Journal of Nonlinear and Convex Analysis
Volume 5, Number 1, 2004, 121–130

Yokohama Publishers
ISSN 1880-5221 ONLINE JOURNAL

QUALITATIVE APPROACHES TO BOUNDARY VALUE PROBLEMS OF FUZZY DIFFERENTIAL EQUATIONS BY THEORY OF ORDINARY DIFFERENTIAL EQUATIONS

SEIJI SAITO

Dedicated to the memory of Dr. Michihiro Nagase

Abstract. In this paper we introduce a parametric representation of fuzzy numbers and definitions of the differential and integral of fuzzy functions. Moreover we show that fuzzy numbers mean bounded continuous curves of the two-dimensional metric space. Finally we consider boundary value problems concerning fuzzy differential equations and fuzzy boundary conditions. By applying Schauder's fixed point theorem and the contraction principle the existence and uniqueness theorems are given in a similar way to the theory of ordinary differential equations.

1. Introduction

There have been many fruitful results on the representations of fuzzy numbers, differentials and integrals of fuzzy functions (see, e.g., in Goetschel-Voxman [7, 8], Dubois-Prade [2, 3, 4, 5], Puri-Ralescue [13], Furukawa [6], Kaleva [10, 11]). They established fundamental results concerning differentials, integrals and fuzzy differential equations of fuzzy functions which map $\mathbf{R}$ to a set of fuzzy numbers. However, it seems to be difficult to apply these results to all the practical and significant problems. In this study we introduce a parametric representation(see [15]) corresponding to the representation of fuzzy numbers due to Goetschel-Voxman so that it is easy to solve fuzzy differential equations.

In Buckley [1], Kaleva [10, 11], Park [12] and Song [19], etc. various types of conditions were obtained for the existence and the uniqueness of solutions to fuzzy differential equations. By the parametric representation stability theory of fuzzy differential equations can be easily treated in an analogous way with the real analysis (see [14]). In Section 2 we denote a fuzzy number x by (x_1, x_2), where x_1, x_2 are endpoints of the $\alpha-$cut set of the membership function μ_x, respectively. Then we consider a metric space which includes the set of fuzzy numbers and also note that x_1, x_2 are continuous. In Section 3 we give definitions of the differential and integral of fuzzy functions and sufficient conditions for fuzzy functions to be differentiable or integrable. In Section 4 we treat a fuzzy differential equation $x^{''} = f(t, x, x^{'})$ with fuzzy boundary conditions $x(a) = A, x(b) = B$,where f is a fuzzy valued function defined on $J = [a, b]$ in the set of real numbers $\mathbf{R}$ and A, B are fuzzy numbers. Moreover we discuss the existence and the uniqueness of solutions for the fuzzy boundary problems.

2000 *Mathematics Subject Classification.* Primary 34B15, Secondary 04A72.

Key words and phrases. fuzzy number, fuzzy differential equation, ordinary differential equation, Schauder's fixed point theorem, contraction principle.

2. Parametric Representation of Fuzzy Numbers

Denote $I = [0,1]$. The following definition means that fuzzy numbers are identified with membership functions.

Definition 1. Consider a set of fuzzy numbers with bounded supports as follows.

$$\mathcal{F}_{\mathbf{b}}^{st} = \{\mu : \mathbf{R} \to I \text{ satisfying (i)-(iv) below}\}.$$

(i) There exists a unique $m \in \mathbf{R}$ such that $\mu(m) = 1$.
(ii) The set $supp(\mu) = cl(\{\xi \in \mathbf{R} : \mu(\xi) > 0\})$ is bounded in $\mathbf{R}$.
(iii) One of the following conditions holds:
(a) μ is strictly fuzzy convex on $supp(\mu)$, i.e.,

$$\mu(c\xi_1 + (1-c)\xi_2) > \min[\mu(\xi_1), \mu(\xi_2)]$$

for $\xi_1, \xi_2 \in supp(\mu),\ 0 < c < 1$;
(b) $\mu(m) = 1$ and $\mu(\xi) = 0$ for $\xi \neq m$.
(iv) μ is upper semi-continuous on $\mathbf{R}$.

