Dw^t|\det D\phi|\, dx, \tag{3.17}$$

for all u, $w \in W_0^{1,2}(\Omega)$.

Then we have the following (cf. [6, §3].)

Theorem 3.18. *Let Ω be an open connected subset of $\mathbb{R}^n$ satisfying (3.6). Let $\phi \in \Phi(\Omega)$. Then the following statements hold.*

(i) *Let $\phi \in \Phi(\Omega)$. The function Q_ϕ of $(W_0^{1,2}(\Omega))^2$ to $\mathbb{R}$ defined by*

$$Q_\phi[u_1, u_2] \equiv \int_\Omega Du_1 (D\phi)^{-1}(D\phi)^{-t} Du_2^t |\mathrm{det} D\phi| \, dx \qquad \forall u_1, u_2 \in W_0^{1,2}(\Omega),$$

is a scalar product in $W_0^{1,2}(\Omega)$, which makes $W_0^{1,2}(\Omega)$ a Hilbert space, which we denote by the symbol $w_{0,\phi}^{1,2}(\Omega)$. Moreover, the following inequality holds

$$\frac{\mathrm{ess\,inf}_\Omega |\mathrm{det} D\phi|}{\| |D\phi| \|^2_{L^\infty(\Omega)}} \le \eta[Q_\phi] \le \left\| \left| (D\phi)^{-1} \right| \right\|^2_{L^\infty(\Omega)} \|\mathrm{det} D\phi\|_{L^\infty(\Omega)} ,$$

where

$$\eta[Q_\phi] \equiv \inf \left\{ \frac{\int_\Omega \left|Du(D\phi)^{-1}\right|^2 |\mathrm{det} D\phi| \, dx}{\int_\Omega |Du|^2 \, dx} : u \in W_0^{1,2}(\Omega) \setminus \{0\} \right\} .$$

In particular, $Q_\phi \in \mathcal{Q}\left(\left(w_0^{1,2}(\Omega) \right)^2, \mathbb{R} \right)$ (cf. (2.2).)

(ii) *The operator $\mathcal{J}_\phi$ is linear and continuous.*

(iii) *The operator Δ_ϕ is a linear homeomorphism of $W_0^{1,2}(\Omega)$ onto $W^{-1,2}(\Omega)$.*

(iv) *The operator $T_\phi \equiv -\Delta_\phi^{(-1)} \circ \mathcal{J}_\phi \circ \mathcal{I}$ is compact and selfadjoint in $w_{0,\phi}^{1,2}(\Omega)$.*

(v) *If the pair $(\lambda, v) \in \mathbb{R} \times \left(W_0^{1,2}(\phi(\Omega)) \setminus \{0\} \right)$ satisfies equation*

$$v = -\lambda \Delta^{(-1)} \circ \mathcal{J} \circ \mathcal{I}[v] , \tag{3.19}$$

then $\lambda > 0$ and the pair $(\mu \equiv \lambda^{-1}, u \equiv v \circ \phi)$ belongs to $]0, +\infty[\times W_0^{1,2}(\Omega)$ and satisfies equation

$$\mu u = T_\phi u . \tag{3.20}$$

Conversely, if $(\mu, u) \in \mathbb{R} \times \left(W_0^{1,2}(\Omega) \setminus \{0\} \right)$ satisfies equation (3.20), then $\mu > 0$ and the pair $(\lambda \equiv \mu^{-1}, v \equiv u \circ \phi^{(-1)})$ belongs to $\mathbb{R} \times \left(W_0^{1,2}(\phi(\Omega)) \setminus \{0\} \right)$ and satisfies equation (3.19).

(vi) *$J^+[T_\phi] = \mathbb{N} \setminus \{0\}$, $J^-[T_\phi] = \emptyset$, and equation (3.20) has a decreasing sequence $\{\mu_j[\phi]\}_{j \in \mathbb{N} \setminus \{0\}}$ of eigenvalues in $]0, +\infty[$, and $\mu_j[\phi] = \lambda_j^{-1}[\phi]$ (cf. Theorem 3.5 (ii).)*

We are now ready to prove the following.

