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CONTROL PROBLEMS GOVERNED BY FUNCTIONAL EVOLUTION INCLUSIONS WITH YOUNG MEASURES

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ABSTRACT. We study some Bolza-type problems governed by two classes of functional evolution inclusions where the controls are Young measures. In particular, we present some variational properties of the value function associated to these control problems, and we show that the lower value function is a viscosity subsolution of the associated Hamilton-Jacobi-Bellman equation in these classes of evolution inclusions.

1. INTRODUCTION AND BACKGROUND

The pioneering works concerning control problems governed by ordinary differential equations and evolution inclusions with Young measures are developed in [17], [16]. In the same spirit, we consider in this paper some dynamic control problems governed by functional evolution inclusions (FEI) [14], [21] where the controls are Young measures. Essentially, we present some variational properties of the value function of these dynamics and we show that the lower value function associated to a continuous cost function is a viscosity subsolution of the corresponding Hamilton-Jacobi-Bellman equation. This shed a new light in the study of the viscosity solutions for the dynamics governed by ordinary differential equations. Here we also extend a number of results in the literature dealing with undelayded evolution inclusions and ordinary differential equations.

In the sequel, $(\Omega, \mathcal{S}, P)$ is a probability space, $\mathcal{L}([0,1])$ is the σ-algebra of Lebesgue-measurable sets in $[0,1]$. For any Polish space $X, \mathcal{B}(X)$ denotes the Borel tribe of X and $\mathcal{Y}(\Omega, \mathcal{S}, P; X)$ denotes the set of Young measures defined on X. For the sake of completeness, we summarize some useful facts concerning Young measures. Let X be a Polish space and let $\mathcal{C}^b(X)$ be the space of all real-valued bounded continuous functions defined on X. Let $\mathcal{M}^1_+(X)$ be the set of all Borel probability measures on X equipped with the narrow topology. A Young measure $\lambda : \Omega \to \mathcal{M}^1_+(X)$ is, by definition, a *scalarly measurable* mapping from Ω into $\mathcal{M}^1_+(X)$, that is, for every $f \in \mathcal{C}^b(X)$, the mapping

$$\omega \mapsto \langle f, \lambda_\omega \rangle := \int_X f(x)\, d\lambda_\omega(x)$$

is $\mathcal{S}$-measurable. Let us denote by $\mathcal{Y}(\Omega, \mathcal{S}, P; X)$ the space of all Young measures defined on X. A sequence (λ^n) in $\mathcal{Y}(\Omega, \mathcal{S}, P; X)$ *stably converges* to a Young measure $\lambda \in \mathcal{Y}(\Omega, \mathcal{S}, P; X)$ if the following holds

$$\lim_n \int_A [\int_X f(x)\, d\lambda^n_\omega(x)]\, dP(\omega) = \int_A [\int_X f(x)\, d\lambda_\omega(x)]\, dP(\omega)$$

for every $A \in \mathcal{S}$ and for every $f \in \mathcal{C}^b(X)$. Finally, we recall, for the sake of completeness the following result concerning the fiber product lemma for Young measures. See ([17], Theorem 2.3.1). For more on Young measures, see e.g. [2], [17], [40], [41], and the references therein.

Proposition 1.1. *Assume that S and T are Polish spaces. Let (μ^n) be a sequence in $\mathcal{Y}(\Omega, \mathcal{S}, P; S)$ and (ν^n) be a sequence in $\mathcal{Y}(\Omega, \mathcal{S}, P; T)$. Assume that*
(i) (μ^n) converges in probability to $\mu^\infty \in \mathcal{Y}(\Omega, \mathcal{S}, P; S)$,
(ii) (ν^n) stably converges to $\nu^\infty \in \mathcal{Y}(\Omega, \mathcal{S}, P; T)$.
Then $(\mu^n \otimes \nu^n)$ stably converges to $\mu^\infty \otimes \nu^\infty$.

For the sake of completeness, let us mention a general result of convergence for Young measures ([17], Corollary 2.3.2) that we need in the statement of next results.

Proposition 1.2. *Assume that S and T are Polish spaces. Let (u^n) be sequence of $\mathcal{S}$-measurable mappings from Ω into S such that (u^n) converges in probability to a $\mathcal{S}$-measurable mapping u^∞ from Ω into S and (v^n) be a sequence of $\mathcal{S}$-measurable mappings from Ω into T such that (v^n) stably converges to $\nu^\infty \in \mathcal{Y}(\Omega, \mathcal{S}, P; T)$. Let $h : \Omega \times S \times T \to \mathbf{R}$ be a Carathéodory integrand such that the sequence $(h(., u_n(.), v_n(.))$ is uniformly integrable. Then the following result holds*

$$\lim_{n\to\infty} \int_\Omega h(\omega, u^n(\omega), v^n(\omega))\, dP(\omega) = \int_\Omega [\int_T h(\omega, u^\infty(\omega), t)\, d\nu^\infty_\omega(t)]\, dP(\omega).$$

2. Functional evolution inclusions governed by a nonconvex sweeping process

We consider a multifunction $C : [0, 1] \to E(E = \mathbf{R}^d)$ and we assume that
(H_1): For each $t \in [0, 1]$, $C(t)$ is a nonempty closed subset in $\mathbf{R}^d$ that is ρ-prox-regular [23], [37] for some fixed $\rho \in [0, \infty]$,
(H_2): $C(t)$ varies in an absolutely continuous way, that is, there exists an absolutely continuous function $v : [0, 1] \to \mathbf{R}$ such that

$$|d(x, C(t)) - d(y, C(s))| \leq ||x - y|| + |v(t) - v(s)|$$

for all $x, y \in \mathbf{R}^d$ and $s, t \in [0, 1]$.

Recall that a nonempty subset S of $\mathbf{R}^d$ is ρ-proximal regular [37] or equivalently ρ-proximally smooth [23] when every nonzero proximal normal to S can be realized by a ρ-ball. This is equivalent to say that for every $x \in S$, and for every $0 \neq v \in N^p(S; x)$,

$$\langle \frac{v}{||v||}, x' - x\rangle \leq \frac{1}{2\rho}||x' - x||^2$$

for all $x' \in S$ where $N^p(S; x)$ is the proximal normal cone [23], [37] of S at the point $x \in S$. Actually, $N^p(S; x)$ coincides with the Clarke normal cone $N(S; x)$ of S at the point $x \in S$. We make the convention $\frac{1}{\rho} = 0$ for $\rho = +\infty$. Recall that for $\rho = +\infty$, the ρ-proximal regularity of S is equivalent to the convexity of S. One of the key properties of proximally regular sets S is that for all $x_i \in S$ and all $v_i \in N^p(S; x_i)$ with $||v_i|| \leq \rho$ $(i = 1, 2)$ one has ([37])

$$\text{(Hypomonotonicity): } \langle v_1 - v_2, x_1 - x_2\rangle \geq -||x_1 - x_2||^2.$$

Let $r > 0$ be a finite delay, $\mathcal{C}_0 = \mathcal{C}_{\mathbf{R}^d}([-r, 0])$ be the Banach space of all continuous $\mathbf{R}^d$-valued functions defined on $[-r, 0]$ equipped with the norm $||.||_0$ of uniform convergence. For any $t \in [0,1]$, let $\tau(t) : \mathcal{C}_{\mathbf{R}^d}([-r,t]) \to \mathcal{C}_0$ defined by $(\tau(t)u)(s) = u(t+s)$, for all $s \in [-r,0]$ and for all $u \in \mathcal{C}_{\mathbf{R}^d}([-r,t])$.

We summarize first a preliminary result that is a combined effort of the techniques of Theorem 2.1 and Proposition 3.1 in [21] regarding the compactness of the solutions set for a FEI governed by a nonconvex sweeping process. However this result needs a careful look because we deal with the uniqueness of solutions and the compactness of the solutions set for a new class of FEI. Compare with Proposition 3.2 in [19] dealing with convex sweeping process.

Proposition 2.1. *Assume that* (H_1), (H_2) *are satisfied and the sets* $C(t)$ *are compact. Let* $\varphi \in \mathcal{C}_{\mathbf{R}^d}([-r,0])$ *be given with* $\varphi(0) \in C(0)$ *and let* $g : [0,1] \times \mathcal{C}_0 \to \mathbf{R}^d$ *be such that:*

(i) *for every* $u \in \mathcal{C}_0, g(.,u)$ *is Lebesgue-measurable on* $[0,1]$,

(ii) *for every* $\eta > 0$ *there exists a nonnegative Lebesgue-integrable function* $l_\eta(.)$ *defined on* $[0,1]$ *such that* $||g(t,u) - g(t,v)|| \leq l_\eta(t)||u-v||_0$ *for all* $t \in [0,1]$ *and for all* $u, v \in \overline{B}_{\mathcal{C}_0}(0,\eta) \times \overline{B}_{\mathcal{C}_0}(0,\eta)$,

(iii) *there exist nonnegative Lebesgue-integrable functions* $p(.)$ *and* $q(.)$ *on* $[0,1]$ *such that* $||g(t,u)|| \leq p(t) + q(t)||u(0)||$ *for all* $(t,u) \in [0,1] \times \mathcal{C}_0$.

Then there exists a unique continuous function $u_\varphi : [-r,1] \to \mathbf{R}^d$ *such that its restriction on* $[-r,0]$ *is equal to* φ *and its restriction to* $[0,1]$ *is absolutely continuous, (i.e* $u_\varphi(t) = \varphi(0) + \int_0^t \dot{u}_\varphi(s)\,ds$, *for all* $t \in [0,1]$ *with* $\dot{u}_\varphi \in L^1_{\mathbf{R}^d}([0,1])$) *and satisfies*

$$(P_\tau) \begin{cases} \dot{u}_\varphi(t) \in -N(C(t); u_\varphi(t)) + g(t, \tau(t)u_\varphi) \text{ a.e. } t \in [0,1], \\ u_\varphi(s) = \varphi(s), \forall s \in [-r,0]; u_\varphi(t) \in C(t), \forall t \in [0,1], \end{cases}$$

with $||\dot{u}_\varphi(t)|| \leq \gamma(t)$ *for a.e.* $t \in [0,1]$, *where* γ *is a nonnegative Lebesgue-integrable function which depends only on* $\dot{v}$, p, q *and* $C[0,1] := \cup_{t\in[0,1]} C(t)$. *Consequently, if* $\mathcal{K}$ *denotes the compact set in* $\mathcal{C}_{\mathbf{R}^d}([-r,0])$ *with* $\varphi(0) \in C(0)$ *for all* $\varphi \in \mathcal{K}$, *then the solution sets* $\{u_\varphi : \varphi \in \mathcal{K}\}$ *is relatively compact for the topology of uniform convergence.*