Remark 1. The above condition (iiia) is stronger than one in the usual case where μ is fuzzy convex. From (iiia) it follows that $\mu(\xi)$ is strictly increasing in $\xi \in (\min supp(\mu), m)$ and strictly decreasing in $\xi \in (m, \max supp(\mu))$. This condition plays an important role in Theorem 1.

We introduce the following parametric representation of $\mu \in \mathcal{F}_{\mathbf{b}}^{st}$ as

$$x_1(\alpha) = \min L_\alpha(\mu),$$
$$x_2(\alpha) = \max L_\alpha(\mu)$$

for $0 < \alpha \le 1$ and

$$L_\alpha(\mu) = \{\xi \in \mathbf{R} : \mu(\xi) \ge \alpha\},$$
$$x_1(0) = \min supp(\mu),$$
$$x_2(0) = \max supp(\mu).$$

Let $C(I)$ be the set of continuous functions from I to $\mathbf{R}$. The following theorem is given in [14], where the continuity of functions x_1, x_2 on I are proved.

Theorem 1. *Denote the left-, right-end points of the $\alpha-$cut set of $\mu \in \mathcal{F}_{\mathbf{b}}^{st}$ by x_1, x_2, respectively. Here $x_1, x_2 : I \to \mathbf{R}$. The following properties* (i)-(iii) *hold.*

(i) $x_1, x_2 \in C(I)$.
(ii) $\max x_1(\alpha) = x_1(1) = m = \min x_2(\alpha) = x_2(1)$.
(iii) *One of the following statements holds:*
(a) *x_1 is non-decreasing and x_2 is non-increasing. There exists a positive $c \le 1$ such that $x_1(\alpha) < x_2(\alpha)$ for $0 \le \alpha < c$ and that $x_1(\alpha) = m = x_2(\alpha)$ for $c \le \alpha \le 1$;*
(b) *$x_1(\alpha) = x_2(\alpha) = m$ for $0 \le \alpha \le 1$.*

Conversely, under the above conditions (i) -(iii)*, if we denote*

$$\mu(\xi) = \sup\{\alpha \in I : x_1(\alpha) \le \xi \le x_2(\alpha)\}$$

then $\mu \in \mathcal{F}_{\mathbf{b}}^{st}$.

In what follows denote $\mu = (x_1, x_2)$ for $\mu \in \mathcal{F}^{st}_{\mathbf{b}}$. The parametric representation of μ is very useful in analyzing binary operations of fuzzy numbers and qualitative behaviors of fuzzy differential equations. From the extension principle of Zadeh, it follows that

$$\begin{aligned} &\mu_{x+y}(\xi) \\ &= \max_{\xi=\xi_1+\xi_2} \min_{i=1,2} (\mu_i(\xi_i)) \\ &= \max\{\alpha \in I : \xi = \xi_1 + \xi_2, \xi_i \in L_\alpha(\mu_i)\} \\ &= \max\{\alpha \in I : \xi \in [x_1(\alpha) + y_1(\alpha), x_2(\alpha) + y_2(\alpha)]\}, \end{aligned}$$

where μ_1, μ_2 are membership functions of $x, y \in \mathcal{F}^{st}_{\mathbf{b}}$, respectively. Thus we get $x + y = (x_1 + y_1, x_2 + y_2)$. In the similar way $x - y = (x_1 - y_2, x_2 - y_1)$.

Denote a metric by

$$d(x, y) = \sup_{\alpha \in I} (|x_1(\alpha) - y_1(\alpha)| + |x_2(\alpha) - y_2(\alpha)|)$$

for $x = (x_1, x_2), y = (y_1, y_2) \in \mathcal{F}^{st}_{\mathbf{b}}$. We show that $\mathbf{R} \subset \mathcal{F}^{st}_{\mathbf{b}}$ and that $\mathcal{F}^{st}_{\mathbf{b}}$ is a complete metric space in $C(I)^2$ in [9].

Remark 2. From the above Condition (i) a fuzzy number $x = (x_1, x_2)$ means a bounded continuous curve of $\mathbf{R}^2$ and $x_1(\alpha) \le x_2(\alpha)$ for $\alpha \in I$.

In the following example we illustrate typical three types of fuzzy numbers.