Theorem 3.21. *Let Ω be a connected open subset of $\mathbb{R}^n$ such that (3.6) holds. Let $\mathcal{X}$ be a normed space continuously imbedded in $\mathrm{Lip}(\Omega)$. Let F be a finite nonempty subset of $\mathbb{N} \setminus \{0\}$. Let*

$$\mathcal{A}_\Omega[F] \equiv \{ \phi \in \mathcal{A}_\Omega \cap \mathcal{X}^n : \lambda_l[\phi] \notin \{\lambda_j[\phi] : j \in F\} \, \forall l \in \mathbb{N} \setminus (F \cup \{0\}) \} .$$

Then the following statements hold.

(i) *The set $\mathcal{A}_\Omega[F]$ is open in $\mathcal{X}^n$. The map $P_F[\cdot]$ of the set $\mathcal{A}_\Omega[F]$ to the space $\mathcal{L}\left(W_0^{1,2}(\Omega), W_0^{1,2}(\Omega) \right)$ which takes $\phi \in \mathcal{A}_\Omega[F]$ to the orthogonal projection* *of* $w^{1,2}_{0,\phi}(\Omega)$ *onto the (finite dimensional) subspace of* $w^{1,2}_{0,\phi}(\Omega)$ *generated by the set*

$$\left\{u \in W^{1,2}_0(\Omega) :\, -\Delta\left[u \circ \phi^{(-1)}\right] = \lambda_j[\phi]\mathcal{J} \circ \mathcal{I}\left[u \circ \phi^{(-1)}\right] \text{ for some } j \in F\right\} ,$$

is real analytic.

(ii) *Let* $s \in \{1, \ldots, |F|\}$. *The function* $\Lambda_{F,s}$ *of* $\mathcal{A}_\Omega[F]$ *to* $\mathbb{R}$ *defined by*

$$\Lambda_{F,s}[\phi] \equiv \sum_{j_1,\ldots,j_s \in F \ j_1 < \cdots < j_s} \lambda_{j_1}[\phi] \cdots \lambda_{j_s}[\phi] \qquad \forall \phi \in \mathcal{A}_\Omega[F]$$

is real analytic.

Proof. Clearly, the gradient operator D is linear and continuous from $\mathcal{X}^n$ to $(L^\infty(\Omega))^{n^2}$. Since linear and bilinear continuous operators are real analytic, we conclude that Δ_ϕ, and $\mathcal{J}_\phi$, and Q_ϕ are real analytic from $\mathcal{A}_\Omega \cap \mathcal{X}^n$ to the space $\mathcal{L}\left(w^{1,2}_0(\Omega), w^{-1,2}(\Omega)\right)$, and to $\mathcal{L}\left(L^2(\Omega), w^{-1,2}(\Omega)\right)$, and to $\mathcal{Q}\left(\left(w^{1,2}_0(\Omega)\right)^2, \mathbb{R}\right)$, respectively. Since the map which takes an operator into its inverse is real analytic on the set of invertible operators in $\mathcal{L}\left(w^{1,2}_0(\Omega), w^{-1,2}(\Omega)\right)$ (cf. *e.g.*, Hille and Phillips [3, Thms. 4.3.2, 4.34]), it follows that the map $\phi \mapsto (Q_\phi, T_\phi)$ is real analytic from $\mathcal{A}_\Omega \cap \mathcal{X}^n$ to

$$\text{(3.22)} \quad \mathcal{O}_\Omega \equiv \left\{(Q,T) \in \mathcal{Q}\left(\left(w^{1,2}_0(\Omega)\right)^2, \mathbb{R}\right) \times \mathcal{K}\left(w^{1,2}_0(\Omega), w^{1,2}_0(\Omega)\right) :\right.$$
$$\left. T \text{ is selfadjoint with respect to } Q\right\},$$

which is a subset of the linear space $\mathcal{B}_s\left(\left(w^{1,2}_0(\Omega)\right)^2, \mathbb{R}\right) \times \mathcal{L}\left(w^{1,2}_0(\Omega), w^{1,2}_0(\Omega)\right)$. By Theorem 3.18 (vi), the set $\mathcal{A}_\Omega[F]$ coincides with the set

$$\{\phi \in \mathcal{A}_\Omega \cap \mathcal{X}^n :\, (Q_\phi, T_\phi) \in \mathcal{A}[F]\} ,$$

