Journal of Nonlinear and Convex Analysis
Volume 5, Number 1, 2004, 153–155

Yokohama Publishers
ISSN 1880-5221 ONLINE JOURNAL

A NOTE ON DIFFERENCE SEQUENCE SPACES

SATIT SAEJUNG

Abstract. In this short note we prove that the difference sequence spaces, appearing in [4] are actually isometrically isomorphic to Musielak-Orlicz sequence spaces endowed with the Luxemburg norm. Thus, all results in [4] are easily deduced from known results. Furthermore, the characterizations are obtained under weaker assumptions.

1. Introduction

A convex function $\varphi : \mathbb{R} \to \mathbb{R}_+ = [0, \infty)$ is called an *Orlicz function* if it vanishes at zero and is even on the whole line $\mathbb{R}$ and is not identically equal to zero. Denote by l the space of all real sequences $x = (x_n)_{n=1}^{\infty}$. For a given *Musielak-Orlicz function* Φ, i.e. a sequence (φ_n) of Orlicz functions, the *Musielak-Orlicz sequence space* l_Φ is the space

$$l_\Phi := \Big\{ x \in l : \sum_{n=1}^{\infty} \varphi_n(\lambda x_n) < \infty \text{ for some } \lambda > 0 \Big\}$$

equipped with the *Luxemburg norm* defined by

$$\|x\| = \inf \Big\{ \lambda > 0 : \sum_{n=1}^{\infty} \varphi_n(x_n/\lambda) \le 1 \Big\}.$$

It is known that $l_\Phi := (l_\Phi, \| \cdot \|)$ is a Banach space (see [5]).

We say that a Musielak-Orlicz function $\Phi = (\varphi_n)$ *satisfies the δ_2-condition* ($\Phi \in \delta_2$) if there exist constants $K \ge 2$, $u_0 > 0$ and a sequence (c_n) of positive numbers such that $\sum_{n=1}^{\infty} c_n < \infty$ and the inequality $\varphi_n(2u) \le K\varphi_n(u) + c_n$ holds for every $n \in \mathbb{N}$ and $u \in \mathbb{R}$ satisfying $\varphi_n(u) \le u_0$ (see [2]).

We also say that a Musielak-Orlicz function $\Phi = (\varphi_n)$ *satisfies the $(*)$-condition* if for any $\varepsilon \in (0, 1)$ there exists a $\delta > 0$ such that, for all $n \in \mathbb{N}$ and $u \in \mathbb{R}$, $\varphi_n((1 + \delta)u) \le 1$ whenever $\varphi_n(u) \le 1 - \varepsilon$ (see [3]).

2. Difference Sequence Spaces

For a given Musielak-Orlicz sequence space l_Φ, we define a *difference sequence space* $l_\Phi(\triangle)$ by

$$l_\Phi(\triangle) := \left\{ (x_n)_{n=1}^{\infty} : (\triangle x_n)_{n=1}^{\infty} \in l_\Phi \right\}.$$

Here $(\triangle x_n)_{n=1}^{\infty} = (x_1, x_2 - x_1, x_3 - x_2, \dots)$. For $(x_n)_{n=1}^{\infty} \in l_\Phi(\triangle)$, we also define

$$\|(x_n)\|_\triangle = \|(\triangle x_n)\|.$$

In general, the spaces $l_\Phi(\triangle)$ and l_Φ need not be the same. However, we have

2000 *Mathematics Subject Classification.* 46B20, 46E30.
Key words and phrases. difference sequence space, Musielak-Orlicz sequence space.