Example 1. Consider the following $L-R$ fuzzy number $x \in \mathcal{F}^{st}_{\mathbf{b}}$ with a membership function as follows:

$$\mu_x(\xi) = \begin{cases} L(\frac{m-\xi}{l})_+ & \text{for } \xi \le m \\ R(\frac{\xi-m}{r})_+ & \text{for } \xi > m \end{cases}$$

where $m \in \mathbf{R}, l > 0, r > 0$ and L, R are into mappings defined on $\mathbf{R}_+ = [0, \infty)$. Let $L(\xi)_+ = \max(L(\xi), 0)$ etc. We identify μ_x with $x = (x_1, x_2)$. Since there exist L^{-1} and R^{-1}, we have $x_1(\alpha) = m - L^{-1}(\alpha)l$ and $x_2(\alpha) = m + R^{-1}(\alpha)r$.

Let $L(\xi) = -c_1\xi + 1$, where $c_1 > 0$. We illustrate the following cases (i)-(iii).

(i) Let $R(\xi) = -c_2\xi + 1$, where $c_2 > 0$. Then $c_2 l(x_2 - m) = c_1 r(m - x_1)$.
(ii) Let $R(\xi) = -c_2\sqrt{\xi} + 1$, where $c_2 > 0$. Then $c_2 l(x_2 - m)^2 = c_1 r^2(m - x_1)$.
(iii) Let $R(\xi) = -c_2\xi^2 + 1$, where $c_2 > 0$. Then $c_2^2 l^2(x_2 - m) = c_1^2 r(x_1 - m)^2$.

3. Differential and Integral of Fuzzy Functions

Let J be an interval in $\mathbf{R}$. Denote an $\mathcal{F}^{st}_{\mathbf{b}}-$valued function $x : J \to \mathcal{F}^{st}_{\mathbf{b}}$ by $x(t) = (x_1, x_2)(t)$. Here x_1, x_2 are functions defined on $J \times I$ to $\mathbf{R}$ and $x_1(t, \alpha), x_2(t, \alpha)$ are left- and right-end points of the $\alpha-$cut set of the membership function $\mu_{x(t)}$ for the function x at t, respectively. The $\alpha-$cut set $[x_1(t, \alpha), x_2(t, \alpha)] \subset \mathbf{R}$ can be identified by a point $(x_1(t, \alpha), x_2(t, \alpha))^T \in \mathbf{R}^2$, then it can be seen that

$$x(t) = \{(x_1(t, \alpha), x_2(t, \alpha))^T \in \mathbf{R}^2 : \alpha \in I\}$$

for $t \in J$.

Define the continuity and differentiability of fuzzy-valued function (see [10, 11]).

Definition 2. A function $x : J \to \mathcal{F}^{st}_{\mathbf{b}}$ is continuous at $t \in J$ if

$$\lim_{h\to 0} d(x(t+h), x(t)) = 0.$$

Let $x(t) = (x_1, x_2)(t)$ for $t \in J$. The function x is said to be differentiable at $t \in J$ if for any $\alpha \in I$ there exist $\dfrac{\partial x_1}{\partial t}(t,\alpha), \dfrac{\partial x_2}{\partial t}(t,\alpha)$ such that $\dfrac{\partial x_1}{\partial t}(t,\alpha) \le \dfrac{\partial x_2}{\partial t}(t,\alpha)$ and $\mu_{x'(t)} \in \mathcal{F}^{st}_{\mathbf{b}}$, where $\mu_{x'(t)}(\xi) = \sup\{\alpha \in I : \frac{\partial x_1}{\partial t}(t,\alpha) \le \xi \le \frac{\partial x_2}{\partial t}(t,\alpha)\}$. The function x is said to be differentiable on J if x is differentiable at any $t \in J$. Denote $\dfrac{dx}{dt} = x^{'} = (\dfrac{\partial x_1}{\partial t}, \dfrac{\partial x_2}{\partial t})$ and it is said to be the derivative of x.

We consider the following definition of the integral of $\mathcal{F}^{st}_{\mathbf{b}}-$ valued functions.