where $\mathcal{A}[F]$ has been introduced in (2.5) for $H = w^{1,2}_0(\Omega)$. Since $\mathcal{A}[F]$ is open in $\mathcal{O}_\Omega$, and $\phi \mapsto (Q_\phi, T_\phi)$ is continuous on $\mathcal{A}_\Omega \cap \mathcal{X}^n$, we conclude that $\mathcal{A}_\Omega[F]$ is open in $\mathcal{X}^n$. Then statement (i) follows by Theorem 2.18. Since $\lambda_j[\phi] = \mu_j^{-1}[T_\phi]$, we have that

$$\text{(3.23)} \qquad \Lambda_{F,s}[\phi] = \frac{M_{F,|F|-s}[T_\phi]}{M_{F,|F|}[T_\phi]} \qquad s = 1, \ldots, |F| ,$$

where $M_{F,0}[T_\phi] \equiv 1$. Then statement (ii) follows by Theorem 2.30. □

Corollary 3.24. *Let* Ω *be a connected open subset of* $\mathbb{R}^n$ *such that (3.6) holds. Let* F *be a finite nonempty subset of* $\mathbb{N} \setminus \{0\}$. *Let*

$$\text{(3.25)} \qquad \Theta_\Omega[F] \equiv \{\phi \in \mathcal{A}_\Omega[F] :\, \lambda_j[\phi] \text{ have a common value } \lambda_F[\phi] \ \forall j \in F\} .$$

Then the real analytic functions

$$\left(\begin{pmatrix} |F| \\ 1 \end{pmatrix}^{-1} \Lambda_{F,1}[\cdot]\right)^{\frac{1}{1}}, \dots, \left(\begin{pmatrix} |F| \\ |F| \end{pmatrix}^{-1} \Lambda_{F,|F|}[\cdot]\right)^{\frac{1}{|F|}},$$

of $\mathcal{A}_\Omega[F]$ *to* $\mathbb{R}$ *coincide on* $\Theta_\Omega[F]$ *with the function which takes* ϕ *to* $\lambda_F[\phi]$.

We note that a version of Corollary 3.24 for ϕ's in the Schauder class of functions of class C^k with Hölder continuous derivatives of order k, for $k \geq 2$, has been deduced in [7, Thm. 3.34] by different avenues.

We conclude this section by computing the derivatives of the functions $\Lambda_{F,s}[\cdot]$ at a point $\tilde{\phi} \in \Theta_\Omega[F]$.

Let Ω be an open subset of $\mathbb{R}^n$. As customary, we denote by $W^{2,2}(\Omega)$ the Sobolev space of distributions in Ω with derivatives of order less or equal to 2 in $L^2(\Omega)$, and by $W^{1,\infty}(\Omega)$ the space of distributions in Ω with derivatives of order less or equal to 1 in $L^\infty(\Omega)$. Then we have the following technical Lemma.

Lemma 3.26. *Let* Ω *be a connected open subset of* $\mathbb{R}^n$ *such that (3.6) holds. Let* $\mathcal{X}$ *be a normed space continuously imbedded in* $\mathrm{Lip}(\Omega)$. *Let* F *be a finite nonempty subset of* $\mathbb{N} \setminus \{0\}$. *Let* $\tilde{\phi} \in \Theta_\Omega[F]$. *Let* $\tilde{u}_1$, $\tilde{u}_2$ *be two eigenvectors corresponding to the eigenvalue* $\lambda_F^{-1}[\tilde{\phi}]$ *of the operator* $T_{\tilde{\phi}}$. *Then we have that*

$$\begin{aligned}
(3.27)\quad & Q_{\tilde{\phi}}\left[d_{|\phi=\tilde{\phi}}\left[\Delta_\phi^{(-1)} \circ \mathcal{J}_\phi \circ \mathcal{I}\right][\psi]\tilde{u}_1, \tilde{u}_2\right] = \\
& -\lambda_F^{-1}[\tilde{\phi}] \int_{\tilde{\phi}(\Omega)} D\tilde{v}_1 \left[D\left(\psi \circ \tilde{\phi}^{(-1)}\right) + D\left(\psi \circ \tilde{\phi}^{(-1)}\right)^t\right] D\tilde{v}_2^t \, dy \\
& - \int_{\tilde{\phi}(\Omega)} \tilde{v}_1 \tilde{v}_2 \mathrm{div}\left(\psi \circ \tilde{\phi}^{(-1)}\right) dy + \lambda_F^{-1}[\tilde{\phi}] \int_{\tilde{\phi}(\Omega)} D\tilde{v}_1 D\tilde{v}_2^t \mathrm{div}\left(\psi \circ \tilde{\phi}^{(-1)}\right) dy\,,
\end{aligned}$$

