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CONVEX INTERIOR MAPPING THEOREMS FOR CONTINUOUS NONSMOOTH FUNCTIONS AND OPTIMIZATION*

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Abstract. We present a convex interior mapping theorem for (not necessarily locally Lipschitz) continuous maps by using unbounded approximate Jacobians. As an application we derive a general Lagrange multiplier rule for a constrained optimization problem involving both equality and inequality constraints, and continuous functions.

1. Introduction

One of the key results of classical variational analysis is the interior mapping theorem for differentiable functions. It asserts that if $f : \mathbb{R}^n \to \mathbb{R}^m$ is stritcly differentiable and the derivative $\nabla f(x)$ is surjective then $f(x) \in \text{int} f(\mathbb{R}^n)$, the interior of $f(\mathbb{R}^n)$. The interior mapping theorem plays an important role in optimization and particularly, in the derivation of a Lagrange multiplier rule which dominates many issues of mathematics, engineering, and economics. Over the years, a great deal of attention has been focused on generalizing this result for functions which are not necessarily differentiable at all of its points (see [3, 4, 14, 15, 21]). Pourciau [14, 15] extended this theorem for Lipschitz continuous functions using the Clarke generalized Jacobian, and consequently derived Lagrange multiplier conditions for optimization problems involving *equality constraints.* A more subtle interior mapping theorem is needed for deriving multiplier conditions for optimization problems involving both equality and inequality constraints. Pourciau [14, 15] obtained such an interior mapping theorem, called the *convex interior mapping theorem.* It states for locally Lipschitz maps that if every matrix $M \in \partial^c f(x)$ is surjective, then $f(x) \in \text{int} f(C)$, where $x \in \overline{C} \subseteq \mathbb{R}^n$ and $\overline{C}$ is the closure of a convex set C, and $\partial^c f(x)$ is the Clarke generalized Jacobian of f at x. As a consequence Lagrange multiplier conditions for a constrained nonsmooth Lipschitz optimization problem were presented.

The purpose of this paper is to establish a convex interior mapping theorem for (not necessarily Lipschitz) continuous functions using unbounded approximate Jacobians and to derive Lagrange multiplier conditions for constrained optimization problems. These approximate Jacobians, which enjoy rich calculus ([7, 8, 9, 10, 11]), always exist for continuous functions and yield sharp conditions for locally Lipschitz functions (see [20]). The techniques of recession directions of unbounded approximate Jacobians and of partial approximate Jacobians play a central device

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in deriving the convex interior mapping theorem. We then apply the generalized convex interior mapping theorem to obtain a general multiplier rule for optimization problems with continuous data.

The outline of the paper is as follows. Section 2 presents approximate Jacobian based basic calculus and preliminary results on recession directions which allow us to describe results involving unbounded approximate Jacobians. Section 3 establishes a convex interior mapping theorem. Section 4 provides convex interior mapping theorems using partial approximate Jacobians. Section 5 presents necessary optimality conditions for a general constrained nonsmooth optimization problem.

2. Approximate Jacobians

We begin this section by presenting the definition of the approximate Jacobian and its associated basic calculus.

Let $f := (f_1, f_2, ..., f_m) : \mathbb{R}^n \to \mathbb{R}^m$ be a continuous function. We recall that a closed set of $(m \times n)$-matrices $\partial f(x) \subseteq L(\mathbb{R}^n, \mathbb{R}^m)$ is said to be an *approximate Jacobian* of f at x if for every $u \in \mathbb{R}^n$ and $v \in \mathbb{R}^m$, we have

$$(vf)^+(x, u) \leq \sup_{M \in \partial f(x)} \langle v, M(u) \rangle,$$

where vf is the real function $\sum_{i=1}^m v_i f_i$ and $(vf)^+(x, u)$ is the upper Dini directional derivative of the function vf at x in the direction u, that is

$$(vf)^+(x, u) := \limsup_{t \downarrow 0} \frac{(vf)(x + tu) - (vf)(x)}{t}$$

Let $F : \mathbb{R}^n \rightrightarrows L(\mathbb{R}^n, \mathbb{R}^m)$ be a set-valued map. If at every x the set $F(x)$ is an approximate Jacobian of f at x, then we say that F is an approximate Jacobian map of f. When $m = 1$, an approximate Jacobian $\partial f(x)$ of f at x is also called a generalized subdifferential of f at x.