Proof. By using Theorem 2.1 and the remark of Lemma 1.3 in [21], it is easily seen that, for each $\varphi \in \mathcal{C}_{\mathbf{R}^d}([-r,0])$ with $\varphi(0) \in C(0)$, there is at least one solution u of the problem (P_τ) with $||\dot{u}_\varphi(t)|| \leq \gamma(t)$ for a.e. $t \in [0,1]$, where γ is a nonnegative Lebesgue-integrable function which depends only on $\dot{v}$, p, q and $C[0,1]$. In order to prove the uniqueness, let us assume that x and y are two solutions of (P_τ). Then we have, for a.e. $t \in [0,1]$

$$g(t, \tau(t)x) - \dot{x}(t) \in N(C(t); x(t)),$$

and

$$g(t, \tau(t)y) - \dot{y}(t) \in N(C(t); y(t)).$$

Put $m(t) = ||g(t,\tau(t)x)|| + ||g(t,\tau(t)y)||$, by ([21], Prop. 1.1), we have, for a.e. $t \in [0,1]$

$$||g(t,\tau(t)x) - \dot{x}(t)|| \leq ||\dot{x}(t)|| + m(t) \leq \dot{v}(t) + m(t),$$

and

$$||g(t,\tau(t)y - \dot{y}(t)|| \leq ||\dot{y}(t)|| + m(t) \leq \dot{v}(t) + m(t).$$

Then for a.e. $t \in [0,1]$

$$\frac{\rho}{|\dot{v}(t)| + m(t)}(g(t, \tau(t)x) - \dot{x}(t)) \in N_{C(t)}(x(t)),$$

and

$$||\frac{\rho}{|\dot{v}(t)| + m(t)}(g(t, \tau(t)x) - \dot{x}(t))|| \leq \rho,$$

and similarly

$$\frac{\rho}{|\dot{v}(t)| + m(t)}(g(t, \tau(t)y) - \dot{y}(t)) \in N_{C(t)}(y(t)),$$

and

$$||\frac{\rho}{|\dot{v}(t)| + m(t)}(g(t, \tau(t)y) - \dot{y}(t))|| \leq \rho.$$

By virtue of the hypomonocity property of the normal cone, we have for a.e. $t \in [0,1]$

$$\langle g(t, \tau(t)x) - \dot{x}(t) - (g(t, \tau(t)y) - \dot{y}(t)), x(t) - y(t)\rangle$$

$$\geq -\rho^{-1}(|\dot{v}(t)| + m(t))||x(t) - y(t)||^2.$$

Let $\eta > 0$ be such that $\eta > sup_{t\in[0,1]}(||\tau(t)x||_0 + ||\tau(t)y||_0)$. Using Lipschitz property (i) and integrating on $[0,t]$, from the inequality above we derive

$$\begin{aligned}||x(t) - y(t)||^2 \leq 2\int_0^t l_\eta(s)||\tau(s)x - \tau(s)y||_0||x(s) - y(s)||\, ds \\ + 2\int_0^t \rho^{-1}(|\dot{v}(s)| + m(s))||x(s) - y(s)||^2\, ds.\end{aligned}$$

Now we repeat some arguments in ([19], Prop. 2.2). Indeed, as $x = y = \varphi$ on $[-r,0]$, by the preceding inequality, we see that

$$\begin{aligned}||x - y||_t^2 \leq 2\int_0^t l_\eta(s)||x - y||_s^2\, ds + 2\int_0^t \rho^{-1}(|\dot{v}(s)| + m(s))||x - y||_s^2\, ds \\ = 2\int_0^t (l_\eta(s) + \rho^{-1}(|\dot{v}(s)| + m(s)))||x - y||_s^2\, ds,\end{aligned}$$

where $||.||_t$ denotes the norm of $\mathcal{C}_t := \mathcal{C}_{\mathbf{R}^d}([-r,t])$ As $t \mapsto ||x - y||_t$ is continuous, by applying Gronwall's lemma we conclude that

$$||x - y||_t = 0,\ \forall t \in [0,1].$$

Hence $x = y$.

The relative compactness of $\{u_\varphi : \varphi \in \mathcal{K}\}$ follows because $u_\varphi(t) \in C(t)$ for all $t \in C(t)$ and $||\dot{u}_\varphi(t)|| \leq \gamma(t)$ for all $\varphi \in \mathcal{K}$ and for a.e. $t \in [0,1]$. □

Remark. The growth condition (ii) allows to recover the case when τ is the zero mapping that corresponds to undelayed evolution inclusion and the known case $||g(t,u)|| \leq m$ for all $(t,u) \in [0,1] \times C_0$ for some positive constant m.

3. Functional evolution inclusion governed by a m-accretive operator

For simplicity, we consider here a functional evolution inclusion in the finite dimensional space $E = \mathbf{R}^d$, althought the results given below hold for a reflexive Banach space such that its strong dual is uniformly convex. See [14] for details. Recal that a multivalued operator $A(t) : E \rightrightarrows E, (t \in [0,1])$ is *m-accretive*, if, for each $t \in [0,1]$ and each $\lambda > 0$, $R(I_E + \lambda A(t)) = E$, and for each $x_1 \in D(A(t)), x_2 \in D(A(t)), y_1 \in A(t)x_1, y_2 \in A(t)x_2$, we have

(j) $||x_1 - x_2|| \leq ||(x_1 - x_2) + \lambda(y_1 - y_2)||$

where $D(A(t)) := \{x \in E : A(t)x \neq \emptyset\}$. If $A(t)$ is m-accretive, then, for every $\lambda > 0$,

(jj) $\frac{1}{\lambda}||x - J_\lambda A(t)x|| = ||A_\lambda(t)x|| \leq |A(t)x|_0 := \inf_{y \in A(t)x} ||y||, \ \forall x \in D(A(t))$,

where $J_\lambda A(t)x = (I_E + \lambda A(t))^{-1}x$. We refer to [1], [12], [42] for the theory of accretive operators We will consider the following assumptions regarding the m-accretive operator $A(t)$.

(H_1) There exist a continuous function $\rho : [0,1] \to E$ and a nondecreasing function $L : [0,\infty[\to [0,\infty[$ such that

$$||J_\lambda A(t)x - J_\lambda A(s)x|| \leq \lambda|\rho(t) - \rho(s)|L(||x||),$$

for all $\lambda \in]0,1]$, for all $(t,s) \in [0,1] \times [0,1]$, and for all $x \in E$.

(H_2) For every $s > 0$, there exists $\delta(s) > 0$ such that

$$||J_\lambda A(0)x - x|| \leq \lambda\delta(s),$$

for all $\lambda \in]0,1]$ and for all $x \in \overline{D(A(0))}$ with $||x|| \leq s$.

(H_3) (a) For every $L^2_E([0,1])$-mapping $u : [0,1] \to E$ satisfying $u(t) \in D(A(t))$ for all $t \in [0,1]$, the multifunction $t \to A(t)u(t)$ is Lebesgue-measurable, (b) for every $x \in E$ and for every $\lambda > 0$, $t \mapsto (I_E + \lambda A(t))^{-1}x$ is Lebesgue-measurable, (c) there exists $\overline{g} \in L^2_E([0,1])$ such that $t \mapsto (I_E + \lambda A(t))^{-1}\overline{g}(t)$ belongs to $L^2_E([0,1])$ for for all $\lambda > 0$.

Remarks. 1) Assumption (H_1) is similar to that adopted by [12], [25] in the study of quasi-autonomuous evolution equations. By ([25], Lemma 3.1), (H_1) implies that the sets $D(A(t))$ are constant, i.e $\overline{D(A(t))} := D$ for all $t \in [0,1]$.

2) In view of (jj) (H_2) is satisfied if, $0 \in D(A(0)) = \overline{D(A(0))}$ and $A(0)$ satisfies the following boundedness type condition, namely, for any closed ball $\overline{B}(0,\eta)$, the set $\{|A(0)x|_0 : x \in \ D(A(0)) \cap \overline{B}(0,\eta)\}$ is bounded in $\mathbf{R}$. In particular, (H_2) is satisfied if $A(0) : D(A(0)) \to E$ is convex compact-valued upper semicontinuous multifunction, because the set $\{A(0)x : x \in \ D(A(0)) \cap \overline{B}(0,\eta)\}$ is compact.

Let us recall and summarize the following existence and uniqueness result ([14], Theorem 2.4 and Prop. 2.9) which is a variant of Proposition 2.1.

Proposition 3.1. *Suppose that $A(t) : E \rightrightarrows E; t \in [0,1]$, is a closed convex-valued m-accretive operator and assumptions $(H_1), (H_2), (H_3)$ are satisfied. Let $\mathcal{K}$ denotes the compact set in $\mathcal{C}([-r,0])$ with $\varphi(0) \in \overline{D}$ for all $\varphi \in \mathcal{K}$. Let $\varphi \in \mathcal{K}$ and let $g : [0,1] \times \mathcal{C}_0 \to \mathbf{R}^d$ be a Carathéodory mapping such that:*

(i) *there exits $\eta > 0$ such that $||g(t,u) - g(t,v)|| \leq \eta||u - v||_{\mathcal{C}_{\mathbf{R}^d}([-r,0])}$, for all $t \in [0,1]$ and for all $u, v \in \mathcal{C}_{\mathbf{R}^d}([-r,0])$,*

(ii) *there exists* $K > 0$ *such that* $||g(t,u)|| \leq K$, *for all* $(t,u) \in [0,1] \times \mathcal{C}_{\mathbf{R}^d}([-r,0])$.

Then there exists a unique continuous function $u_\varphi : [-r,1] \to \mathbf{R}^d$ *such that its restriction on* $[-r,0]$ *is equal to* φ *and its restriction to* $[0,1]$ *is absolutely continuous, (i.e.* $u_\varphi(t) = \varphi(0) + \int_0^t \dot{u}_\varphi(s)\,ds$, *for all* $t \in [0,1]$ *with* $\dot{u}_\varphi \in L^1_{\mathbf{R}^d}([0,1])$*) and satisfies*

$$(P_\tau)\begin{cases} \dot{u}_\varphi(t) \in -A(t)u_\varphi(t) + g(t,\tau(t)u_\varphi) \ \text{a.e. } t \in [0,1], \\ u_\varphi(s) = \varphi(s), \forall s \in [-r,0]; u_\varphi(t) \in \overline{D}, \forall t \in [0,1]. \end{cases}$$

Moreover, the solution sets $\{u_\varphi : \varphi \in \mathcal{K}\}$ *is relatively compact for the topology of uniform convergence and* $||\dot{u}_\varphi(t)|| \leq m$, *for all* $\varphi \in \mathcal{K}$ *and for a.e.* $t \in [0,1]$, *where* m *is a positive constant which depends only on* $\mathcal{K}(0)$, ρ, L, K.