for all $\psi \in \mathcal{X}^n$, *where* $\tilde{v}_1 = \tilde{u}_1 \circ \tilde{\phi}^{(-1)}$, $\tilde{v}_2 = \tilde{u}_2 \circ \tilde{\phi}^{(-1)}$. *If we further assume that* $\tilde{v}_1$, $\tilde{v}_2 \in W^{2,2}(\tilde{\phi}(\Omega))$, *then the right hand side of (3.27) equals*

$$-\lambda_F^{-1}[\tilde{\phi}] \int_{\tilde{\phi}(\Omega)} \mathrm{div}\left[\left(\psi \circ \tilde{\phi}^{(-1)}\right) D\tilde{v}_1 D\tilde{v}_2^t\right] dy\,, \tag{3.28}$$

for all $\psi \in (\mathrm{Lip}(\Omega) \cap L^\infty(\Omega))^n$.

Proof. To shorten our notation, we set $\tilde{\lambda} \equiv \lambda_F[\tilde{\phi}]$. By standard Calculus in Banach space, and by the obvious equalities $\Delta_{\tilde{\phi}}[\tilde{u}_1] = -\tilde{\lambda}\mathcal{J}_{\tilde{\phi}} \circ \mathcal{I}[\tilde{u}_1]$, $Q_{\tilde{\phi}}[u,w] = -\Delta_{\tilde{\phi}}[u][w]$ for all u, $w \in W_0^{1,2}(\Omega)$, the left hand side of (3.27) equals

$$\begin{aligned}
& Q_{\tilde{\phi}}\left[\Delta_{\tilde{\phi}}^{(-1)} \circ \left(d_{|\phi=\tilde{\phi}}\left[\mathcal{J}_\phi \circ \mathcal{I}\right][\psi]\right)[\tilde{u}_1], \tilde{u}_2\right] + Q_{\tilde{\phi}}\left[\left(d_{|\phi=\tilde{\phi}}\Delta_\phi^{(-1)}[\psi]\right) \circ \mathcal{J}_{\tilde{\phi}} \circ \mathcal{I}[\tilde{u}_1], \tilde{u}_2\right] \\
& \quad = -\left(d_{|\phi=\tilde{\phi}}\left[\mathcal{J}_\phi \circ \mathcal{I}\right][\psi][\tilde{u}_1]\right)[\tilde{u}_2] - \Delta_{\tilde{\phi}}\left[\left(d_{|\phi=\tilde{\phi}}\Delta_\phi^{(-1)}[\psi]\right) \circ \mathcal{J}_{\tilde{\phi}} \circ \mathcal{I}[\tilde{u}_1]\right][\tilde{u}_2] \\
& \quad = -\left(d_{|\phi=\tilde{\phi}}\left[\mathcal{J}_\phi \circ \mathcal{I}\right][\psi][\tilde{u}_1]\right)[\tilde{u}_2] \\
& \qquad\qquad + \Delta_{\tilde{\phi}}\left[\Delta_{\tilde{\phi}}^{(-1)} \circ d_{|\phi=\tilde{\phi}}\left[\Delta_\phi\right][\psi] \circ \Delta_{\tilde{\phi}}^{(-1)} \circ \mathcal{J}_{\tilde{\phi}} \circ \mathcal{I}[\tilde{u}_1]\right][\tilde{u}_2] \\
& \quad = -\left(d_{|\phi=\tilde{\phi}}\left[\mathcal{J}_\phi \circ \mathcal{I}\right][\psi][\tilde{u}_1]\right)[\tilde{u}_2] - \tilde{\lambda}^{-1}\left(d_{|\phi=\tilde{\phi}}\left[\Delta_\phi[\psi]\right][\tilde{u}_1]\right)[\tilde{u}_2]\,,
\end{aligned}$$

for all $\psi \in \mathcal{X}^n$. We now compute $\left(d_{|\phi=\tilde{\phi}}\left[\mathcal{J}_\phi \circ \mathcal{I}\right][\psi][\tilde{u}_1]\right)[\tilde{u}_2]$. By standard Calculus, it is easy to see that