As it was noted in the papers [7, 8, 9, 10], for locally Lipschitz functions, the Clarke generalized Jacobian and the coderivative [17] are examples of approximate Jacobians. However, a locally Lipschitz function may admit an approximate Jacobian whose convex hull is strictly contained in the Clarke Jacobian. This implies in particular that any necessary optimality conditons expressing in terms of approximate Jacobians will not only be valid for the Clarke Jacobian but also often yield sharper conditions. Other examples of approximate Jacobians include Warga's unbounded derivate containers [21], Ioffe's fan-prederivative [6] and the quasidifferential of Demyanov and Rubinov [2].

It is known (see [8]) that when $m = 1$ and x is a local minimum point of the continuous function f, then

$$0 \in \overline{co} \partial f(x)$$

where $\partial f(x)$ is a generalized subdifferential of f at x and $\overline{co}$ stands for the closed convex hull. When we are looking for a local minimum of f on a convex subset $C \subseteq \mathbb{R}^n$ the above rule can be generalized as follows.

Proposition 2.1. *Let C be a convex set in $\mathbb{R}^n$ and let $f : \mathbb{R}^n \to \mathbb{R}$ be continuous. If $x \in C$ is a local minimum point of f on C and if $\partial f(x)$ is a generalized subdifferential* *of f at x, then*

$$\sup_{\xi\in\partial f(x)} \langle \xi, u\rangle \geq 0 \ , \quad \forall\ u \in T(C,x)$$

where $T(C,x) := \overline{\{t(c-x) : x \in C, t \geq 0\}}$ *, is the tangent cone to C at x.*

Proof. It suffices to show the inequality for those $u \in T(C,x)$ of the form $u = c - x$ where $c \in C$. Suppose to the contrary that the inequality does not hold for some $u = c - x,\ c \in C$, that is

$$\sup_{\xi\in\partial f(x)} \langle \xi, c-x\rangle < 0.$$

It follows that

$$f^+(x, c-x) = \limsup_{t\downarrow 0} \frac{f\left(x + t(c-x)\right) - f(x)}{t} < 0.$$

Hence for t sufficiently small, we derive

$$f\left(x + t(c-x)\right) - f(x) < 0,$$

which contradicts the hypothesis. □

Recall that for a nonempty subset $A \subseteq \mathbb{R}^n$, a vector $u \in \mathbb{R}^n$ is said to be a recession direction of A if there is a sequence of positive numbers $\{t_i\}$ converging to 0 and a sequence $\{a_i\} \subseteq A$ such that $u = \lim\limits_{i\to\infty} t_i a_i$. The collection of all recession directions of A is called the *recession cone* of A and denoted by A_∞. Recession cones play an important role in the study of unbounded sets and unbounded functions. The interested reader is referred to [13, 17] for properties and uses of recession cones of nonconvex sets. Given a cone $K \subseteq \mathbb{R}^n$ and $0 < \epsilon < 1$, the $\epsilon-$conic neighborhood of K is denoted by K^ϵ and is defined by

$$K^\epsilon := \{x + \epsilon\|x\| y \ : \ y \in B_n(0,1),\ x \in K\}$$

where $B_n(0,1)$ denotes the closed ball in $\mathbb{R}^n$ centered at 0 with radius 1.

We recall further that a set valued map $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is said to be upper semicontinuous at x if for every $\epsilon > 0$, there is some $\delta > 0$ such that

$$F(x') \subseteq F(x) + \epsilon B_m(0,1), \quad \text{for} \quad x' \in x + \delta B_n(0,1).$$

We are now able to provide some calculus rules that were established in [7, 8, 9, 11].

(1) **Differentiability**. Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be continuous. If it is Gâteaux differentiable at x, then every approximate Jacobian of f at x contains the Gâteaux derivative of f at x in its closed convex hull. Moreover, f is Gâteaux differentiable at x if and only if it admits a singleton approximate Jacobian at this point.

(2) **Chain rule**. Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be continuous and $g : \mathbb{R}^m \to \mathbb{R}$ be continuously differentiable at some point $f(x) \in \mathbb{R}^m$. If ∂f is an approximate Jacobian map of f which is upper semicontinuous at x and if $\nabla g(f(x)) \neq 0$, then for every $\epsilon > 0$, the closure of the set

$$\nabla g(f(x)) \circ [\partial f(x) + (\partial f(x))^\epsilon_\infty]$$

is an approximate Jacobian of $g \circ f$ at x.