Proposition 2.1. *$l_\Phi(\triangle)$ and l_Φ are isometrically isomorphic.*

Proof. Define $T : l_\Phi(\triangle) \to l_\Phi$ by $T(x_n) = (\triangle x_n)$ for $(x_n) \in l_\Phi(\triangle)$. It is easy to see that T is an isomorphism. Moreover, $\|T(x_n)\| = \|(x_n)\|_\triangle$ for all $(x_n) \in l_\Phi(\triangle)$ and this implies that T is an isometry. □

Let us now recall some known characterizations:

Theorem 2.2. (1) [6, Theorem 4.1] *The Musielak-Orlicz sequence space l_Φ is rotund if and only if $\Phi \in \delta_2$, each φ_i vanishes only at zero, and there exists a sequence $\{a_i\} \subset [0, \infty)$ such that φ_i is strictly convex on $[0, a_i]$ for all $i \in \mathbb{N}$ and $\varphi_j(a_j) + \varphi_k(a_k) \geq 1$ for every $j \neq k$.*
(2) [1, Theorem 2] *If, in addition, $\Phi = (\varphi_i)$ satisfies the $(*)$-condition and each φ_i vanishes only at zero, then l_Φ has property* (H) *if and only if $\Phi \in \delta_2$.*

So we can characterize all geometric properties of the difference sequence space $l_\Phi(\triangle)$ via characterizations of the corresponding properties of l_Φ.

Example 2.3. [4] Let $\mathbf{p} = \{p_n\}_{n=1}^{\infty} \subset [1, \infty)$. Define

$$l(\triangle, \mathbf{p}) = \Big\{(x_n) : |x_1| + \sum_{n=1}^{\infty} |\lambda(x_{n+1} - x_n)|^{p_n} < \infty \text{ for some } \lambda > 0\Big\}$$

equipped with the norm defined by

$$\|(x_n)\|_\triangle = \inf\Big\{\lambda > 0 : \Big|\frac{x_1}{\lambda}\Big| + \sum_{n=1}^{\infty} \Big|\frac{x_{n+1} - x_n}{\lambda}\Big|^{p_n} \leq 1\Big\}.$$

It is easy to see that $l(\triangle, \mathbf{p}) = l_\Phi(\triangle)$ where $\Phi = (\varphi_n)$ is defined by $\varphi_1(u) = |u|$ and $\varphi_n(u) = |u|^{p_{n-1}}$ for all $n \geq 2$ and $u \in \mathbb{R}$. As consequences of Theorem 2.2, we obtain

(1) $l(\triangle, \mathbf{p})$ is strictly convex if and only if $\limsup_{n\to\infty} p_n < \infty$ and $p_n > 1$ for all n.
(2) $l(\triangle, \mathbf{p})$ has property (H) if and only if $\limsup_{n\to\infty} p_n < \infty$.

Let us note that the above results are obtained without the assumption that $\limsup_{n\to\infty} p_n < \infty$ as was the case in [4].

Acknowledgement

This work was supported by Thailand Research Fund under grant BRG/01/2544. The author would like to thank Professor Sompong Dhompongsa for his helpful discussion and comments. This work was carried out while the author was at the University of Newcastle. The author would like to thank the School of Mathematical and Physical Sciences and Professor Brailey Sims for their kind hospitality during all his stay.

References

[1] Y. Cui, H. Hudzik and R. Płuciennik, *Banach-Saks property in some Banach sequence spaces*, Annales Polonici Math., **65**(1997), 193–202.

[2] A. Kamińska, *Rotundity of sequence Musielak-Orlicz spaces*, Bull. Polish Acad. Sci. Math. **29** (1981), 137–144.

[3] A. Kamińska, *Uniform rotundity of Musielak-Orlicz sequene spaces*, J. Approx. Theory, **47**(1986), 302–322.

[4] A. Kananthai, M. Musaleen, W. Sanhan and S. Suantai, *On property* (H) *and rotundity of difference sequence spaces*, J. Nonlinear Convex Anal. **3**(2002), 401–409.

[5] J. Musielak, "Orlicz spaces and modular spaces," Lecture Notes in Math. No. 1034, Springer-Verlag, Berlin, 1983.

[6] S. Saejung and S. Dhompongsa, *Extreme points in Musielak-Orlicz sequence spaces*, Acta Math. Vietnamica, **7:2**(2002), 219–229.

Manuscript received May 5, 2003

Satit Saejung
Department of Mathematics, Faculty of Science, Chiang Mai University, Chiang Mai, 50200, Thailand
E-mail address: `g4365178@cm.edu`