$$\left[\left(d_{|\phi=\tilde{\phi}}\left(\mathrm{det}D\phi\right)[\psi]\right) \circ \tilde{\phi}^{(-1)}\right] \mathrm{det}D\tilde{\phi}^{(-1)} = \mathrm{div}\left(\psi \circ \tilde{\phi}^{(-1)}\right), \tag{3.29}$$

and that the map of $A \equiv \{f \in L^\infty(\Omega) : \mathrm{ess\,inf}_\Omega |f| > 0\}$ to $L^\infty(\Omega)$ which takes f to $|f|$ is differentiable, and that for all $f \in A$, its differential at f is the map of $L^\infty(\Omega)$ to itself which maps h to $\mathrm{sgn}(f)h$. Then by (3.29), and by changing the variable with the map $\tilde{\phi}$ (cf. Reshetnyak [13, Thm. 2.2, p. 99]), we obtain

$$\begin{aligned}&\left(d_{|\phi=\tilde{\phi}}\left[\mathcal{J}_\phi \circ \mathcal{I}\right][\psi][\tilde{u}_1]\right)[\tilde{u}_2] \\ &\qquad = \int_\Omega \tilde{u}_1\tilde{u}_2 d_{|\phi=\tilde{\phi}}\left(|\mathrm{det}D\phi|\right)[\psi]\,dx = \int_{\tilde{\phi}(\Omega)} \tilde{v}_1\tilde{v}_2 \mathrm{div}\left(\psi \circ \tilde{\phi}^{(-1)}\right)dy\,.\end{aligned}$$

We now compute $d_{|\phi=\tilde{\phi}}\left[\Delta_\phi[\psi]\right][\tilde{u}_1][\tilde{u}_2]$. To shorten our notation, we find convenient to set $G_\phi \equiv (D\phi)^{-1}(D\phi)^{-t}$. Then by definition of Δ_ϕ, we obtain

$$\begin{aligned}d_{|\phi=\tilde{\phi}}\left[\Delta_\phi[\psi]\right][\tilde{u}_1][\tilde{u}_2] = &-\int_\Omega D\tilde{u}_1\left(d_{|\phi=\tilde{\phi}}G_\phi[\psi]\right)D\tilde{u}_2^t|\mathrm{det}D\tilde{\phi}|\,dx \\ &-\int_\Omega D\tilde{u}_1 G_{\tilde{\phi}} D\tilde{u}_2^t d_{|\phi=\tilde{\phi}}\left(|\mathrm{det}D\phi|\right)[\psi]\,dx\,.\end{aligned} \tag{3.30}$$

By equality (3.29), we have

$$\int_\Omega D\tilde{u}_1 G_{\tilde{\phi}} D\tilde{u}_2^t d_{|\phi=\tilde{\phi}}\left(|\mathrm{det}D\phi|\right)[\psi]\,dx = \int_{\tilde{\phi}(\Omega)} D\tilde{v}_1 D\tilde{v}_2^t \mathrm{div}\left(\psi \circ \tilde{\phi}^{(-1)}\right)dy\,. \tag{3.31}$$

We notice that

$$\left[d_{|\phi=\tilde{\phi}}G_\phi[\psi]\right] \circ \tilde{\phi}^{(-1)} = -D\left(\tilde{\phi}^{(-1)}\right)\left[D\left(\psi \circ \tilde{\phi}^{(-1)}\right) + D\left(\psi \circ \tilde{\phi}^{(-1)}\right)^t\right]D\left(\tilde{\phi}^{(-1)}\right)^t. \tag{3.32}$$

Then, by another change of variables, we obtain

$$\begin{aligned}&\int_\Omega D\tilde{u}_1\left(d_{|\phi=\tilde{\phi}}G_\phi[\psi]\right)D\tilde{u}_2^t|\mathrm{det}D\tilde{\phi}|\,dx \\ &\qquad = -\int_{\tilde{\phi}(\Omega)} D\tilde{v}_1\left[D\left(\psi \circ \tilde{\phi}^{(-1)}\right) + D\left(\psi \circ \tilde{\phi}^{(-1)}\right)^t\right]D\tilde{v}_2^t\,dy\,.\end{aligned} \tag{3.33}$$