It is interesting to note that if $\partial f(x)$ is bounded, then $(\partial f(x))_\infty = \{0\}$ and hence $(\partial f(x))^\epsilon_\infty = \{0\}$, and the chain rule above becomes : $\nabla g(f(x)) \circ \partial f(x)$ is an approximate Jacobian of gof at x. The condition $\nabla g(f(x)) \neq 0$ in this case is in fact superfluous (see [9]). When $\partial f(x)$ is unbounded, which is often the case when f is not locally Lipschitz, neither condition $\nabla g(f(x)) \neq 0$, nor the term $(\partial f(x))^\epsilon_\infty$ in the expression of the chain rule can be neglected.

(3) **Mean value condition:** Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be continuous and let $\partial f(x)$ be an approximate Jacobian of f at x for every $x \in [a,b]$ where $a, b \in \mathbb{R}^n$. Then

$$f(b) - f(a) \in \overline{co}(\partial f([a,b])(b-a)).$$

Now let $f : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}^m$ be continuous in both variables $(x,y) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$. An approximate Jacobian $\partial_x f(x,y) \subseteq L(\mathbb{R}^{n_1}, \mathbb{R}^m)$ of the function $x \mapsto f(x,y)$ where $y \in \mathbb{R}^{n_2}$ is fixed, is called a partial approximate Jacobian of f at (x,y) with respect to x. A partial approximate Jacobian of f at (x,y) with respect to y is similarly defined. Note that if $\partial_x f(x,y)$ and $\partial_y f(x,y)$ are partial approximate Jacobians of f at (x,y), then it is not necessary that the set

$$(\partial_x f(x,y), \partial_y f(x,y)) := \{MN \ : \ M \in \partial_x f(x,y), \ N \in \partial_y f(x,y)\} \subseteq L(\mathbb{R}^{n_1 \times n_2}, \mathbb{R}^m)$$

is an approximate Jacobian of f at (x,y). For instance if f is a continuous function which is not Gâteaux differentiable at 0 and possesses partial derivatives $\frac{\partial f(0,0)}{\partial x}$ and $\frac{\partial f(0,0)}{\partial y}$ (in the classical sense) at this point. Then these partial derivatives are partial approximate Jacobians of f at 0. However, the singleton $\left\{\left(\frac{\partial f(0,0)}{\partial x}, \frac{\partial f(0,0)}{\partial y}\right)\right\}$ cannot be an approximate Jacobian of f at 0 since f is not Gâteaux differentiable at this point.

The relationship between approximate Jacobians and partial approximate Jacobians is seen in the following results.

For a subset $Q \subseteq L(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}, \mathbb{R}^m)$, we denote by $\mathrm{proj}_x Q$ the set of matrices $M \in L(\mathbb{R}^{n_1}, \mathbb{R}^m)$ such that $(MN) \in Q$ for some $N \in L(\mathbb{R}^{n_2}, \mathbb{R}^m)$. The notation $\mathrm{proj}_y Q$ is defined in a similar way.

We need first the following standard result on the continuity of sup-functions.

Lemma 2.1. *Let $F : \mathbb{R}^n \rightrightarrows L(\mathbb{R}^n, \mathbb{R}^m)$ be a set-valued map with closed values, which is upper semicontinuous at x. Then for each $u \in \mathbb{R}^n$ and $v \in \mathbb{R}^m$, the sup-function*

$$h(y) := \sup_{M \in F(y)} \langle v, M(u) \rangle$$

is a real-valued upper semicontinuous function at x.

Proof. We first observe that

$$|\langle v, M(u) \rangle| \leq \|v\| \cdot \|M\| \cdot \|u\|$$

for every $u \in \mathbb{R}^n, v \in \mathbb{R}^m$ and $M \in L(\mathbb{R}^n, \mathbb{R}^m)$ and so

$$\sup_{\|M\| \leq 1} \langle v, M(u) \rangle \quad \leq \|u\| \cdot \|v\|.$$

Let $\epsilon > 0$. Then by the upper semicontinuity of F, there exists some $\delta > 0$ such that $F(x') \subseteq F(x) + \epsilon B_{m\times n}(0,1)$ whenever $x' \in x + \delta B_n(0,1)$. Then, it follows that