4. Control problems governed by a nonconvex sweeping process with Young measures

We assume that (S, d_S) and (Z, d_Z) are two compact metric spaces. Let $k(Z)$ be the set of all compact subsets of Z, $\Gamma : [0,1] \to k(Z)$ be a compact valued Lebesgue-measurable multifunction and $\mathcal{M}^1_+(Z)$ be the set of all probability Radon measures on Z. It is well-known that $\mathcal{M}^1_+(S)$ (resp. $\mathcal{M}^1_+(Z)$) is a compact metrizable space for the $\sigma(\mathcal{C}(S)', \mathcal{C}(S))$ (resp. $\sigma(\mathcal{C}(Z)', \mathcal{C}(Z))$) topology. Let us consider a mapping $f : [0,1] \times \mathcal{C}_0 \times S \times Z \to E$ satisfying:

(i) for every $t \in [0,1]$, $f(t,.,.,.)$ is continuous on $\mathcal{C}_0 \times S \times Z$,

(ii) for every $(u,s,z) \in \mathcal{C}_0 \times S \times Z$, $f(.,u,s,z)$ is Lebesgue-measurable on $[0,1]$,

(iii) there is a Lebesgue-integrable function c such that $||f(t,u,s,z)|| \leq c(t)$, for all (t,u,s,z) in $[0,1] \times \mathcal{C}_0 \times S \times Z$,

(iv) there exists a Lipschitz constant $\eta > 0$ such that

$$||f(t,u_1,s,z) - f(t,u_2,s,z)|| \leq \eta\, ||u_1 - u_2||_0$$

for all $(t,u_1,s,z), (t,u_2,s,z) \in [0,1] \times \mathcal{C}_0 \times S \times Z$.

Let $\mathcal{H}$ be a subset of $\mathcal{Y}(\Omega, \mathcal{S}, P; S)$. We consider the solutions sets of the two following functional evolution inclusions

$$(\mathcal{I}_{\mathcal{K},\mathcal{H},\mathcal{O}})\begin{cases} \dot{u}_{\varphi,\mu,\zeta}(t) \in -N(C(t); u_{\varphi,\mu,\zeta}(t)) \\ + \int_S f(t,\tau(t)u_{\varphi,\mu,\zeta}, s, \zeta(t))\,\mu_t(ds), \\ u_{\varphi,\mu,\zeta}(s) = \varphi(s), \forall s \in [-r,0], \end{cases}$$

where $\varphi \in \mathcal{K}, \mu \in \mathcal{H}$, and ζ belongs to the set S_Γ of all original controls, i.e. Lebesgue-measurable selections of Γ, and

$$(\mathcal{I}_{\mathcal{K},\mathcal{H},\mathcal{R}})\begin{cases} \dot{u}_{\varphi,\mu,\nu}(t) \in -N(C(t); u_{\varphi,\mu,\nu}(t)) \\ + \int_{\Gamma(t)} [\int_S f(t,\tau(t)u_{\varphi,\mu,\nu}, s, z)\mu_t(ds)]\,\nu_t(dz), \\ u_{\varphi,\mu,\nu}(s) = \varphi(s), \forall s \in [-r,0], \end{cases}$$

where $\varphi \in \mathcal{K}, \mu \in \mathcal{H}$ and ν belongs to the set $\mathcal{R}$ of relaxed controls, i.e. Lebesgue-measurable selections of the multifunction

$$\Sigma(t) := \{\nu \in \mathcal{M}^1_+(Z) : \nu(\Gamma(t)) = 1\}.$$

Taking Proposition 2.1 into account, for each $(\varphi, \mu, \zeta) \in \mathcal{K} \times \mathcal{H} \times \mathcal{O}$ (resp. $(\varphi, \mu, \nu) \in \mathcal{K} \times \mathcal{H} \times \mathcal{R}$), there is a unique continuous solution $u_{\varphi,\mu,\zeta}$ (resp. $u_{\varphi,\mu,\nu}$) of $(\mathcal{I}_{\mathcal{K},\mathcal{H},\mathcal{O}})$

(resp. $(\mathcal{I}_{\mathcal{K},\mathcal{H},\mathcal{R}})$). Indeed, it is sufficient to observe that, for each $(\mu,\nu)\in\mathcal{H}\times\mathcal{R}$, the function

$$(t,u)\in[0,1]\times\mathcal{C}_0\to f_{\mu,\nu}(t,u):=\int_{\Gamma(t)}[\int_S f(t,u,s,z)\mu_t(ds)]\,\nu_t(dz)$$

inherits the properties the function g given in Proposition 2.1 and so for each $(\varphi,\mu,\nu)\in\mathcal{K}\times\mathcal{H}\times\mathcal{R}$ the evolution inclusion

$$\dot{u}(t)\in -N(C(t),u(t))+f_{\mu,\nu}(t,\tau(t)u),t\in[0,1];\ u(s)=\varphi(s),\forall s\in[-r,0],$$

admits a unique continuous solution.

We aim to present a Bolza-type problem for the two above functional evolution inclusions. In particular, we show the continuous dependence of the solutions $u_{\varphi,\mu,\nu}$ with respect to the data $(\varphi,\mu,\nu)\in\mathcal{K}\times\mathcal{H}\times\mathcal{R}$ and also the compactness with respect to the topology of uniform convergence of the solutions set $\{u_{\varphi,\mu,\nu}:(\varphi,\mu,\nu)\in\mathcal{K}\times\mathcal{H}\times\mathcal{R}\}$. This allows to study the semicontinuity property of the value function associated to a given bounded lower semicontinuous (resp. upper semicontinuous) function $l:E\to\mathbf{R}$, namely

$$(\varphi,\mu,t)\mapsto\inf_{\nu\in\mathcal{R}} l(u_{\varphi,\mu,\nu}(t))$$

and

$$(\varphi,\mu,t)\mapsto\sup_{\nu\in\mathcal{R}} l(u_{\varphi,\mu,\nu}(t))$$

respectively. See [3], [15], [17], [28], [29], [43], [44] for other related results. The results we present in this section are essentially a continuation of the work developed in [16] dealing with an undelayed evolution inclusion governed by nonconvex sweeping process with one class of relaxed controls (i.e. Young measures). For further results on nonconvex sweeping process and related results, see [4], [5], [6] [7], [8], [9], [10], [11], [20] [21], [22], [27], [38] and the references therein. For sweeping process, evolution equations and related results, see [12], [20], [15], [18], [19], [25], [30], [31], [32], [33], [34], [35], [39].

Theorem 4.1. *Assume that the hypotheses and notations of Proposition 2.1 are satisfied and $\mathcal{H}$ is compact for the convergence in probability, $I:[0,1]\times E\times S\times Z\to\mathbf{R}$ is a L^1- bounded Carathéodory integrand, (that is, $I(t,.,.,.)$ is continuous on $E\times S\times Z$, for all $t\in[0,1]$ and $I(.,x,s,z)$ is Lebesgue-measurable on $[0,1]$, for all $(x,s,z)\in E\times S\times Z$) which satisfies the condition: there is a positive Lebesgue-integrable function h such that $|I(t,x,s,y)|\le h(t)$, for all $(t,x,s,z)\in[0,1]\times E\times S\times Z$. Let us consider the control problems*

$$(P_{\mathcal{K},\mathcal{H},\mathcal{O}}):\inf_{(\varphi,\mu,\zeta)\in\mathcal{K}\times\mathcal{H}\times\mathcal{O}}\int_0^1[\int_S I(t,u_{\varphi,\mu,\zeta}(t),s,\zeta(t))\,\mu_t(ds)]\,dt,$$

and

$$(P_{\mathcal{K},\mathcal{H},\mathcal{R}}):\inf_{(\varphi,\mu,\nu)\in\mathcal{K}\times\mathcal{H}\times\mathcal{R}}\int_0^1[\int_Z[\int_S I(t,u_{\varphi,\mu,\nu}(t),s,z))\,\mu_t(ds)]\nu_t(dz)]\,dt,$$

where $u_{\varphi,\mu,\zeta}$ (resp. $u_{\varphi,\mu,\nu}$) is the unique solution associated to (φ,μ,ζ) (resp. (φ,μ,ν)) to the FEI $(\mathcal{I}_{\mathcal{K},\mathcal{H},\mathcal{O}})$ (resp. $(\mathcal{I}_{\mathcal{K},\mathcal{H},\mathcal{R}})$. Then one has $\inf(P_{\mathcal{K},\mathcal{H},\mathcal{O}})=\min(P_{\mathcal{K},\mathcal{H},\mathcal{R}})$.

Proof. Claim 1. The graph of the mapping $(\varphi, \mu, \nu) \mapsto u_{\varphi,\mu,\nu}$ defined on the compact space $\mathcal{K} \times \mathcal{H} \times \mathcal{R}$ with value in the Banach space $\mathcal{C}_E([-r,1])$ of all continuous mappings from $[-r,1]$ into E endowed with the sup norm is compact.

Let $(\varphi^n, \mu^n, \nu^n)$ be a sequence in $\mathcal{K} \times \mathcal{H} \times \mathcal{R}$. As $\mathcal{K}$ is compact in $C_E([-r,0])$, we may suppose that (φ^n) converges uniformly on $[-r,0]$ to a function $\varphi^\infty \in C_E([-r,0])$. As $\mathcal{H} \subset \mathcal{Y}(\Omega, \mathcal{S}, P; S)$ is compact for the convergence in probability, we may assume that (μ^n) converges in probability to a Young measure $\mu^\infty \in \mathcal{H}$. As $\mathcal{R}$ is compact for the stable convergence, we may suppose that (ν^n) stably converges to a Young measure ν^∞ with $\nu_t^\infty(\Gamma(t)) = 1$ a.e.. In view of Proposition 2.1, the sequence $(u_{\varphi^n,\mu^n,\zeta^n})$ is relatively compact in $C_E([-r,1])$, hence we may suppose that $(u_{\varphi^n,\mu^n,\nu^n})$ converges uniformly on $[-r,1]$ to a continuous function $u^\infty \in C_E([-r,1])$ with $u^\infty = \varphi^\infty$ on $[-r,0]$ and $\varphi^\infty(0) \in C(0)$, and $(\dot{u}_{\varphi^n,\mu^n,\nu^n})$ $\sigma(L^1, L^\infty)$ converges to $\dot{u}^\infty \in L_E^1([0,1])$. So, for every $t \in [0,1]$, $(\tau(t)u_{\varphi^n,\mu^n,\nu^n})$ converges to $\tau(t)u^\infty$ in the Banach space $\mathcal{C}_0$. By Proposition 1.1, we conclude that $(\delta_{\tau(.)u_{\varphi^n,\mu^n,\nu^n}} \otimes \mu^n \otimes \nu^n)$ stably converges to $\delta_{\tau(.)u^\infty} \otimes \mu^\infty \otimes \nu^\infty$. Let $h \in L_E^\infty([0,1])$. As the mapping f is L^1-bounded by (iii), so is the integrand $(t,u,s,z) \mapsto \langle h(t), f(t,u,s,z)\rangle$ defined on $[0,1] \times \mathcal{C}_0 \times S \times Z$. Let us put

$$v^n(t) = \int_Z [\int_S f(t, \tau(t)u_{\varphi^n,\mu^n,\nu^n}, s, z)\mu_t^n(ds)]\, \nu_t^n(dz)],\ \forall t \in [0,1],$$

and

$$v^\infty(t) = \int_Z [\int_S f(t, \tau(t)u^\infty, s, z)\mu_t^\infty(ds)]\, \nu_t^\infty(dz)],\ \forall t \in [0,1].$$