By the above equalities, it follows that (3.27) holds. We now consider the case in which we further assume that $\tilde{v}_1,\ \tilde{v}_2 \in W^{2,2}(\tilde{\phi}(\Omega))$. To shorten our notation, we set $\omega \equiv (\omega_s)_{s=1,\dots,n}$ where $\omega_s = \psi_s \circ \tilde{\phi}^{(-1)}$ and $\psi = (\psi_s)_{s=1,\dots,n}$. Since $l_\Omega[\tilde{\phi}] > 0$, then $\tilde{\phi}^{(-1)}$ is Lipschitz continuous on $\tilde{\phi}(\Omega)$, and thus ω is also Lipschitz continuous on $\tilde{\phi}(\Omega)$, and the fuctions ω_s have essentially bounded first order distributional derivatives. Since $\psi \in (L^\infty(\Omega))^n$, the Lipschitz continuity of $\tilde{\phi}^{(-1)}$ in $\tilde{\phi}(\Omega)$ ensures that $\omega \in \left(L^\infty(\tilde{\phi}(\Omega))\right)^n$. Then we have $\omega \in \left(W^{1,\infty}(\tilde{\phi}(\Omega))\right)^n$, and $\omega D\tilde{v}_r^t \in W^{1,2}(\tilde{\phi}(\Omega))$ for $r = 1, 2$. Now we note that

$$\begin{aligned}&D\tilde{v}_1\left(D\omega + D\omega^t\right)D\tilde{v}_2^t \\ &\qquad = \mathrm{div}\left((\omega D\tilde{v}_1^t)D\tilde{v}_2 + (\omega D\tilde{v}_2^t)D\tilde{v}_1 - (D\tilde{v}_1 D\tilde{v}_2^t)\omega\right)\end{aligned} \tag{3.34}$$

$$- \left[(\omega D\tilde{v}_1^t)\Delta\tilde{v}_2 + (\omega D\tilde{v}_2^t)\Delta\tilde{v}_1\right] + (D\tilde{v}_1 D\tilde{v}_2^t)\mathrm{div}\omega\,,$$

where Δ denotes the Laplacian (in the sense of distributions) applied to a function of $W^{2,2}(\tilde{\phi}(\Omega))$. Next, we show that

$$A[v,w] \equiv \int_{\tilde{\phi}(\Omega)} \mathrm{div}\left[(\omega Dv^t)Dw - (DvDw^t)\omega\right]\,dy = 0\,, \tag{3.35}$$

for all $(v,w) \in \left(W_0^{1,2}(\tilde{\phi}(\Omega)) \cap W^{2,2}(\tilde{\phi}(\Omega))\right) \times W^{2,2}(\tilde{\phi}(\Omega))$. To do so, we first show that A vanishes on pairs $(\beta_1,\beta_2) \in \mathcal{D}(\tilde{\phi}(\Omega)) \times \left(C^\infty(\tilde{\phi}(\Omega)) \cap W^{2,2}(\tilde{\phi}(\Omega))\right)$. Possibly multiplying β_2 by a function of $\mathcal{D}(\tilde{\phi}(\Omega))$ equal to 1 on an open neighborhood of the support of β_1, we can assume that $\beta_2 \in \mathcal{D}(\tilde{\phi}(\Omega))$. The function ω is Lipschitz continuous, and thus it can be extended to the colosure of $\tilde{\phi}(\Omega)$, and then to all of $\mathbb{R}^n$ as a Lipschitz continuous function (cf. *e.g.* Troianiello [15, Thm. 1.2, p. 12].) Then by applying the Divergence Theorem to a ball containing the supports of β_1 and of β_2 in the interior, one realizes that $A[\beta_1,\beta_2] = 0$. Now we note that the integrand in (3.35) equals

$$\sum_{r,s=1}^{n} \left\{ \frac{\partial v}{\partial y_s}\frac{\partial}{\partial y_r}\left(\omega_s \frac{\partial w}{\partial y_r}\right) - \frac{\partial v}{\partial y_s}\frac{\partial}{\partial y_r}\left(\omega_r \frac{\partial w}{\partial y_s}\right)\right\}\,,$$