$$\begin{aligned}
\limsup_{x'\to x} h(x') &= \limsup_{x'\to x} \sup_{M\in F(x')} \langle v, M(u)\rangle \\
&\le \limsup_{x'\to x} \sup_{M\in F(x)+\epsilon B_{m\times n}(0,1)} \langle v, M(u)\rangle \\
&\le \sup_{M\in F(x)} \langle v, M(u)\rangle + \sup_{N\in \epsilon B_{m\times n}(0,1)} \langle v, N(u)\rangle \\
&\le h(x) + \epsilon\|u\|\cdot\|v\|.
\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, we conclude that

$$\limsup_{x'\to x} h(x') \le h(x)$$

which shows that h is upper semicontinuous at x. $\square$

Proposition 2.2. *Let $f : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}^m$ be continuous. If $\partial f(x,y) \subseteq L(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}, R^m)$ is an approximate Jacobian of f at (x,y), then $\overline{proj_x}\partial f(x,y)$ and $\overline{proj_y}\partial f(x,y)$ are partial approximate Jacobians of f at (x,y). Conversely, if $\partial_x f(x,y)$ and $\partial_y f(x,y)$ are partial approximate Jacobians of f at (x,y), and if the set valued map $\partial_y f(\cdot,\cdot) : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \rightrightarrows L(\mathbb{R}^{n_2}, \mathbb{R}^m)$ is upper semicontinuous at (x,y), then the set $(\partial_x f(x,y),\ \partial_y f(x,y))$ is an approximate Jacobian of f at (x,y).*

Proof. For the first part let $u \in \mathbb{R}^{n_1}$ and $w \in \mathbb{R}^m$. Then

$$\begin{aligned}
(wf(\cdot,y))^+(x,u) &= \limsup_{t\downarrow 0} \frac{(wf)((x,y)+t(u,0)) - (wf)(x,y)}{t} \\
&\le \sup_{(MN)\in\partial f(x,y)} \langle w, (MN)(u,0)\rangle \\
&\le \sup_{M\in \mathrm{proj}_x \partial f(x,y)} \langle w, M(u)\rangle.
\end{aligned}$$

This shows that $\overline{\mathrm{proj}_x}\partial f(x,y)$is an approximate Jacobian of the function $f(\cdot,y)$ at x.

To prove the second part, let $(u,v) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ and $w \in \mathbb{R}^m$. Then

$$\begin{aligned}
(wf)^+((x,y),(u,v)) &= \limsup_{t\downarrow 0} \frac{(wf)(x+tu, y+tv) - (wf)(x,y)}{t} \\
\le \limsup_{t\downarrow 0} &\frac{(wf)(x+tu, y+tv) - (wf)(x+tu,y)}{t} \\
+ \limsup_{t\downarrow 0} &\frac{(wf)(x+tu, y) - (wf)(x,y)}{t} \\
\le \limsup_{t\downarrow 0} &\frac{(wf)(x+tu, y+tv) - (wf)(x+tu,y)}{t} + \sup_{M\in\partial_x f(x,y)} \langle w, M(u)\rangle.
\end{aligned}$$

To estimate the first term of the last sum, let us apply the mean value theorem to the function $f(x+tu,\cdot)$ on the interval $[y, y+tv]$. Thus, for $\epsilon > 0$, there is some $N_t \in \overline{co}\partial_y f(x+tu, [y, y+tv])$ such that

$$(wf)(x+tu, y+tv) - (wf)(x+tu, y) \le \langle w, N_t(tv)\rangle + t\epsilon.$$

This and Lemma 2.1 yield

$$\begin{aligned}
&\limsup_{t\downarrow 0} \frac{(wf)(x+tu, y+tv) - (wf)(x+tu, y)}{t} \\
&\le \limsup_{t\downarrow 0} \sup_{N\in \overline{co}\partial_y f(x+tu,[y,y+tv])} (\langle w, N(v)\rangle + \epsilon) \\
&\le \limsup_{t\downarrow 0} \sup_{N\in \partial_y f(x+tu,[y,y+tv])} (\langle w, N(v)\rangle + \epsilon) \\
&\le \sup_{N\in\partial_y f(x,y)} \langle w, N(v)\rangle + \epsilon.
\end{aligned}$$

Summing up the above inequalities and taking into account the fact that ϵ is arbitrary, we obtain

$$\begin{aligned}
(wf)^+((x,y),(u,v)) &\le \sup_{M\in\partial_x f(x,y)} \langle w, M(u)\rangle + \sup_{N\in\partial y f(x,y)} \langle w, N(v)\rangle \\
&\le \sup_{(MN)\in(\partial_x f(x,y),\partial_y f(x,y))} \langle w, (MN)(u,v)\rangle
\end{aligned}$$

which shows that $(\partial_x f(x,y), \partial_y f(x,y))$ is an approximate Jacobian of f at (x,y). $\square$

Finally let us state a special case of the standard min-max theorem which is needed in the sequel.