Definition 3. Let $x : J \to \mathcal{F}^{st}_{\mathbf{b}}$ be $x(t) = (x_1, x_2)(t)$ for $t \in J$. The function x is said to be integrable over $[t_1, t_2]$, if x_1, x_2 are Riemann integrable over $[t_1, t_2]$. Then we define the integral as follows:

$$\int_{t_1}^{t_2} x(s)ds = \{(\int_{t_1}^{t_2} x_1(s,\alpha)ds, \int_{t_1}^{t_2} x_2(s,\alpha)ds)^T \in \mathbf{R}^2 : \alpha \in I\}.$$

Remark 3. Let x be an $\mathcal{F}^{st}_{\mathbf{b}}-$valued function and $t \in J$.

(i) If $x : J \to \mathcal{F}^{st}_{\mathbf{b}}$ is differentiable at t, we get the integral over $[t_1, t_2] \subset J$ as follows:

$$\int_{t_1}^{t_2} x^{'}(s)ds + x(t_1) = x(t_2).$$

(ii) If $x : J \to \mathcal{F}^{st}_{\mathbf{b}}$ is integrable over $[t_1, t_2]$, then we have $\int_{t_1}^{t_2} x(s)ds \in \mathcal{F}^{st}_{\mathbf{b}}$ and

$$d(\int_{t_1}^{t_2} x(s)ds, 0) \le \int_{t_1}^{t_2} d(x(s), 0)ds.$$

(iii) If x, y are integrable $\mathcal{F}^{st}_{\mathbf{b}}-$valued functions and $t_1, t_2 \in J$, then

$$\int_{t_1}^{t_2} (x(s) + y(s))ds = \int_{t_1}^{t_2} x(s)ds + \int_{t_1}^{t_2} y(s)ds.$$

4. Fuzzy Boundary Value Problems

Let $J = [a, b] \subset \mathbf{R}$. In this section we consider the following fuzzy differential equation with fuzzy boundary conditions

$$\frac{d^2x}{dt^2}(t) = f(t, x, x^{'}),\ x(a) = A,\ x(b) = B, \tag{4.1}$$

where $t \in J,\ x = (x_1, x_2) \in \mathcal{F}^{st}_{\mathbf{b}}, A = (A_1, A_2), B = (B_1, B_2) \in \mathcal{F}^{st}_{\mathbf{b}}$. Then we get the following boundary value problem of ordinary differential equations

$$\begin{aligned}
&\frac{d^2x_1}{dt^2}(t) = f_1(t, x_1, x_2, x_1^{'}, x_2^{'}),\\
&\frac{d^2x_2}{dt^2}(t) = f_2(t, x_1, x_2, x_1^{'}, x_2^{'}),\\
&x_1(a) = A_1, \quad x_2(a) = A_2,\\
&x_1(b) = B_1, \quad x_2(b) = B_2.
\end{aligned}$$

In case where functions f_1, f_2 satisfy Conditions (i) - (iii) of Theorem 1, then so are solutions of (4.1).

Putting $y_1 = x_1', y_2 = x_2'$ we have

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ f_1(t, x_1, x_2, y_1, y_2) \\ f_2(t, x_1, x_2, y_1, y_2) \end{pmatrix}.$$

The parameter $\alpha \in I$ is fixed. Then, denoting $z = (x_1, x_2, y_1, y_2)^T \in \mathbf{R}^4$, we get

$$\frac{dz}{dt}(t) = Mz + F(t, z),\ \mathcal{L}(z) = c. \tag{4.2}$$

Here

$$M = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, F(t, z) = \begin{pmatrix} 0 \\ 0 \\ f_1(t, z) \\ f_2(t, z) \end{pmatrix}, c = \begin{pmatrix} A_1(\alpha) \\ A_2(\alpha) \\ B_1(\alpha) \\ B_2(\alpha) \end{pmatrix} \tag{4.3}$$

for $\alpha \in I$ and $\mathcal{L}$ is a bounded linear operator from $C(J)^4$ to $\mathbf{R}^4$ as follows:

$$\mathcal{L}(z) = (x_1(a, \alpha), x_2(a, \alpha), x_1(b, \alpha), x_2(b, \alpha))^T.$$