an expression which does not contain second order derivatives in v. Hence, A defines a bilinear and continuous map of $W_0^{1,2}(\tilde{\phi}(\Omega)) \times W^{2,2}(\tilde{\phi}(\Omega))$. Since such form vanishes on $\mathcal{D}(\tilde{\phi}(\Omega)) \times \left(C^\infty(\tilde{\phi}(\Omega)) \cap W^{2,2}(\tilde{\phi}(\Omega))\right)$, which is a dense subset of its domain, we conclude that (3.35) holds. Since $\tilde{v}_1,\ \tilde{v}_2 \in W_0^{1,2}(\tilde{\phi}(\Omega))$, $\omega \in \left(W^{1,\infty}(\tilde{\phi}(\Omega))\right)^n$, we have

$$\int_{\tilde{\phi}(\Omega)} \tilde{v}_1\tilde{v}_2 \mathrm{div}\omega\,dy = -\int_{\tilde{\phi}(\Omega)} (\omega D\tilde{v}_1^t)\tilde{v}_2\,dy - \int_{\tilde{\phi}(\Omega)} (\omega D\tilde{v}_2^t)\tilde{v}_1\,dy\,. \tag{3.36}$$

Then by combining equalities $\Delta\tilde{v}_r = -\tilde{\lambda}\tilde{v}_r$ for $r = 1,2$ with (3.34), (3.35), with the membership of $\tilde{v}_r$ in $W_0^{1,2}(\tilde{\phi}(\Omega)) \cap W^{2,2}(\tilde{\phi}(\Omega))$ for $r = 1,2$, and with (3.36), we obtain

$$\begin{aligned}\int_{\tilde{\phi}(\Omega)} D\tilde{v}_1\left(D\omega + D\omega^t\right)D\tilde{v}_2^t &= -\tilde{\lambda}\int_{\tilde{\phi}(\Omega)} \tilde{v}_1\tilde{v}_2\mathrm{div}\omega\,dy \\ &\quad + \int_{\tilde{\phi}(\Omega)} D\tilde{v}_1 D\tilde{v}_2^t \mathrm{div}\omega\,dy + \int_{\tilde{\phi}(\Omega)} \mathrm{div}\left[(D\tilde{v}_1 D\tilde{v}_2^t)\omega\right]\,dy.\end{aligned} \tag{3.37}$$

Then by equality (3.27), equality (3.28) follows. □

We then have the following.

Theorem 3.38. *Let Ω be a connected open subset of $\mathbb{R}^n$ such that (3.6) holds. Let $\mathcal{X}$ be a normed space continuously imbedded in* Lip(Ω). *Let F be a finite nonempty subset of $\mathbb{N} \setminus \{0\}$. Let $\Theta_\Omega[F]$ be as in (3.25). Let $\tilde{\phi} \in \Theta_\Omega[F]$. Let $\tilde{v}_1,\dots,\tilde{v}_{|F|}$ be an orthonormal basis of the eigenspace associated to the eigenvalue $\lambda_F[\tilde{\phi}]$ of $-\Delta$ in*

$W_0^{1,2}(\tilde{\phi}(\Omega))$, *where the orthonormality is taken with respect to the scalar product of* $w_0^{1,2}(\tilde{\phi}(\Omega))$ *(cf. (3.4).) Then we have*

$$(3.39) \quad d_{|\phi=\tilde{\phi}}\left(\Lambda_{F,s}\right)[\psi]$$
$$= -\lambda_F^s[\tilde{\phi}] \begin{pmatrix} |F|-1 \\ s-1 \end{pmatrix} \sum_{l=1}^{|F|} \left\{ \int_{\tilde{\phi}(\Omega)} \left[\lambda_F[\tilde{\phi}]\tilde{v}_l^2 - |D\tilde{v}_l|^2 \right] \operatorname{div}\left(\psi \circ \tilde{\phi}^{(-1)}\right) dy \right.$$
$$\left. + \int_{\tilde{\phi}(\Omega)} D\tilde{v}_l \left[D\left(\psi \circ \tilde{\phi}^{(-1)}\right) + D\left(\psi \circ \tilde{\phi}^{(-1)}\right)^t \right] D\tilde{v}_l^t \, dy \right\},$$

for all $\psi \in \mathcal{X}^n$, $s = 1, \ldots, |F|$. *If we further assume that* $\tilde{v}_l \in W^{2,2}(\tilde{\phi}(\Omega))$ *for* $l = 1, \ldots, |F|$, *then the right hand side of (3.39) equals*