Lemma 2.2. *Let $v_0 \in \mathbb{R}^m$, let $D \subseteq \mathbb{R}^n$ be a nonempty compact set and let $Q \subseteq L(R^n, R^m)$ be a nonempty set. Then we have*

$$\sup_{M\in Q} \inf_{u\in D} \langle v_0, M(u)\rangle = \inf_{u\in D} \sup_{M\in Q} \langle v_0, M(u)\rangle.$$

Proof. The function $(u, M) \mapsto \langle v_0, Mu\rangle$ from $\mathbb{R}^n \times L(\mathbb{R}^n, \mathbb{R}^m)$ to $\mathbb{R}\cup\{\infty\}$ being linear in each of variables u and M and D being compact, the lemma is obtained from the standard minimax theorem. $\square$

3. A Generalized Convex Interior Mapping Theorem

In this section we shall present a convex interior mapping theorem for (not necessarily locally Lipschitz) continuous functions by using approximate Jacobians.

Let $C \subseteq \mathbb{R}^n$ be a nonempty subset and let M be an $m \times n$ matrix. We say that M is surjective on C at $a \in \overline{C}$ if

$$M(a) \in \mathrm{int} M(C),$$

where $\mathrm{int}M(C)$ denotes the interior of the image $M(C)$ of C under M. The following local surjectivity result is crucial in establishing the main result of this section.

Lemma 3.1. *Let $C \subseteq \mathbb{R}^n$ be a nonempty convex set with $0 \in \overline{C}$. Let $F : \mathbb{R}^n \rightrightarrows L(\mathbb{R}^n, \mathbb{R}^m)$ be a set-valued map with closed values, which is upper semicontinuous at 0. If every element of the set $\overline{co}F(0) \cup co((F(0))_\infty \backslash \{0\})$ is surjective on C at 0, then there exist some $k > 0$ and $\delta > 0$ such that*

$$B_m(0,1) \subseteq kM[B_n(0,1) \cap (C - x)],$$

for every $x \in B_n(0,\delta) \cap \overline{C}$ and for every

$$M \in \bigcup_{y\in B_n(0,\delta)} \overline{co}\left[F(y) + (F(y))^\delta_\infty\right].$$

Proof. Suppose to the contrary that the conclusion is not true. Thus, for each $k \geq 1$ and $\delta = 1/k$ there exist $x_k \in B_n(0,1/k) \cap \overline{C}$, $v_k \in B_m(0,1)$ and $M_k \in \bigcup_{y \in B_n(0,1/k)} \overline{co}\left[F(y) + (F(y))_\infty^{1/k}\right]$ such that

$$v_k \notin kM_k[B_n(0,1) \cap (C - x_k)]. \tag{3.1}$$

Without loss of generality we may assume that

$$\lim_{k\to\infty} v_k = v_0 \in B_m(0,1). \tag{3.2}$$

We claim that by taking a subsequence if necessary, it can be assumed that either

$$\lim_{k\to\infty} M_k = M_0 \in \overline{co}F(0), \tag{3.3}$$

or

$$\lim_{k\to\infty} t_k M_k = M_* \in co\left[(F(0))_\infty \backslash \{0\}\right] \tag{3.4}$$

where $\{t_k\}$ is some sequence of positive numbers converging to 0.