Then, by the very property of the stable convergence (cf. Proposition 1.2), we get

$$\lim_{n\to\infty} \int_0^1 \langle h(t), v^n(t)\rangle\, dt = \int_0^1 \langle h(t), v^\infty(t)\rangle\, dt.$$

Now, using the weak convergence in $L_E^1([0,1])$ of $(\dot{u}_{\varphi^n,\mu^n,\nu^n})$ to $\dot{u}^\infty$ and the preceding limit, we conclude that the sequence $(\dot{u}_{\varphi^n,\mu^n,\nu^n} - v^n)$ weakly converges in $L_E^1([0,1])$ to $\dot{u} - v^\infty$. As $u_{\varphi^n,\mu^n,\nu^n}$ is the solution of the corresponding FEI, we have

$$(*) \qquad \dot{u}_{\varphi^n,\mu^n,\nu^n}(t) - v^n(t) \in -N(C(t); u_{\varphi^n,\mu^n,\nu^n}(t))\ a.e.\ t \in [0,1],$$

with $u_{\varphi^n,\mu^n,\nu^n}(s) = \varphi^n(s)$ for all $s \in [-r,0]$. In view of [38], this inclusion is equivalent to

$$(**) \qquad \dot{u}_{\varphi^n,\mu^n,\nu^n}(t) - v^n(t) \in -\psi(t)\partial[d_{C(t)}](u_{\varphi^n,\mu^n,\nu^n}(t))\ \ a.e.\ t \in [0,1],$$

where $\psi(t) = 2c(t) + \dot{v}(t)$ for all $t \in [0,1]$, and $\partial[d_{C(t)}]$ denotes the subdifferential of the distance function $d_{C(t)} : x \mapsto d(x, C(t))$. Since $(u_{\varphi^n,\mu^n,\nu^n})$ converges uniformly to $u^\infty(.)$, by (**) and by virtue of a closure-type lemma in ([13], Theorem VI-4), we get

$$\dot{u}^\infty(t) \in -\psi(t)\partial[d_{C(t)}](u^\infty(t)) + \int_Z [\int_S f(t, \tau(t)u^\infty, s, z)\mu_t^\infty(ds)]\, \nu_t^\infty(dz),$$

with $u^\infty(s) = \varphi^\infty(s)$, for all $s \in [-r,0]$ and $u^\infty(t) \in C(t)$ for all $t \in [0,1]$. So, we have necessarily $u^\infty(.) = u_{\varphi^\infty,\mu^\infty,\nu^\infty}(.)$, where $u_{\varphi^\infty,\mu^\infty,\nu^\infty}$ is the unique continuous solution (Cf. Proposition 2.1) of the FEI

$$(\mathcal{I}_{\mathcal{K},\mathcal{H},\mathcal{R}}) \begin{cases} \dot{u}_{\varphi^\infty,\mu^\infty,\nu^\infty}(t) \in -N(C(t); u_{\varphi^\infty,\mu^\infty,\nu^\infty}(t)) \\ + \int_{\Gamma(t)}[\int_S f(t, \tau(t)u_{\varphi^\infty,\mu^\infty,\nu^\infty}, s, z)\mu_t^\infty(ds)]\, \mu_t^\infty(dz), \\ u_{\varphi^\infty,\mu^\infty,\nu^\infty}(s) = \varphi^\infty(s), \forall s \in [-r, 0]. \end{cases}$$

Claim 2. $\inf(P_{\mathcal{K},\mathcal{H},\mathcal{R}}) = \min(P_{\mathcal{K},\mathcal{H},\mathcal{R}})$.

As $\mathcal{O}$ is dense in $\mathcal{R}$ for the stable topology (see e.g. [14], [17], [21]), it is enough to show that the mapping

$$\Psi : (\varphi, \mu, \nu) \to \int_0^1 [\int_Z [\int_S I(t, u_{\varphi,\mu,\nu}(t), s, z)\, \mu_t(ds)]\nu_t(dz)]\, dt$$

is continuous on $\mathcal{K}\times\mathcal{H}\times\mathcal{R}$. This fact follows easily from Claim 1 and the arguments therein. Indeed let $(\varphi^n, \mu^n, \nu^n)$ be a sequence in $\mathcal{K} \times \mathcal{H} \times \mathcal{R}$ with $\varphi^n \to \varphi \in \mathcal{K}$ in the Banach space $\mathcal{C}_0$, $\mu^n \to \mu \in \mathcal{H}$ in probability, and $\nu^n \to \nu \in \mathcal{R}$ stably. Then $(u_{\varphi^n,\mu^n,\nu^n})$ converges uniformly on $[-r, 1]$ to a continuous function $u_{\varphi,\mu,\nu} \in C_E([-r, 1])$ with $u_{\varphi,\mu,\nu} = \varphi$ on $[-r, 0]$, and

$$\begin{cases} \dot{u}_{\varphi,\mu,\nu}(t) \in -N(C(t); u_{\varphi,\mu,\nu}(t)) \\ + \int_{\Gamma(t)}[\int_S f(t, \tau(t)u_{\varphi,\mu,\nu}, s, z)\mu_t(ds)]\, \nu_t(dz), \ \ a.e.\ t \in [0, 1]. \end{cases}$$

The continuity of Ψ follows because $(\delta_{u_{\varphi^n,\mu^n,\nu^n}} \otimes \mu^n \otimes \nu^n)$ stably converges to $\delta_{u_{\varphi,\mu,\nu}} \otimes \mu \otimes \nu$ and I is a L^1-bounded Carathéodory integrand. □

Now we are able to present two variational properties for the value function associated to a cost function mentioned above. We only deal with the lower semicontinuous cost with the upper semicontinuous case being analoguous.

Proposition 4.1. *Assume that the hypotheses of Theorem 4.1 are satisfied. Let* $u_{\varphi,\mu,\nu}$ *be the unique solution of*

$$(\mathcal{I}_{\mathcal{K},\mathcal{H},\mathcal{R}}) \begin{cases} \dot{u}_{\varphi,\mu,\nu}(t) \in -N(C(t); u_{\varphi,\mu,\nu}(t)) \\ + \int_{\Gamma(t)}[\int_S f(t, \tau(t)u_{\varphi,\mu,\nu}, s, z)\mu_t(ds)]\, \nu_t(dz), \\ u_{\varphi,\mu,\nu}(s) = \varphi(s), \forall s \in [-r, 0]; u_{\varphi,\mu,\nu}(t) \in C(t), \forall t \in [0, 1]. \end{cases}$$

Let $l : E \to \mathbf{R}$ *be a bounded lower semicontinuous function and let* V_l *the value function*

$$V_l : (t, \varphi, \mu) \mapsto \inf_{\nu\in\mathcal{R}} l(u_{\varphi,\mu,\nu}(t)), \forall (t, \varphi, \mu) \in [0, 1] \times \mathcal{K} \times \mathcal{H}.$$

Let (l^i) *be an increasing sequence of bounded Lipshitzean functions defined on* E *converging pointwisely to* l, *let* (φ^i) *be a sequence in* $\mathcal{K}$ *converging uniformly to* φ, *let* (μ^i) *be a sequence in* $\mathcal{H}$ *converging in probability to* μ *and let* (t^i) *be a sequence in* $[0, 1]$ *converging to* t, *then we have*

$$\liminf_i V_{l^i}(t^i, \varphi^i, \mu^i) \geq V_l(t, \varphi, \mu).$$

Proposition 4.2. *Assume that the hypotheses of Theorem 4.1 are satisfied. Let* $u_{\varphi,\mu,\nu}$ *be the unique solution of*

$$(\mathcal{I}_{\mathcal{K},\mathcal{H},\mathcal{R}}) \begin{cases} \dot{u}_{\varphi,\mu,\nu}(t) \in -N(C(t); u_{\varphi,\mu,\nu}(t)) \\ + \int_{\Gamma(t)}[\int_S f(t, \tau(t)u_{\varphi,\mu,\nu}, s, z)\mu_t(ds)]\, \nu_t(dz), \\ u_{\varphi,\mu,\nu}(s) = \varphi(s), \forall s \in [-r, 0]; u_{\varphi,\mu,\nu}(t) \in C(t), \forall t \in [0, 1]. \end{cases}$$

Let $h : [0,1] \times E \times S \times Z \to \mathbf{R}^+$ be a L^1-bounded normal integrand, that is $h(t,.,.,.)$ is lower semicontinuous on $E \times S \times Z$ and h is $\mathcal{L}([0,1]) \times \mathcal{B}(E) \times S \times Z)$-measurable and is dominated by a positive Lebesgue-integrable function and let

$$W_h(\varphi,\mu) = \inf_{\nu \in \mathcal{R}} \int_0^1 [\int_{\Gamma(t)} [\int_S h(t, u_{\varphi,\mu,\nu}(t), s, z)\mu_t(ds)]\, \nu_t(dz)]dt.$$

Let (h^i) be an increasing sequence of L^1-bounded Carathéodory integrands defined on $[0,1] \times E \times S \times Z$ such that $h = \sup_i h^i$, let (φ^i) be a sequence in $\mathcal{K}$ converging uniformly to φ, let (μ^i) be a sequence in $\mathcal{H}$ converging in probability to $\mu \in \mathcal{H}$. Then we have

$$\liminf_i W_{h^i}(\varphi^i, \mu^i) \geq W_h(\varphi,\mu),$$

where for all i,

$$W_{h^i}(\varphi^i, \mu^i) := \inf_{\nu \in \mathcal{R}} \int_0^1 [\int_{\Gamma(t)} [\int_S h^i(t, u_{\varphi^i,\mu^i,\nu}(t), s, z)\mu^i_t(ds)]\nu_t(dz)]dt.$$

Proof. We omit the details of the proofs since they are very similar to these of ([17], Prop.3.1.4-3.1.5). We only observe that the function I_h where

$$I_h(\varphi,\mu,\nu) = \int_0^1 [\int_Z [\int_S h(t, u_{\varphi,\mu,\nu}(t), s, z)\mu_t(ds)]\, \nu_t(dz)]dt.$$

is lower semicontinuous on the compact space $\mathcal{K} \times \mathcal{H} \times \mathcal{R}$ since it is the supremum of the continuous functions I_{h^i}

$$I_{h^i}(\varphi,\mu,\nu) = \int_0^1 [\int_Z [\int_S h^i(t, u_{\varphi,\mu,\nu}(t), s, z)\mu_t(ds)]\, \nu_t(dz)]dt.$$

so that W_h is lower semicontinuous on the compact space $\mathcal{K} \times \mathcal{H}$. □

Remarks. The conclusions of Theorem 4.1 and Propositions 4.2-4.3 may fail if one assumes that $\mathcal{H}$ is only compact for the stable topology (instead of the topology of convergence in probablity) because the fiber product theorem is not valid: the sequence $(\delta_{\tau(.)u_{\varphi^n,\mu^n,\mu^n}} \otimes \mu^n \otimes \nu^n)$ does not necessarily stably converges to $\delta_{\tau(.)u^\infty} \otimes \mu^\infty \otimes \nu^\infty$. See [41] for a counter example.