$$(3.40) \qquad -\lambda_F^s[\tilde{\phi}] \begin{pmatrix} |F|-1 \\ s-1 \end{pmatrix} \sum_{l=1}^{|F|} \int_{\tilde{\phi}(\Omega)} \operatorname{div}\left[\left(\psi \circ \tilde{\phi}^{(-1)}\right) |D\tilde{v}_l|^2 \right] dy\,,$$

for all $\psi \in (\mathrm{Lip}(\Omega) \cap L^\infty(\Omega))^n$.

Proof. We set $\tilde{u}_l = \tilde{v}_l \circ \tilde{\phi}$, for all $l = 1, \ldots, |F|$. We first consider case $|F| > 1$. By equalities (2.32) and (3.23), it follows that

$$d_{|\phi=\tilde{\phi}}\left(\Lambda_{F,s}\right)[\psi] = \left\{ d_{|\phi=\tilde{\phi}} M_{F,|F|-s}\left[T_\phi\right][\psi] M_{F,|F|}\left[T_{\tilde{\phi}}\right] \right.$$
$$\left. - M_{F,|F|-s}\left[T_{\tilde{\phi}}\right] d_{|\phi=\tilde{\phi}} M_{F,|F|}\left[T_\phi\right][\psi] \right\} \lambda_F^{2|F|}[\tilde{\phi}]$$
$$= \left[\begin{pmatrix} |F|-1 \\ |F|-s-1 \end{pmatrix} \lambda_F^{s+1-2|F|}[\tilde{\phi}] - \begin{pmatrix} |F| \\ s \end{pmatrix} \begin{pmatrix} |F|-1 \\ |F|-1 \end{pmatrix} \lambda_F^{s+1-2|F|}[\tilde{\phi}] \right]$$
$$\cdot \lambda_F^{2|F|}[\tilde{\phi}] \sum_{l=1}^{|F|} Q_{\tilde{\phi}}\left[d_{|\phi=\tilde{\phi}}\left(T_\phi\right)[\psi][\tilde{u}_l], \tilde{u}_l \right]$$
$$= -\lambda_F^{s+1}[\tilde{\phi}] \begin{pmatrix} |F|-1 \\ s-1 \end{pmatrix} \sum_{l=1}^{|F|} Q_{\tilde{\phi}}\left[d_{|\phi=\tilde{\phi}}\left(T_\phi\right)[\psi][\tilde{u}_l], \tilde{u}_l \right].$$

Then we can conclude by Lemma 3.26. Case $|F| = 1$ can be treated similarly. □

Concerning the statement of Lemma 3.26, we note that if $\tilde{\phi}(\Omega)$ is of class $C^{1,1}$, then by standard elliptic regularity theory, we have $\tilde{v}_r \in W^{2,2}(\tilde{\phi}(\Omega))$ and $D\tilde{v}_r = \frac{\partial \tilde{v}_r}{\partial \nu}\nu$ on $\partial\left(\tilde{\phi}(\Omega)\right)$ for $r = 1, 2$, where ν denotes the exterior unit normal to $\partial\left(\tilde{\phi}(\Omega)\right)$ (cf. *e.g.*, Troianiello [15, Thm. 3.29, p. 195].) Moreover, by the Divergence Theorem, the integral in (3.28) would equal $\int_{\partial\tilde{\phi}(\Omega)} \frac{\partial \tilde{v}_1}{\partial \nu} \frac{\partial \tilde{v}_2}{\partial \nu} \left(\psi \circ \tilde{\phi}^{(-1)}\right) \cdot \nu^t \, d\sigma$ where $d\sigma$ denotes the $(n-1)$-dimensional area element of $\partial\left(\tilde{\phi}(\Omega)\right)$. A corresponding remark holds of course for Theorem 3.38 and formula (3.40).

Furthermore, we note that if we assume that Ω is of class $C^{1,1}$, and that $\tilde{\phi} \in \mathcal{A}_\Omega$ has continuous partial derivatives in Ω satisfying a Lipschitz condition in Ω, then $\tilde{\phi}(\Omega)$ is of class $C^{1,1}$ (cf. *e.g.*, [7, Lem. 2.4].)

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