Let us first see that (3.3) or (3.4) leads to a contradiction. If (3.3) holds, then by the surjectivity of M_0 there is some $\epsilon > 0$ and $k_0 \geq 1$ such that

$$v_0 + B_m(0,\epsilon) \subseteq k_0 M_0[B_n(0,1) \cap C]. \tag{3.5}$$

Moreover, there is $k_1 \geq k_0$ such that

$$\|M_k - M_0\| < \epsilon/4 \quad \text{for } k \geq k_1. \tag{3.6}$$

We want to show that there is $k_2 \geq k_1$ such that

$$v_0 + B_m(0,\epsilon/2) \subseteq k_0 M_0[B_n(0,1) \cap (C - x_k)] \quad \text{for } k \geq k_2. \tag{3.7}$$

Indeed if this is not the case, then one may assume that for each x_k there is some $b_k \in B_m(0,\epsilon/2)$ satisfying

$$v_0 + b_k \notin k_0 M_0[B_n(0,1) \cap (C - x_k)].$$

Since that set $B_n(0,1) \cap (C - x_k)$ is convex, there exists some $\xi_k \in \mathbb{R}^m$ with $\|\xi_k\| = 1$ such that

$$\langle \xi_k,\ v_0 + b_k\rangle \ \leq \langle \xi_k, k_0 M_0(x)\rangle \quad \forall\ x \in B_n(0,1) \cap (C - x_k).$$

Using subsequences if needed, one may again assume that

$$\lim_{k\to\infty} b_k = b_0 \in B_m(0,\epsilon/2),$$
$$\lim_{k\to\infty} \xi_k = \xi_0 \text{ with } \|\xi_0\| = 1.$$

It follows then

$$\langle \xi_0, v_0 + b_0\rangle \leq \langle \xi_0, k_0 M_0(x)\rangle \quad \text{for all } x \in B_n(0,1) \cap C.$$

The point $v_0 + b_0$ being an interior point of the set $v_0 + B_m(0,\epsilon)$, the obtained inequality contradicts (3.5). Thus (3.7) holds for some $k_2 \geq k_1$. Now using (3.6) and (3.7) we derive the following inclusions for $k \geq k_2$:

$$\begin{aligned} v_0 + B_m(0,\frac{\epsilon}{2}) &\subseteq k_0 M_0[B_n(0,1) \cap (C - x_k)] \\ &\subseteq k_0 \left\{M_k[B_n(0,1) \cap (C - x_k)] + (M_0 - M_k)[B_n(0,1) \cap (C - x_k)]\right\} \end{aligned}$$

$$\subseteq k_0 M_k[B_n(0,1) \cap (C - x_k)] + B_m(0, \epsilon/4). \tag{3.8}$$

This gives us

$$v_0 + B_m(0, \frac{\epsilon}{4}) \subseteq k_0 M_k[B_n(0,1) \cap (C - x_k)], \quad \text{for } k \geq k_2. \tag{3.9}$$

We choose now $k \geq k_2$ so large that $v_k \in v_0 + B_m(0, \epsilon/4)$. Then (3.9) yields

$$v_k \in k M_k[B_n(0,1) \cap (C - x_k)], \tag{3.10}$$

which contradicts (3.1).

Now we assume (3.4). Again, since M_* is surjective, relations (3.5), (3.6), (3.7) and (3.8) remain true when we replace M_0 by M_* and M_k by $t_k M_k$. Then relation (3.9) becomes

$$v_0 + B_m(0, \frac{\epsilon}{4}) \subseteq k_0 t_k M_k[B_n(0,1) \cap (C - x_k)] \quad \text{for } k \geq k_2.$$

By choosing $k \geq k_2$ sufficiently large so that $v_k \in v_0 + B_m(0, \epsilon/4)$ and $0 < t_k \leq 1$, we arrive at the same contradiction as (3.10).

The proof will be then completed if we show that either (3.3) or (3.4) holds.

Let

$$M_k \in \overline{co}\left[F(y_k) + (F(y_k))_\infty^{1/k}\right] \quad \text{for some } y_k \in B_n(0, 1/k).$$

Since F is upper semicontinuous at 0, there is $k_0 \geq 1$ such that

$$(F(y_k))_\infty \subseteq (F(0))_\infty, \qquad k \geq k_0.$$

We may assume without loss of generality that this inclusion is true for all $k = 1, 2, \ldots$. Thus, for each $k \geq 1$, there exist $M_{kj} \in F(y_k), N_{kj} \in (F(0))_\infty$, P_{kj} and P_k with

$$\|P_{kj}\| \leq 1, \ \|P_k\| \leq 1, \text{ and } \lambda_{kj} \in [0,1], \ j = 1, \ldots, nm+1$$

such that $\sum_{j=1}^{mn+1} \lambda_{kj} = 1$ and

$$M_k = \sum_{j=1}^{mn+1} \lambda_{kj}(M_{kj} + N_{kj} + \frac{1}{k}\|N_{kj}\|P_{kj}) + \frac{1}{k}P_k.$$