In this case we get the fundamental matrix

$$X(t) = e^{tM} = \begin{pmatrix} 1 & 0 & t & 0 \\ 0 & 1 & 0 & t \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

with $X(0) = E$, where E is the identity matrix. Let U satisfy

$$\mathcal{L}(X(\cdot)z_0) = \begin{pmatrix} 1 & 0 & a & 0 \\ 0 & 1 & 0 & a \\ 1 & 0 & b & 0 \\ 0 & 1 & 0 & b \end{pmatrix} z_0 = Uz_0$$

for $z_0 \in \mathbf{R}^4$. It follows that

$$U^{-1} = \frac{1}{b-a}\begin{pmatrix} b & 0 & -a & 0 \\ 0 & b & 0 & -a \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}.$$

We denote a norm in $\mathbf{R}^4$ by $\| z \| = |x_1| + |x_2| + |y_1| + |y_2|$ and $\| U \| = \sup\limits_{\|z\|=1} \| Uz \|$. Then $\| U \| = \max(2, a+b)$ and $\| U^{-1} \| = \dfrac{b+1}{b-a}$.

In what follows we give the existence and the uniqueness theorems by applying Schauder's fixed point theorem or the contraction principle as in the similar way as in [16] and [17]. Let $r > 0$. Denote a set of continuous functions from J to $\mathcal{F}_{\mathbf{b}}^{st}$ by $C(J; \mathcal{F}_{\mathbf{b}}^{st})$ and $D = \{z = (x, y) : x = (x_1, x_2), y = (y_1, y_2) \in C(J; \mathcal{F}_{\mathbf{b}}^{st})\}$. Denote a subset in D by

$$S = \{z = (x_1, x_2, y_1, y_2) : (x, y) \in D, d_\infty(z, 0) \le r\}.$$

Here

$$d_\infty(z, \bar{z}) = \sup_{t \in J} d(x(t), \bar{x}(t)) + \sup_{t \in J} d(y(t), \bar{y}(t))$$

for $z = (x, y), \bar{z} = (\bar{x}, \bar{y}) \in S$. Then the following functions

$$\mu_{x(t)}(\xi) = \sup\{\alpha \in I : x_1(t, \alpha) \le \xi \le x_2(t, \alpha)\}$$
$$\mu_{y(t)}(\xi) = \sup\{\alpha \in I : y_1(t, \alpha) \le \xi \le y_2(t, \alpha)\}$$

are membership functions of fuzzy numbers $x(t), y(t)$ in $\mathcal{F}_{\mathbf{b}}^{st}$ for $t \in J$, respectively. Then it can be seen that S is a convex and closed subset in D.

In the similar way of discussion as the theory of ordinary differential equations it follows that $z \in S$ is a continuous solution of (4.2) if and only if

$$z(t) = X(a)U^{-1}(c - \mathcal{L}(q_z)) + \int_a^t Mz(s)ds + \int_a^t F(s, z(s))ds$$

for $t \in J$, where

$$q_z(t) = \int_a^t X(t)X^{-1}(s)F(s, z(s))ds.$$

Putting

$$Q = \int_a^b \max_{d(z,0) \le r} (b - s + 1)d(f(s, z), 0)ds,$$

we have $d_\infty(q_z, 0) \le Q$ for $z \in S$. By Schauder's fixed point theorem we get the existence of solutions for (4.1).

Theorem 2. *Assume that positive numbers R, r satisfy $R < e^{-(b-a)}$ and*

$$r > \frac{Q \parallel \mathcal{L} \parallel (b+1) \parallel U^{-1} \parallel}{e^{-(b-a)} - R}.$$

Let f satisfy

$$\int_a^b \max_{d(z,0) \le r} d(f(s, z), 0)ds \le rR.$$

If $A = (A_1, A_2), B = (B_1, B_2) \in \mathcal{F}_{\mathbf{b}}^{st}$ satisfy

$$d(A, 0) + d(B, 0) \le \frac{r(e^{-(b-a)} - R)}{(b+1) \parallel U^{-1} \parallel} - \parallel \mathcal{L} \parallel Q,$$

then (4.1) *has at least one solution in S.*

Proof. Let $u \in S$ and $\alpha \in I$ be fixed. Consider the following boundary linear problem

$$\frac{dz}{dt} = Mz(t) + F(t,u), \mathcal{L}(z) = c.$$

Then there exists a unique solution z_u of the above problem such that

$$\begin{aligned} z_u(t) &= X(t)U^{-1}(c - \mathcal{L}(q_u)) + q_u(t) \\ &= X(a)U^{-1}(c - \mathcal{L}(q_u)) + \int_a^t Mz_u(s)ds + \int_a^t F(s,u(s))ds \end{aligned}$$

for $t \in J$. Denote a mapping by $[\mathcal{V}(u)](t) = z_u(t)$ for $u \in S, t \in J$. Then the solution z of (4.1) means that z is a fixed point of $\mathcal{V}$ in S.