We shall establish some new properties of the lower value function of a control problem governed by a nonconvex sweeping process in the particular case when τ is the zero mapping. Briefly we will deal with an undelayed evolution inclusion governed by nonconvex sweeping process. Our aim is to produce a viscosity type solution to this kind of evolution inclusion where the controls are Young measures. This shed a new light on the classical Hamilton-Jacobi-Bellman equation associated to ordinary differential equations. For this purpose we will impose some restrictions. However, the reader will see that these conditions are necessary even for classical ODE. We will assume that

(K_1) $f : [0,1] \times E \times S \times Z \to E$ is bounded, say, $||f(t,x,s,z)|| \leq M$, for all $(t,x,s,z) \in [0,1] \times E \times S \times Z$, continuous on $[0,1] \times E \times S \times Z$ and uniformly Lipschitzean in $x \in E$,

(K_2) $J : [0,1] \times E \times S \times Z \to \mathbf{R}$ is bounded, say, $|J(t,x,s,z)| \leq N$, for all $(t,x,s,z) \in [0,1] \times E \times S \times Z$, continuous on $[0,1] \times E \times S \times Z$,

(K_3) $C : [0,1] \to E$ is a k-Lipschitzean multifunction with nonempty compact ρ-proximal values in E, that is

$$|d(x, C(t)) - d(y, C(s))| \leq ||x - y|| + k|t - s|$$

$\forall (x, y \in E$ and $\forall (s,t) \in [0,1]$,

(K_4) Assume further that $\mathcal{H}$ and $\mathcal{R}$ are the set of all Lebesgue-measurable mappings from $[0,1]$ into $\mathcal{M}^1_+(S)$ and $\mathcal{M}^1_+(Z)$ respectively. In particular, $\mathcal{H}$ and $\mathcal{R}$ are compact for the stable topology in the space of Young measures $\mathcal{Y}([0,1], S)$ and $\mathcal{Y}([0,1], Z)$ respectively.

By virtue of Theorem 1.5 in [21], for each $\tau \in [0,1]$ and each $x \in C(\tau)$, for each $\mu \in \mathcal{H}$ and for each $\nu \in \mathcal{R}$, there is a unique absolutely continuous solution $u_{x,\mu,\nu}$ of the evolution inclusion

$$\begin{cases} \dot{u}_{x,\mu,\nu}(t) \in -N(C(t); u_{x,\mu,\nu}(t)) \\ + \int_Z [\int_S f(t, u_{x,\mu,\nu}(t), s, z) \mu_t(ds)]\, \nu_t(dz), \; a.e.\ t \in [\tau, 1], \\ u_{x,\mu,\nu}(\tau) = x \in C(\tau). \end{cases}$$

Furthermore, $||\dot{u}_{x,\mu,\nu}(t)|| \leq k + 2M$ for all (t, x, μ, ν). See e.g. ([21], Prop. 1.1). Note that $\tau \in [0,1]$ denotes here the intermediate time, and there is no risk of confusion with the mapping τ given in the above functional evolution inclusions. Before going further, it is worthy to recall [38] that $u_{x,\mu,\nu}$ is solution of the above evolution inclusion iff

$$\begin{cases} \dot{u}_{x,\mu,\nu}(t) \in -(k + 2M)\partial[d_{C(t}](u_{x,\mu,\nu}(t)) \\ + \int_Z [\int_S f(t, u_{x,\mu,\nu}(t), s, z) \mu_t(ds)]\, \nu_t(dz), \; a.e.\ t \in [\tau, 1], \\ u_{x,\mu,\nu}(t) \in C(t), \forall t \in [\tau, 1], \\ u_{x,\mu,\nu}(\tau) = x \in C(\tau). \end{cases}$$

The following is a dynamic programming principle theorem is similar to Theorem 3.2.1 in [17], using the fiber product lemma for Young measures (see Proposition 1.1 or [17], Theorem 2.3.1).

Theorem 4.2. *Assume that* $(K_1), (K_2), (K_3), (K_4)$ *are satisfied. Let us consider the lower value function*

$$U_J(\tau, x) := \sup_{\nu \in \mathcal{R}} \inf_{\mu \in \mathcal{H}} \{ \int_\tau^1 [\int_Z [\int_S J(t, u_{x,\mu,\nu}(t), s, z)\, \mu_t(ds)] \nu_t(ds)]\, dt \},$$

where $u_{x,\mu,\nu}$ *is the unique trajectory solution of*

$$\begin{cases} \dot{u}_{x,\mu,\nu}(t) \in -N(C(t); u_{x,\mu,\nu}(t)) \\ + \int_Z [\int_S f(t, u_{x,\mu,\nu}(t), s, z) \mu_t(ds)]\, \nu_t(dz), \; a.e.\ t \in [\tau, 1], \\ u_{x,\mu,\nu}(\tau) = x \in C(\tau). \end{cases}$$

Then for any $\sigma \in]0,1[$ *with* $\tau + \sigma < 1$,

$$U_J(\tau, x) = \sup_{\nu \in \mathcal{R}} \inf_{\mu \in \mathcal{H}} \{ \int_\tau^{\tau+\sigma} [\int_S J(u_{x,\mu,\nu}(t), s, z)\, \mu_t(ds)] \nu_t(ds)]\, dt \\ + U_J(\tau + \sigma, u_{x,\mu,\nu}(\tau + \sigma)) \},$$

where

$$U_J(\tau+\sigma, u_{x,\mu,\nu}(\tau+\sigma)) = \sup_{\gamma\in\mathcal{R}} \inf_{\beta\in\mathcal{H}} \int_{\tau+\sigma}^1 [\int_Z [\int_S J(v_{x,\beta,\gamma}(t), s, z)\, \beta_t(ds)]\gamma_t(ds)]\, dt,$$

where $v_{x,\beta,\gamma}$ denotes the trajectory solution of the above dynamic associated to the controls $(\beta,\gamma) \in \mathcal{H}\times\mathcal{R}$ with intial condition $v_{x,\beta,\gamma}(\tau+\sigma) = u_{\varphi,\mu,\nu}(\tau+\sigma)$.

We will use a technical result which extends Lemma 2.7 in [16].

Lemma 4.1. *Assume that $(K_1), (K_2), (K_3), (K_4)$ are satisfied. Let $(t_0, x_0) \in [0,1]\times C(t_0)$. Assume that $\Lambda_1 : [0,1]\times E\times \mathcal{M}^1_+(S)\times\mathcal{M}^1_+(Z) \to \mathbf{R}$ is a continuous integrand and $\Lambda_2 : [0,1]\times E\times\mathcal{M}^1_+(Z)\to\mathbf{R}$ is an upper semicontinuous integrand such that, for any bounded subset B of E, $\Lambda_2|_{[0,1]\times B\times\mathcal{M}^1_+(Z)}$ is bounded, and assume that $\Lambda := \Lambda_1+\Lambda_2$ satisfies the following condition*

$$\inf_{\mu\in\mathcal{M}^1_+(S)} \sup_{\nu\in\mathcal{M}^1_+(Z)} \Lambda(t_0, x_0, \mu, \nu) < -\eta < 0 \text{ for some } \eta > 0.$$

Then there is $\overline{\mu}\in\mathcal{M}^1_+(S)$ and $\sigma > 0$ such that

$$\sup_{\nu\in\mathcal{R}} \int_{t_0}^{t_0+\sigma} \Lambda(t, u_{x_0,\overline{\mu},\nu}(t), \overline{\mu}, \nu_t)dt < -\sigma\eta/2,$$

where $u_{x_0,\overline{\mu},\nu}$ denotes the unique trajectory solution of

$$\begin{cases} \dot u_{x_0,\overline{\mu},\nu}(t) \in -N(C(t); u_{x_0,\overline{\mu},\nu}(t)) + \int_Z[\int_S f(t, u_{x_0,\overline{\mu},\nu}(t), z)\overline{\mu}(ds)]\,\nu_t(dz) \\ u_{x_0,\overline{\mu},\nu}(t_0) = x_0 \in C(t_0), \end{cases}$$

associated with the controls $(\overline{\mu},\nu)\in\mathcal{M}^1_+(S)\times\mathcal{R}$.

Proof. By hypothesis,

$$\inf_{\mu\in\mathcal{M}^1_+(S)} \sup_{\nu\in\mathcal{M}^1_+(Z)} \Lambda(t_0, x_0, \mu, \nu) < -\eta < 0,$$

that is,

$$\inf_{\mu\in\mathcal{M}^1_+(S)} \sup_{\nu\in\mathcal{M}^1_+(Z)} [\Lambda_1(t_0, x_0, \mu, \nu) + \Lambda_2(t_0, x_0, \nu)] < -\eta < 0.$$

As the function Λ_1 is continuous, the function

$$\mu \mapsto \sup_{\nu\in\mathcal{M}^1_+(Z)} \Lambda_1(t_0, x_0, \mu, \nu)$$

is continuous on $\mathcal{M}^1_+(S)$, so is the function

$$\mu \mapsto \sup_{\nu\in\mathcal{M}^1_+(Z)} [\Lambda_1(t_0, x_0, \mu, \nu) + \Lambda_2(t_0, x_0, \nu)] = \sup_{\nu\in\mathcal{M}^1_+(Z)} \Lambda_1(t_0, x_0, \mu, \nu) + \sup_{\nu\in\mathcal{M}^1_+(Z)} \Lambda_2(t_0, x_0, \nu).$$

Hence there exists $\overline{\mu}\in\mathcal{M}^1_+(S)$ such that

$$\sup_{\nu\in\mathcal{M}^1_+(Z)} \Lambda(t_0, x_0, \overline{\mu}, \nu) = \inf_{\mu\in\mathcal{M}^1_+(S)} \sup_{\nu\in\mathcal{M}^1_+(Z)} \Lambda(t_0, x_0, \mu, \nu).$$