If all the sequences $\{\lambda_{kj}M_{kj}\}_{k\geq 1}$, $\{\lambda_{kj}N_k\}_{k\geq 1}$, $j = 1, \ldots, mn+1$ are bounded, then so is the sequence $\{M_k\}$. By passing to subsequences if necessary we may assume

$$\lim_{k\to\infty} M_k = M_0, \qquad \lim_{k\to\infty} \lambda_{kj} = \lambda_{0j}$$

$$\lim_{j\to\infty} \lambda_{kj}N_{kj} = N_{0j}, \quad \lim_{k\to\infty} \lambda_{kj}M_{kj} = M_{0j}$$

for $j = 1, \ldots, mn+1$. Since $(F(0))_\infty$ is a closed cone, we have

$$N_{0j} \in (F(0))_\infty, \ \sum_{j=1}^{nm+1} N_{0j} \in co(F(0))_\infty.$$

Moreover, we also have $\sum_{j=1}^{nm+1} \lambda_{0j} = 1$.

Decompose the sum $\sum\limits_{j=1}^{nm+1} \lambda_{kj}M_{kj}$ into two sums: The first sum $\sum_1$ consists of those terms with $\{M_{kj}\}_{k\geq 1}$ bounded, and the second sum $\sum_2$ consists of those terms with $\{M_{kj}\}_{k\geq 1}$ unbounded. Then the limits λ_{0j} with j in the second sum are all zero and the corresponding limits M_{0j} are recession directions of $F(0)$. Hence, $\sum_1 \lambda_{0j} = 1$ and

$$\lim_{k\to\infty} \sum_1 \lambda_{kj} M_{kj} = \sum_1 M_{0j} \in coF(0)$$

by the upper semicontinuity of F at 0, and

$$\lim_{k\to\infty} \sum_2 \lambda_{kj} M_{kj} = \sum_2 M_{0j} \in co(F(0))_\infty.$$

Thus, $M_0 \in co\ F(0) + co\ (F(0))_\infty \subseteq \overline{co}F(0)$ (see [13]) and (3.3) is fulfilled.

If among the above said sequences there are unbounded ones, then again by taking subsequences instead, we may assume to have one of them, say $\{\lambda_{kj_0} M_{kj_0}\}_{k\geq 1}$ for some $j_0 \in \{1, \ldots, mn+1\}$, with the largest norm. (The same argument works for $\{\lambda_{kj_0} N_{kj_0}\}$). Consider the sequence $\{M_k/\|\lambda_{kj_0} M_{kj_0}\|\}_{k\geq 1}$. It is evident that this sequence is bounded, and we may assume it converges to some matrix M_*. We have then $M_* \in co\ (F(0))_\infty$. Note that the cone $co\ (F(0))_\infty$ is pointed, otherwise $co\ [(F(0))_\infty \backslash \{0\}]$ should contain the zero matrix which is certainly not surjective and this should contradict the hypothesis. As before, we may assume that each term in the sum of $\{M_k/\|\lambda_{kj_0} M_{kj_0}\|\}$ is convergent. Then M_* is a finite sum of elements from $co(F(0))_\infty$. Since at least one of the terms of this sum is non-zero (the term corresponding to the index j_0 has a unit norm), and the cone $co(F(0))_\infty$ is pointed, we deduce that M_* is non-zero, and so (3.4) holds. Hence the proof is complete. □

Lemma 3.2. *Assume that the hypotheses of Lemma 3.1 hold. Then there is a closed convex set D containing 0 with $D\backslash\{0\} \subseteq C$ such that*

$$B_m(0,1) \subseteq kM[B_n(0,1) \cap (D - x)]$$

for every $x \in B_n(0,\delta) \cap D$ *and* $M \in \bigcup_{y \in B_n(0,\delta)} \overline{co}[F(y) + (F(y))^\delta_\infty]$.