We shall prove $\mathcal{V}$ is an into mapping. From the definition of $\mathcal{V}$ and $\| X(t) \| \leq b+1$ for $t \in J$, it follows that, putting $C = d(A,0) + d(B,0)$,

$$\begin{aligned} \| z_u(t) \| &\leq (b+1) \| U^{-1} \| (\| C \| + \| \mathcal{L} \| d_\infty(q_u, 0)) + \int_a^t \| M \| \| z_u(s) \| ds \\ &\quad + \int_a^t \| F(s,u(s)) \| ds \\ &\leq (b+1) \| U^{-1} \| (\| C \| + \| \mathcal{L} \| Q) + \int_a^t \| z_u(s) \| ds + rR \end{aligned}$$

for $t \in J$. By Gronwall's inequality we have

$$\| z_u(t) \| \leq e^{b-a}((b+1) \| U^{-1} \| (\| C \| + \| \mathcal{L} \| Q) + rR) \leq r.$$

Thus we have $d_\infty(z_u, 0) \leq r$ for $u \in S$. It is clear that z_u satisfies Conditions (i)-(iii) in Theorem 1. Therefore $z_u \in S$ for $u \in S$. Thus $\mathcal{V}$ is uniformly bounded.

The continuity of q_u on S means that $\mathcal{V}$ is continuous. The uniform continuity of F leads to the equicontinuity of $\mathcal{V}$ and the compactness of $\mathcal{V}$ is proved by Ascoli-Arzela's theorem. By Schauder's fixed point theorem it follows that there exists at least one solution in S. This completes the proof. □

We illustrate the above theorem by showing the following example. Consider a fuzzy function $f = (f_1, f_2)$ such that

$$f_i(t,x,y,\alpha) = \frac{m_i(\alpha)p_i(t)|x_i(\alpha)|}{|x_i(\alpha)| + 1} + \frac{n_i(\alpha)q_i(t)(|y_i(\alpha)| + y_i(\alpha))}{2}$$

for $t \in J, \alpha \in I, i = 1,2$. Here $m = (m_1, m_2), n = (n_1, n_2), x = (x_1, x_2), y = (y_1, y_2) \in \mathcal{F}_{\mathbf{b}}^{st}$ and continuous real-valued functions $p_i, q_i : J \to \mathbf{R}_+$. Assume that $m_1(0), n_1(0) \geq 0$ and that $f(t,x,y) \in \mathcal{F}_{\mathbf{b}}^{st}$ for $t \in J, (x,y) \in \mathcal{F}_{\mathbf{b}}^{st} \times \mathcal{F}_{\mathbf{b}}^{st}$. Denote $\rho = e^{-(b-a)} - R$ and assume that $\rho > 0$. Denote

$$K(t) = m_2(0) \sum_{i=1}^{2} p_i(t) + n_2(0) \sum_{i=1}^{2} q_i(t)$$

for $t \in J$ and assume that

$$r \geq \frac{1}{R} \int_a^b K(s)ds,$$

$$r > \frac{(b+1) \parallel \mathcal{L} \parallel}{\rho(b-a)} \int_a^b (b-s+1)K(s)ds.$$

If $A = (A_1, A_2), B = (B_1, B_2) \in \mathcal{F}_{\mathbf{b}}^{st}$ satisfy $d(A, 0) + d(B, 0) \leq \frac{r\rho(b-a)}{(b+1)^2} - \parallel \mathcal{L} \parallel \int_a^b (b-s+1)K(s)ds$, then, from the above theorem, (4.1) has at least one solution.

Finally we show the existence and uniqueness theorem concerning solutions of (4.1) by the contraction principle.