As the function $(t,x) \mapsto \sup_{\nu\in\mathcal{M}^1_+(Z)} \Lambda_1(t,x,\overline{\mu},\nu)$ is continuous and the function $(t,x) \mapsto \sup_{\nu\in\mathcal{M}^1_+(Z)} \Lambda_2(t,x,\nu)$ is upper semicontinuous, there is $\zeta > 0$ such that

$$\sup_{\nu\in\mathcal{M}^1_+(Z)} \Lambda(t,x,\overline{\mu},\nu) < -\eta/2,$$

for $0 < t - t_0 \leq \zeta$ and $||x - x_0|| \leq \zeta$. For all $\nu \in \mathcal{R}$, we recall the estimate (see also the proof of Theorem 4.1)

$$||\dot{u}_{x_0,\overline{\mu},\nu}(t)|| \leq k + 2M \quad a.e.$$

Thus for $\sigma > 0$ such that $\sigma(k + 2M) \leq \zeta$, we get

$$||u_{x_0,\overline{\mu},\nu}(t) - u_{x_0,\overline{\mu},\nu}(t_0)|| \leq \zeta,$$

for all $t \in [t_0, t_0 + \sigma]$ and for all $\nu \in \mathcal{R}$. Hence the functions $\Lambda(t, u_{x_0,\overline{\mu},\nu}(t), \overline{\mu}, \nu_t)$ bounded and Lebesgue-measurable on $[t_0, t_0 + \sigma]$. Then by integrating

$$\begin{aligned}\int_{t_0}^{t_0+\sigma} \Lambda(t, u_{x_0,\overline{\mu},\nu}(t), \overline{\mu}, \nu_t)dt &\leq \int_{t_0}^{t_0+\sigma} [\sup_{\nu'\in\mathcal{M}^1_+(Z)} \Lambda(t, u_{x_0,\overline{\mu},\nu'}(t), \overline{\mu}, \nu')]dt \\ &< -\sigma\eta/2 < 0,\end{aligned}$$

for all $\nu \in \mathcal{R}$ and the result follows. □

Theorem 4.3. *(Existence of viscosity solution)*
Assume that $(K_1), (K_2), (K_3), (K_4)$ are satisfied. Let us consider the lower value function

$$U_J(\tau, x) := \sup_{\nu\in\mathcal{R}} \inf_{\mu\in\mathcal{H}} \{\int_\tau^1 [\int_Z [\int_S J(t, u_{x,\mu,\nu}(t), s, z)\, \mu_t(ds)]\nu_t(dz)]\, dt\}.$$

Let us consider the upper Hamiltonian

$$H^+(t,x,y) = \inf_{\mu\in\mathcal{M}^1_+(S)} \sup_{\nu\in\mathcal{M}^1_+(Z)} \{\langle y, \widetilde{f}(t,x,\mu,\nu)\rangle + \widetilde{J}(t,x,\mu,\nu)\},$$

where

$$\widetilde{f}(t,x,\mu,\nu) := \int_Z [\int_S f(t,x,s,z)\mu(ds)]\nu(dz),$$

$$\widetilde{J}(t,x,\mu,\nu) := \int_Z [\int_S J(t,x,s,z)\mu(ds)]\nu(dz),$$

and the perturbed Hamiltonian

$$H(t,x,y) := H^+(t,x,y) + \delta^*(y, -(k+2M)\partial[d_{C(t)}](x)).$$

Then U_J is a viscosity subsolution of the perturbed Hamilton-Jacobi-Bellman equation

$$U_t(t,x) + H^+(t,x,\nabla U(t,x)) + \delta^*(\nabla U(t,x), -(k+2M)\partial[d_{C(t)}](x)) = 0,$$

that is, if for any $\varphi \in \mathcal{C}^1([0,1] \times E)$ for which $U_J - \varphi$ reaches a local maximum at $(t_0, x_0) \in Graph\, C$, then

$$\frac{\partial\varphi}{\partial t}(t_0,x_0) + H^+(t_0,x_0,\nabla\varphi(t_0,x_0))$$

$$+\delta^*(\nabla\varphi(t_0,x_0), -(k+2M)\partial[d_{C(t_0)}](x_0)) \geq 0.$$

Proof. We will make use of some arguments developed in [26], [24], [16], [17]. Assume by contradiction that there exists a $\varphi \in \mathcal{C}^1([0,1]\times E)$ and a point (t_0,x_0) in the graph of C for which

$$\frac{\partial\varphi}{\partial t}(t_0,x_0) + H^+(t_0,x_0,\nabla\varphi(t_0,x_0)) \\ + \delta^*(\nabla\varphi(t_0,x_0), -(k+2M)\partial[d_{C(t_0)}](x_0)) \leq -\eta,$$

for some $\eta > 0$. Applying Lemma 4.1 to $\Lambda := \Lambda_1 + \Lambda_2$, with

$$\Lambda_1 = \widetilde{J} + \langle\nabla\varphi, \widetilde{f}\rangle + \frac{\partial\varphi}{\partial t},$$

and

$$\Lambda_2(t,x) = \delta^*(\nabla\varphi(t,x), -(k+2M)\partial[d_{C(t)}](x)), \quad \forall (t,x) \in [0,1]\times E,$$

provides $\overline{\mu} \in \mathcal{M}^1_+(S)$ and $\sigma > 0$ such that

$$\begin{aligned}\sup_{\nu\in\mathcal{R}}\{\int_{t_0}^{t_0+\sigma} [\int_Z[\int_S J(t,u_{x_0,\overline{\mu},\nu}(t),s,z)\overline{\mu}(ds)]\nu_t(dz)]\,dt \\ + \int_{t_0}^{t_0+\sigma}[\int_Z[\int_S \langle\nabla\varphi(t,u_{x_0,\overline{\mu},\nu}(t)), f(t,u_{x_0,\overline{\mu},\nu}(t),s,z)\rangle\overline{\mu}(ds)]\nu_t(dz)]\,dt \\ + \int_{t_0}^{t_0+\sigma} \delta^*(\nabla\varphi(t,u_{x_0,\overline{\mu},\nu}(t)), -(k+2M)\partial[d_{C(t)}](u_{x_0,\overline{\mu},\nu}(t)))dt \\ + \int_{t_0}^{t_0+\sigma}\frac{\partial\varphi}{\partial t}(t,u_{x_0,\overline{\mu},\nu}(t))dt\} \\ \leq -\sigma\eta/2.\end{aligned}$$

Thus

$$\begin{aligned}(4.3.1)\quad \sup_{\nu\in\mathcal{R}}\inf_{\mu\in\mathcal{H}}\{\int_{t_0}^{t_0+\sigma}[\int_Z[\int_S J(t,u_{x_0,\mu,\nu}(t),s,z)\mu_t(ds)]\nu_t(dz)]\,dt \\ + \int_{t_0}^{t_0+\sigma}[\int_Z[\int_S\langle\nabla\varphi(t,u_{x_0,\mu,\nu}(t)), f(t,u_{x_0,\mu,\nu}(t),s,z)\rangle\mu_t(ds)]\nu_t(dz)]\,dt \\ + \int_{t_0}^{t_0+\sigma}\delta^*(\nabla\varphi(t,u_{x_0,\mu,\nu}(t)), -(k+2M)\partial[d_{C(t)}](u_{x_0,\mu,\nu}(t)))dt \\ + \int_{t_0}^{t_0+\sigma}\frac{\partial\varphi}{\partial t}(t,u_{x_0,\mu,\nu}(t))dt\} \\ \leq -\sigma\eta/2,\end{aligned}$$

where $u_{x_0,\mu,\nu}$ is the trajectory solution associated with the control $(\mu,\nu) \in \mathcal{H}\times\mathcal{R}$ of

$$\begin{cases} \dot{u}_{x_0,\mu,\nu}(t) \in -N(C(t);u_{x_0,\mu,\nu}(t)) \\ +\int_Z[\int_S f(t,u_{x_0,\mu,\nu}(t),s,z)\mu_t(ds)]\,\nu_t(dz), \;\; a.e.\; t\in[0,1], \\ u_{x,\mu,\nu}(t_0) = x_0 \in C(t_0). \end{cases}$$

From Theorem 4.2 (of dynamic programming) (see e.g [17], Theorem 3.2.1) we deduce

$$(4.3.2)\qquad U_J(t_0,x_0)=\sup_{\nu\in\mathcal{R}}\inf_{\mu\in\mathcal{H}}\{\int_{t_0}^{t_0+\sigma}[\int_Z[\int_S J(t,u_{x_0,\mu,\nu}(t),z)\mu_t(ds)]\,\nu_t(dz)]dt + U_J(t_0+\sigma,u_{x_0,\mu,\nu}(t_0+\sigma))\}.$$

Since $U_J-\varphi$ has a local maximum at (t_0,x_0), so for σ small enough

$$(4.3.3)\qquad U_J(t_0,x_0)-\varphi(t_0,x_0) \geq U_J(t_0+\sigma,u_{x_0,\mu,\nu}(t_0+\sigma))-\varphi(t_0+\sigma,u_{x_0,\mu,\nu}(t_0+\sigma)).$$

From (4.3.2) and (4.3.3) we get

$$(4.3.4)\qquad \sup_{\nu\in\mathcal{R}}\inf_{\mu\in\mathcal{H}}\{\int_{t_0}^{t_0+\sigma}[\int_Z[\int_S J(t,u_{x,\mu,\nu}(t),s,z)\mu_t(ds)]\,\nu_t(dz)]dt +\varphi(t_0+\sigma,u_{x_0,\mu,\nu}(t_0+\sigma))-\varphi(t_0,x_0)\}\geq 0.$$

As φ is $\mathcal{C}^1$ and $u_{x_0,\mu,\nu}$ is the trajectory solution of our dynamic

$$\begin{aligned}(4.3.5)\qquad &\varphi(t_0+\sigma,u_{x_0,\mu,\nu}(t_0+\sigma))-\varphi(t_0,x_0)\\ &=\int_{t_0}^{t_0+\sigma}\langle\nabla\varphi(t,u_{x_0,\mu,\nu}(t)),\dot u_{x_0,\mu,\nu}(t)\rangle\,dt\\ &\quad+\int_{t_0}^{t_0+\sigma}\frac{\partial\varphi}{\partial t}(t,u_{x_0,\mu,\nu}(t))\,dt\\ &\leq\int_{t_0}^{t_0+\sigma}[\int_Z[\int_S\langle\nabla\varphi(t,u_{x_0,\mu,\nu}(t)),f(t,u_{x_0,\mu,\nu}(t),s,z)\rangle\mu_t(ds)]\nu_t(dz)]\,dt\\ &\quad+\int_{t_0}^{t_0+\sigma}\delta^*(\nabla\varphi(t,u_{x_0,\mu,\nu}(t)),-(k+2M)\partial[d_{C(t)}](u_{x_0,\mu,\nu}(t)))\,dt\\ &\quad+\int_{t_0}^{t_0+\sigma}\frac{\partial\varphi}{\partial t}(t,u_{x_0,\mu,\nu}(t))\,dt.\end{aligned}$$

Using the estimate (4.3.5) and coming back to (4.3.4), we have a contradiction to (4.3.1). Therefore we must have

$$\frac{\partial\varphi}{\partial t}(t_0,x_0)+H^+(t_0,x_0,\nabla\varphi(t_0,x_0)) +\delta^*(\nabla\varphi(t_0,x_0),-(k+2M)\partial[d_{C(t_0)}](x_0))\geq 0.$$

□

5. Control problems governed by an m-accretive operator with Young measures

In this section the control spaces of Young measures are the same as given in section 4. Let us consider now a closed convex valued m-accretive operator $A(t): E\rightrightarrows E\,(t\in[0,1])$ satisfying the hypotheses (H_1) (H_2) (H_3) of Proposition 3.4 and a compact set $\mathcal{K}$ in $\mathcal{C}([-r,0])$ with $\varphi(0)\in\overline{D}$ for all $\varphi\in\mathcal{K}$ where $D:=D(A(t))$ for all $t\in[0,1]$. Let us consider a mapping $g:[0,1]\times\mathcal{C}_0\times S\times Z\to E$ satisfying:

(i) for every $t \in [0,1], g(t,.,.,.)$ is continuous on $\mathcal{C}_0 \times S \times Z$,
(ii) for every $(u,s,z) \in \mathcal{C}_0 \times S \times Z, g(.,u,s,z)$ is Lebesgue-measurable on $[0,1]$,
(iii) there is a constant $c > 0$ such that $||g(t,u,s,z)|| \leq c$ for all (t,u,s,z) in $[0,1] \times \mathcal{C}_0 \times S \times Z$,
(iv) there exists a Lipschitz constant $\eta > 0$ such that

$$||g(t,u_1,s,z) - g(t,u_2,s,z)|| \leq \eta\, ||u_1 - u_2||_0$$

$\forall (t,u_1,s,z), (t,u_2,s,z) \in [0,1] \times \mathcal{C}_0 \times S \times Z$.
Let $\mathcal{H}$ be a subset of Young measures $\mathcal{Y}([0,1],S)$. We consider the solutions sets of the two following functional evolution inclusions

$$(\mathcal{J}_{\mathcal{K},\mathcal{H},\mathcal{O}}) \begin{cases} \dot{u}_{\varphi,\mu,\zeta}(t) \in -A(t)u_{\varphi,\mu,\zeta}(t) + \int_S g(t,\tau(t)u_{\varphi,\mu,\zeta}, s, \zeta(t))\, \mu_t(ds), \\ u_{\varphi,\mu,\zeta}(s) = \varphi(s), \forall s \in [-r,0];\ u_{\varphi,\mu,\zeta}(t) \in \overline{D}, \forall t \in [0,1], \end{cases}$$

where $\varphi \in \mathcal{K}, \mu \in \mathcal{H}$, and ζ belongs to the set S_Γ of all original controls, i.e. Lebesgue-measurable selections of Γ, and

$$(\mathcal{J}_{\mathcal{K},\mathcal{H},\mathcal{R}}) \begin{cases} \dot{u}_{\varphi,\mu,\nu}(t) \in -A(t)u_{\varphi,\mu,\nu}(t) \\ +\int_{\Gamma(t)}[\int_S g(t,\tau(t)u_{\varphi,\mu,\nu}, s, z)\mu_t(ds)]\, \nu_t(dz), \\ u_{\varphi,\mu,\nu}(s) = \varphi(s), \forall s \in [-r,0];\ u_{\varphi,\mu,\zeta}(t) \in \overline{D}, \forall t \in [0,1], \end{cases}$$

where $\varphi \in \mathcal{K}, \mu \in \mathcal{H}$ and $\nu \in \mathcal{R}$ Taking Proposition 3.1 into account, for each $(\varphi,\mu,\zeta) \in \mathcal{K} \times \mathcal{H} \times \mathcal{O}$ (resp. $(\varphi,\mu,\nu) \in \mathcal{K} \times \mathcal{H} \times \mathcal{R}$), there is a unique continuous solution $u_{\varphi,\mu,\zeta}$ to $(\mathcal{J}_{\mathcal{K},\mathcal{H},\mathcal{O}})$ (resp. $u_{\varphi,\mu,\nu}$ to $(\mathcal{J}_{\mathcal{K},\mathcal{H},\mathcal{R}})$. Indeed, it is enough to observe that, for each $(\mu,\nu) \in \mathcal{H} \times \mathcal{R}$, the function

$$(t,u) \in [0,1] \times \mathcal{C}_0 \to g_{\mu,\nu}(t,u) := \int_{\Gamma(t)} [\int_S g(t,u,s,z)\mu_t(ds)]\, \nu_t(dz)$$

inherits the properties of the function g given in this proposition, so that for each $(\varphi,\mu,\nu) \in \mathcal{K} \times \mathcal{H} \times \mathcal{R}$, there is a unique solution $u_{\varphi,\mu,\nu}$ of the functional evolution inclusion

$$\dot{u}(t) \in -A(t)u(t) + g_{\mu,\nu}(t,\tau(t)u), t \in [0,1];\ u(s) = \varphi(s), \forall s \in [-r,0].$$

Now let us consider a Bolza-type problem for the above functional evolution inclusion and discuss the continuous dependence of the solutions $u_{\varphi,\mu,\nu}$ with respect to the data $(\varphi,\mu,\nu) \in \mathcal{K} \times \mathcal{H} \times \mathcal{R}$ and the compactness (with respect to the topology of uniform convergence) of the solutions set $\{u_{\varphi,\mu,\nu} : (\varphi,\mu,\nu) \in \mathcal{K} \times \mathcal{H} \times \mathcal{R}\}$. This will allow to obtain the semicontinuity property of the value function associated to a given bounded lower semicontinuous (resp. upper semicontinuous) function $l : E \to \mathbf{R}$, namely

$$(\varphi,\mu,t) \mapsto \inf_{\nu \in \mathcal{R}} l(u_{\varphi,\mu,\nu}(t))$$

and

$$(\varphi,\mu,t) \mapsto \sup_{\nu \in \mathcal{R}} l(u_{\varphi,\mu,\nu}(t))$$

respectively and even for the case of when the cost function is an integral functional in the same way as in Proposition 4.2.

Theorem 5.1. *Assume that $\mathcal{H}$ is compact for the convergence in probability, $I : [0,1]\times E\times S\times Z \to \mathbf{R}$ is a L^1- bounded Carathéodory integrand, (that is, $I(t,.,.,.)$ is continuous on $E\times S\times Z$, for all $t\in[0,1]$ and $I(.,x,s,z)$ is Lebesgue-measurable on $[0,1]$, for all $(x,s,z)\in E\times S\times Z$) which satisfies the condition: there is a positive Lebesgue integrable function h such that $|I(t,x,s,y)| \le h(t)$ for all $(t,x,s,z) \in [0,1]\times E\times S\times Z$. Let us consider the control problems*

$$(P_{\mathcal{K},\mathcal{H},\mathcal{O}}) : \inf_{(\varphi,\mu,\zeta)\in\mathcal{K}\times\mathcal{H}\times\mathcal{O}} \int_0^1 [\int_S I(t, u_{\varphi,\mu,\zeta}(t), s, \zeta(t))\, \mu_t(ds)]\, dt$$

and

$$(P_{\mathcal{K},\mathcal{H},\mathcal{R}}) : \inf_{(\varphi,\mu,\nu)\in\mathcal{K}\times\mathcal{H}\times\mathcal{R}} \int_0^1 [\int_{\Gamma(t)} [\int_S I(t, u_{\varphi,\mu,\nu}(t), s, z))\, \mu_t(ds)]\nu_t(dz)]\, dt$$

where $u_{\varphi,\mu,\zeta}$ (resp. $u_{\varphi,\mu,\nu}$) is the unique solution associated to (φ,μ,ζ) (resp. (φ,μ,ν)) to the FEI $(\mathcal{J}_{\mathcal{K},\mathcal{H},\mathcal{O}})$ (resp. $(\mathcal{J}_{\mathcal{K},\mathcal{H},\mathcal{R}})$. Then one has $\inf(P_{\mathcal{K},\mathcal{H},\mathcal{O}}) = \min(P_{\mathcal{K},\mathcal{H},\mathcal{R}})$.

Proof. Claim 1. The mapping $(\varphi,\mu,\nu) \mapsto u_{\varphi,\mu,\nu}$ defined on the compact space $\mathcal{K}\times\mathcal{H}\times\mathcal{R}$ with value in the Banach space $\mathcal{C}_E([-r,1])$ of all continuous mappings from $[-r,1]$ into E endowed with the sup norm has a compact graph.

Let (φ^n,μ^n,ν^n) be a sequence in $\mathcal{K}\times\mathcal{H}\times\mathcal{R}$. As $\mathcal{K}$ is compact in $C_E([-r,0])$, we may suppose that (φ^n) converges uniformly on $[-r,0]$ to a function $\varphi^\infty \in C_E([-r,0])$. As $\mathcal{H}\subset\mathcal{Y}(\Omega,\mathcal{S},P;S)$ is compact for the convergence in probability, we may assume that (μ^n) converges in probability to a Young measure $\mu^\infty\in\mathcal{H}$. As $\mathcal{R}$ is compact for the stable convergence, we may suppose that (ν^n) stably converges to a Young measure ν^∞ with $\nu^\infty_t(\Gamma(t)) = 1$ a.e.. In view of Proposition 3.1, the sequence $(u_{\varphi^n,\mu^n,\nu^n})$ is relatively compact in $C_E([-r,1])$, so we may suppose that $(u_{\varphi^n,\mu^n,\nu^n})$ converges uniformly on $[-r,1]$ to a continuous function $u^\infty\in C_E([-r,1])$ with $u^\infty = \varphi^\infty$ on $[-r,0]$, and $(\dot u_{\varphi^n,\mu^n,\nu^n})$ $\sigma(L^2,L^2)$ converges to $\dot u^\infty \in L^2_E([0,1])$. So, for every $t\in[0,1]$, $(\tau(t)u_{\varphi^n,\mu^n,\nu^n})$ converges to $\tau(t)u^\infty$ in the Banach space $\mathcal{C}_0$. By Proposition 1.1, we conclude that $(\delta_{\tau(.)u_{\varphi^n,\mu^n,\nu^n}}\otimes\mu^n\otimes\nu^n)$ stably converges to $\delta_{\tau(.)u^\infty}\otimes\mu^\infty\otimes\nu^\infty$. Let $h\in L^2_E([0,1])$. As the mapping g is bounded by (iii), the integrand $(t,u,s,z)\mapsto\langle h(t), g(t,u,s,z)\rangle$ defined on $[0,1]\times\mathcal{C}_0\times S\times Z$ is L^1-bounded. Let us set

$$v^n(t) = \int_Z [\int_S g(t,\tau(t)u_{\varphi^n,\mu^n,\nu^n}, s, z)\mu^n_t(ds)]\,\nu^n_t(dz)],\ \forall t\in[0,1]$$

and

$$v^\infty(t) = \int_Z [\int_S g(t,\tau(t)u^\infty, s, z)\mu^\infty_t(ds)]\,\nu^\infty_t(dz)],\ \forall t\in[0,1].$$

Then, by the very property of the stable convergence (cf. Proposition 1.2), we get

$$\lim_{n\to\infty}\int_0^1 \langle h(t), v^n(t)\rangle\, dt = \int_0^1 \langle h(t), v^\infty(t)\rangle\, dt.$$

Now, using the weak $L^2_E([0,1])$-convergence of $(\dot u_{\varphi^n,,\mu_n,\nu_n})$ to $\dot u^\infty$ and the preceding limit, we conclude that the sequence $(\dot u_{\varphi^n,\mu^n,\nu^n} - v^n)$ weakly $L^2_E([0,1])$-converges to

$\dot{u}^\infty - v^\infty$. As $u_{\varphi^n,\mu^n,\nu^n}$ is the solution of the corresponding FEI, we have

$$(^*)\ \dot{u}_{\varphi^n,\mu^n,\nu^n}(t) - v^n(t) \in -A(t)u_{\varphi^n,\mu^n,\nu^n}(t)\ a.e.\ t \in [0,1],$$

with $u_{\varphi^n,\mu^n,\nu^n} = \varphi^n$ on $[-r, 0]$. Since $(u_{\varphi^n,\mu^n,\nu^n})$ converges uniformly to u^∞, by virtue of a closure-type lemma in ([14], Lemma 2.3) we conclude that

$$\dot{u}^\infty(t) \in -A(t)u^\infty(t) + \int_Z [\int_S g(t, \tau(t)u^\infty, s, z)\mu_t^\infty(ds)]\, \nu_t^\infty(dz),$$

with $u^\infty = \varphi^\infty$ on $[-r, 0]$. So, by virtue of the uniqueness of the solution of our FEI (cf. Proposition 3.1), we have $u^\infty(.) = u_{\varphi^\infty,\mu^\infty,\nu^\infty}(.)$, where $u_{\varphi^\infty,\mu^\infty,\nu^\infty}$ is the unique continuous solution of the FEI

$$(\mathcal{I}_{\mathcal{K},\mathcal{H},\mathcal{R}}) \begin{cases} \dot{u}_{\varphi^\infty,\mu^\infty,\nu^\infty}(t) \in -A(t)u_{\varphi^\infty,\mu^\infty,\nu^\infty}(t) \\ + \int_{\Gamma(t)} [\int_S g(t, \tau(t)u_{\varphi^\infty,\mu^\infty,\nu^\infty}, s, z)\mu_t^\infty(ds)]\, \mu_t^\infty(dz). \end{cases}$$

Claim 2. $\inf(P_{\mathcal{K},\mathcal{H},\mathcal{R}}) = \min(P_{\mathcal{K},\mathcal{H},\mathcal{R}})$.

As $\mathcal{O}$ is dense in $\mathcal{R}$ for the stable topology (see e.g. [14], [17], [21]), it is sufficient to show that the mapping

$$\Psi : (\varphi, \mu, \nu) \to \int_0^1 [\int_Z [\int_S I(t, u_{\varphi,\mu,\nu}(t), s, z))\, \mu_t(ds)]\nu_t(dz)]\, dt$$

is continuous on $\mathcal{K}\times\mathcal{H}\times\mathcal{R}$. This fact easily follows from Claim 1 and the arguments therein. Indeed let $(\varphi^n, \mu^n, \nu^n)$ be a sequence in $\mathcal{K} \times \mathcal{H} \times \mathcal{R}$ with $\varphi^n \to \varphi \in \mathcal{K}$ in the Banach space $\mathcal{C}_0$, $\mu^n \to \mu \in \mathcal{H}$ in probability, and $\nu^n \to \nu \in \mathcal{R}$ stably. Then $u_{\varphi^n,\mu^n,\nu^n}$ converges uniformly on $[-r, 1]$ to a continuous function $u_{\varphi,\mu,\nu} \in C_E([-r, 1])$ with $u_{\varphi,\mu,\nu} = \varphi$ on $[-r, 0]$, and

$$\begin{cases} \dot{u}_{\varphi,\mu,\nu}(t) \in -A(t)u_{\varphi,\mu,\nu}(t) \\ + \int_{\Gamma(t)} [\int_S g(t, \tau(t)u_{\varphi,\mu,\nu}, s, z)\mu_t(ds)]\, \nu_t(dz),\ a.e.\ t \in [0,1]. \end{cases}$$

The continuity of Ψ follows because $(\delta_{u_{\varphi^n,\mu^n,\nu^n}} \otimes \mu^n \otimes \nu^n)$ stably converges to $\delta_{u_{\varphi,\mu,\nu}} \otimes \mu \otimes \nu$ and J is a L^1-bounded Carathéodory integrand. □

Let us consider now a viscosity variant of Theorem 4.3 concerning our second dynamic model. For simplicity, we will assume that $A(t) : E \to ck(E) \cup \{\emptyset\}$ $(t \in [0, 1])$ is a convex compact-valued m-accretive operator with closed domain $D(A)$ satisfying the assumption (H_1) of Proposition 3.1 and such that $A : [0, 1] \times D(A) \to ck(E)$ is upper semicontinuous. It is obvious that $(H_2), (H_3)(a), (H_3)(b)$ are satisfied. Indeed, (H_2) is satisfied by using property (jj) for m-accretive operators and the upper semicontinuity assumption of A. $(H_3)(a)$ is obvious: for any Lebesgue-measurable mapping $u : [0, 1] \to D(A)$, the multifunction $t \mapsto A(t)u(t)$ is Lebesgue-measurable because $A : [0, 1] \times D(A) \to ck(E)$ is upper semicontinuous. Let $g : [0, 1] \to E$ be a Lebesgue-measurable mapping and $\lambda > 0$. Then the graph $Gr(h_\lambda)$ of the mapping $h_\lambda : t \mapsto (I_E + \lambda A(t))^{-1}g(t)$ is given by

$$\begin{aligned} Gr(h_\lambda) &= \{(t, y) \in [0, 1] \times D(A) : g(t) \in y + \lambda A(t)y\} \\ &= \{(t, y) \in [0, 1] \times D(A) : d(g(t), y + \lambda A(t, y)) = 0\} \\ &= \{(t, y) \in [0, 1] \times D(A) : \inf_n ||g(t) - (y + \lambda\sigma_n(t, y))|| = 0\} \end{aligned}$$

where (σ_n) is a countable dense $\mathcal{B}([0,1]) \otimes \mathcal{B}(D(A))$-measurable selections of A (see e.g. ([13], Theorem III.9). Since each function $(t,y) \in [0,1] \times E \to ||g(t) - (y + \lambda\sigma_n(t,y))||$ is $\mathcal{L}([0,1]) \otimes \mathcal{B}(E)$-measurable, the graph of h_λ belongs to $\mathcal{L}([0,1]) \otimes \mathcal{B}(E)$. Furthermore, for any continuous function $v : [0,1] \times E \to E$, the function $G : (t,y,x) \mapsto \delta^*(v(t,y), A(t)x)$ is upper semicontinuous on $[0,1] \times E \times D(A)$ (e.g. $A(t) = \partial f_t$ where $f : [0,1] \times E \to \mathbf{R}$ is a convex continuous integrand and ∂f_t denotes the subdifferential of the convex continuous function f_t). Further, in this special case, $(H_3)(c)$ is superfluous because the upper semicontinuity assumption on A allows to apply the classical closure type lemma in ([13], Theorem VI-4) in the proofs of existence results (see [14], Theorem 2.4 for details) and the continuous dependence of the solution $u_{\varphi,\mu,\nu}$ with respect to the data $(\varphi, \mu, \nu) \in \mathcal{K} \times \mathcal{H} \times \mathcal{R}$ given in Theorem 5.1. If $A(t)$ is constant, (H_1) is superfluous.
We will assume that

- (L_1) $g : [0,1] \times E \times S \times Z \to E$ is bounded, continuous, uniformly Lipschitzian in $x \in E$ and the family $(g(.,.,s,z))_{(s,z)\in S\times Z}$ is equicontinuous,
- (L_2) $J : [0,1] \times E \times S \times Z \to \mathbf{R}$ is a bounded, continuous, and the family $(J(,.,.,s,z))_{(s,z)\in S\times Z}$ is equicontinuous.
- (L_3) $\mathcal{H}$ and $\mathcal{R}$ are the set of all Lebesgue-measurable mappings from $[0,1]$ into $\mathcal{M}^1_+(S)$ and $\mathcal{M}^1_+(Z)$ respectively. In particular, $\mathcal{H}$ and $\mathcal{R}$ are compact for the stable topology in the space of Young measures $\mathcal{Y}([0,1], S)$ and $\mathcal{Y}([0,1], Z)$ respectively.

The preceding considerations allow to obtain a variant of Theorem 4.3.

Theorem 5.2. *Assume that $(L_1), (L_2)$ (L_3) are satisfied and $A(t) : E \to ck(E) \cup \{\emptyset\}$ $(t \in [0,1])$ is a convex compact-valued m-accretive operator with closed domain $D(A)$ satisfying (H_1) of Proposition 3.1 and such that $A : [0,1] \times D(A) \to ck(E)$ is upper semicontinuous. Let us consider the lower value function*

$$U_J(\tau, x) = \sup_{\nu\in\mathcal{R}} \inf_{\mu\in\mathcal{H}} \{ \int_\tau^1 [\int_Z [\int_S J(t, u_{x,\mu,\nu}(t), s, z)\, \mu_t(ds)] \nu_t(ds)]\, dt \},$$

where $u_{x,\mu,\nu}$ is the unique trajectory solution of

$$\begin{cases} \dot{u}_{x,\mu,\nu}(t) \in -A(t)u_{x,\mu,\nu}(t) \\ + \int_Z [\int_S g(t, u_{x,\mu,\nu}(t), s, z) \mu_t(ds)]\, \nu_t(dz), \ \ a.e. \ \in [\tau, 1], \\ u_{x,\mu,\nu}(\tau) = x \in D(A). \end{cases}$$

Let us consider the upper Hamiltonian defined on $[0,1] \times D(A) \times E$ by

$$H^+(t,x,y) = \inf_{\mu\in\mathcal{M}^1_+(S)} \sup_{\nu\in\mathcal{M}^1_+(Z)} \{\langle y, \int_Z [\int_S g(t,x,s,z)\mu_t(ds)]\nu_t(dz)\rangle + \int_Z [\int_S J(t,x,s,z)\mu_t(ds)]\nu_t(dz)\}.$$

Then U_J is a viscosity subsolution of the perturbed Hamilton-Jacobi-Bellman equation $U_t(t,x)+H^+(t,x,\nabla U(t,x))+\delta^*(\nabla U(t,x)), -A(t)x) = 0$, that is, if for any $\varphi \in \mathcal{C}^1([0,1]\times E)$ for which $U_J - \varphi$ reaches a local maximum at $(t_0,x_0) \in [0,1]\times D(A)$, then

$$\frac{\partial\varphi}{\partial t}(t_0,x_0) + H^+(t_0,x_0,\nabla\varphi(t_0,x_0)) + \delta^*(\nabla\varphi(t_0,x_0), -A(t_0)x_0) \geq 0.$$

In particular, if A is single-valued, then U_J is a viscosity solution of the above perturbed Hamilton-Jacobi-Bellman equation.

Proof. It is similar to the proof of Theorem 4.3 and Theorem 3.2.1 in [17] using the DP principle ([17], Theorem 3.2.1). Equicontinuity assumptions on J and g are necessary to prove that U_J is a viscosity solution of the associated HJB equation, in the particular case when A is single-valued, by applying Lemma 3.2.2(b) in [17]. For shortness, we omit the details.

□

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