Theorem 3. *Assume that there exists an integrable function* $\ell : J \to \mathbf{R}_+$ *such that*

$$((b+1) \parallel U^{-1} \parallel + \parallel \mathcal{L} \parallel) \int_a^b (b-s+1)\ell(s)ds < 1.$$

Let f *satisfy*

$$d(f(t,z), f(t,\bar{z})) \leq \ell(t)d(z,\bar{z})$$

for $z = (x,y), \bar{z} = (u,v)$ *and* $t \in J$, *where* $x, y, u, v \in \mathcal{F}_{\mathbf{b}}^{st}$. *Then* (4.1) *has one and only one solution for any* c *in* $C^2(J; \mathcal{F}_{\mathbf{b}}^{st})$.

Proof. It follows that for $z = (x,y), \bar{z} = (u,v) \in C(J; \mathcal{F}_{\mathbf{b}}^{st})$ and $t \in J$

$$[\mathcal{V}(z)](t) - [\mathcal{V}(\bar{z})](t) = X(t)U^{-1}(-\mathcal{L}(q_z) + \mathcal{L}(q_{\bar{z}})) + q_z(t) - q_{\bar{z}}(t).$$

Here $\mathcal{V}$ is the mapping defined in Theorem 2. So we get

$$d_\infty(\mathcal{V}(z), \mathcal{V}(\bar{z})) \leq [((b+1) \parallel U^{-1} \parallel + \parallel \mathcal{L} \parallel) \int_a^b (b-s+1)\ell(s)ds] d_\infty(z, \bar{z}),$$

which means that $\mathcal{V}$ is a contraction, i.e., (4.1) has one and only one solution. This completes the proof. □

We illustrate the above theorem by showing the following example. Consider a fuzzy function $f = (f_1, f_2)$ such that

$$\begin{aligned} & f_i(t,x,y,\alpha) \\ &= \frac{1}{2} j_i(\alpha) g_i(t) |x_1(\alpha) + \sin x_1(\alpha)| + k_i(\alpha) h_i(t) |y_1(\alpha)| \\ &\quad + \frac{m_i(\alpha) p_i(t) |x_2(\alpha)|}{|x_2(\alpha)| + 1} + \frac{n_i(\alpha) q_i(t) (|y_2(\alpha)| + y_2(\alpha))}{2} \end{aligned}$$

for $t \in J, \alpha \in I, i = 1, 2$. Here $j = (j_1, j_2), k = (k_1, k_2), m = (m_1, m_2), n = (n_1, n_2), x = (x_1, x_2), y = (y_1, x_2) \in \mathcal{F}_{\mathbf{b}}^{st}$ and continuous real-valued functions $g_i, h_i, p_i, q_i : J \to \mathbf{R}_+$. Assume that $j_1(0), k_1(0), m_1(0), n_1(0) \geq 0$ and that $f(t,x,y) \in \mathcal{F}_{\mathbf{b}}^{st}$ for $t \in J, (x,y) \in \mathcal{F}_{\mathbf{b}}^{st} \times \mathcal{F}_{\mathbf{b}}^{st}$.

Denote

$$\ell(t) = \max(j_2(0) \sum_{i=1}^2 g_i(t), k_2(0) \sum_{i=1}^2 h_i(t), m_2(0) \sum_{i=1}^2 p_i(t), n_2(0) \sum_{i=1}^2 q_i(t))$$

for $t \in J$ and assume that

$$(\frac{(b+1)^2}{b-a} + \parallel \mathcal{L} \parallel) \int_a^b (b-s+1)\ell(s)ds < 1.$$

Then we have for each $\alpha \in I, t \in J, i = 1, 2$ and $x, y, u, v \in \mathcal{F}_{\mathbf{b}}^{st}$,

$$\begin{aligned}
&|f_i(t,x,y,\alpha) - f_i(t,u,v,\alpha)| \\
&\leq j_i(\alpha)g_i(t)\big||x_1(\alpha)| - |u_1(\alpha)|\big| + k_i(\alpha)h_i(t)\big||y_1(\alpha)| - |v_1(\alpha)|\big| \\
&\quad + m_i(\alpha)p_i(t)\big||x_2(\alpha)| - |u_2(\alpha)|\big| + n_i(\alpha)q_i(t)\big||y_2(\alpha)| - |v_2(\alpha)|\big| \\
&\leq j_2(0)g_i(t)|x_1(\alpha) - u_1(\alpha)| + k_2(0)h_i(t)|y_1(\alpha) - v_1(\alpha)| \\
&\quad + m_2(0)p_i(t)|x_2(\alpha) - u_2(\alpha)| + n_2(0)q_i(t)|y_2(\alpha) - v_2(\alpha)|,
\end{aligned}$$

which means that

$$d(f(t,x,y), f(t,u,v)) \leq \ell(t)(d(x,u) + d(y,v)).$$

Thus, for any $A = (A_1, A_2), B = (B_1, B_2) \in \mathcal{F}_{\mathbf{b}}^{st}$, there exists one and only one solution of (4.1) in $C(J; \mathcal{F}_{\mathbf{b}}^{st})$.

Acknowledgement. The author is grateful to the referees for many valuable suggestions.

References

[1] J. Buckley, T. Feuring, Fuzzy Differential Equations, Fuzzy Sets and Systems 110 (2000) , 43-54.

[2] D. Dubois, H. Prade, Operations on Fuzzy Numbers, Internat. J. of Systems 9 (1978), 613-626.

[3] D. Dubois, H. Prade, Towards Fuzzy Differential Calculus Part I : Integration of Fuzzy Mappings, Fuzzy Sets and Systems 8 (1982), 1–17

[4] D. Dubois, H. Prade, Towards Fuzzy Differential Calculus Part II : Integration of Fuzzy Intervals, Fuzzy Sets and Systems 8 (1982), 105-116.

[5] D. Dubois, H. Prade, Towards Fuzzy Differential Calculus Part III: Differentiation, Fuzzy Sets and Systems 8 (1982), 225-233.

[6] N. Furukawa, Mathematical Methods of Fuzzy Optimization(in Japanese), Morikita Pub., Tokyo, Japan, 1999.

[7] Jr.R. Goetschel, W. Voxman, Topological Properties of Fuzzy Numbers, Fuzzy Sets and Systems 9 (1983), 87-99.

[8] Jr.R. Goetschel, W. Voxman, Elementary Fuzzy Calculus, Fuzzy Sets and Systems 18 (1986), 31-43.

[9] H. Ishii, H. Morita and S. Saito, Mathematical Approaches to Decision Making under Uncertainty (in Japanese), Osaka University Publ.(2004).

[10] O. Kaleva, Fuzzy Differential Equations, Fuzzy Sets and Systems 24 (1987), 301-317.

[11] O. Kaleva, The Cauchy Problem for Fuzzy Differential Equation, Fuzzy Sets and Systems 35 (1990), 389-396.

[12] J.Y. Park, H.K. Han, Fuzzy Differential Equations, Fuzzy Sets and Systems 110 (2000), 69-77.

[13] M.L. Puri, D.A. Ralescu, Differential of Fuzzy Functions, J. Math. Anal. Appl. 91 (1983) , 552-558.

[14] S. Saito, On Some Topics of Fuzzy Differential Equations and Fuzzy Optimization Problems via a Parametric Representation of Fuzzy Numbers, "Differential Equations and Applications, Vol. 3" ed. by Yeol Je Cho, Nova Science Publishers, Inc., New York (2004).

[15] S. Saito, H. Ishii, M. Chern, Fuzzy Differentials and Deterministic Approaches to Fuzzy Optimization problems, Dynamical Aspects in Fuzzy Decision Making, Physica-verlag, 2001, 163-186.

[16] S. Saito, M. Yamamoto, Boundary Value Problems of Quasilinear Ordinary Differential Systems on an infinite Interval, Mathematica Japonica 34 (1989), 433-445.

[17] S. Saito, M. Yamamoto, Boundary Value Problems of Quasilinear Ordinary Differential Systems on a finite Interval, Mathematica Japonica 34 (1989), 447-458.

[18] S. Seikkala, On Fuzzy Initial Problems, Fuzzy Sets and Systems 24 (1987) , 319-330.
[19] S. Song, C. Wu, Existence and Uniqueness of Solutions to Cauchy Problem of Fuzzy Differential Equations, Fuzzy Sets and Systems 110 (2000) , 55-67.

Manuscript received April 4, 2003
revised February 23, 2004

Seiji Saito
Graduate School of Information Science and Technology, Osaka University, Suita, Osaka 565-0871, JAPAN
E-mail address: saito@ist.osaka-u.ac.jp