Proof. As noticed in [14, 15], it is easy to construct an increasing (by inclusions) sequence of closed and convex sets $\{D_k\}$ such that $0 \in D_k \subseteq C \cup \{0\}$ and $C \subseteq \overline{[\cup_{k=1}^\infty D_k]}$. We show that for k sufficiently large, every matrix of the set $\overline{co}\ F(0) \cup co\ [(F(0))_\infty \backslash \{0\}]$ is surjective on D_k at 0. Indeed, if this is not the case, then for each $k = 1, 2, \ldots$ there is $M_k \in \overline{co}\ F(0) \cup co\ [(F(0))_\infty \backslash \{0\}]$ such that

$$0 \notin \mathrm{int} M_k(D_k \cap B_n(0,1)).$$

Since $D_k \cap B_n(0,1)$ is convex, using the separation theorem we find $\xi_k \in \mathbb{R}^m$ with $\|\xi_k\| = 1$ such that

$$0 \leq \langle \xi_k, M_k(x) \rangle \quad \text{for } x \in D_k \cap B_n(0,1). \tag{3.11}$$

Without loss of generality we may assume that

$$\lim_{k\to\infty} \xi_k = \xi_0 \quad \text{with } \|\xi_0\| = 1$$

and either

$$\lim_{k\to\infty} M_k = M_0 \in \overline{co}\ F(0) \cup co\ [(F(0))_\infty \backslash \{0\}]$$

or there are $t_k > 0$ such that

$$\lim_{k\to\infty} t_k M_k = M_0 \in co[(F(0))_\infty \backslash \{0\}].$$

In all cases (3.11) yields

$$0 \leq \langle \xi_0, M_0(x)\rangle \quad \text{for } x \in C \cap B_n(0,1).$$

This contradicts the surjectivity of M_0 on C at 0.

Thus, for k sufficiently large, Lemma 3.1 is applicable to the set $D = D_k$ and produces the desired result. □

Now we formulate and prove the main result of the paper.

Theorem 3.3. *Let C be a nonempty convex set in $\mathbb{R}^n$ and let $f : \mathbb{R}^n \to \mathbb{R}^m$ be continuous. Assume that*

i) *$\partial f : \mathbb{R}^n \rightrightarrows L(\mathbb{R}^n, \mathbb{R}^m)$ is an approximate Jacobian map of f which is upper semicontinuous at $a \in \overline{C}$;*

ii) *Every matrix of the set $\overline{co}\partial f(a) \cup co[(\partial f(a))_\infty \backslash \{0\}]$ is surjective on C at a.*

Then $f(a) \in int f(C)$.

Proof. Without loss of generality we may assume that $a = 0$ and $f(a) = 0$. Moreover, by Lemma 3.2, we may also assume that C is closed. We obtain the conclusion by establishing the inclusion

$$B_m\left(0, \frac{\delta}{4k}\right) \subseteq f(B_n(0,\delta) \cap C). \tag{3.12}$$

Suppose that (3.12) is false. Then we can find $\overline{y}$ with $\|\overline{y}\| \leq \dfrac{\delta}{4k}$ such that

$$\overline{y} \notin f(B_n(0,\delta) \cap C).$$

We define a real function $\varphi : \mathbb{R}^n \to \mathbb{R}$ by

$$\varphi(x) := \|\overline{y} - f(x)\| + \frac{2}{\delta}\|\overline{y}\| \cdot \|x\|.$$

It is clear that φ is continuous. Hence it attains its minimum on the compact set $B_n(0,\delta) \cap C$ at some point $\overline{x} \in B_n(0,\delta) \cap C$. We claim that

$$\overline{x} \in (\text{int} B_n(0,\delta)) \cap C. \tag{3.13}$$

In fact, if $\|\overline{x}\| = \delta$, then

$$\varphi(\overline{x}) = \|\overline{y} - f(\overline{x})\| + 2\|\overline{y}\| > \varphi(0) = \|\overline{y}\|$$

because $\overline{x} \in C \cap B_n(0,\delta)$ and $\overline{y} \notin f(B_n(0,\delta) \cap C)$, which is impossible for $\overline{x}$ being a minimum point.

It follows from (3.13) that

$$\text{cone}(C - \overline{x}) = \text{cone}[(B_n(0,1) \cap C) - \overline{x}].$$

Consequently, if $\partial\varphi(\overline{x})$ is a generalized subdifferential of φ at $\overline{x}$, then Proposition 2.1 yields

$$\sup_{\xi \in \partial\varphi(\overline{x})} \langle \xi, u\rangle \geq 0 \quad \text{for all } u \in C - \overline{x}. \tag{3.14}$$

Let us now find a generalized subdifferential of φ at $\overline{x}$. To this purpose, note that $\overline{y} \neq f(\overline{x})$, therefore the function $y \to \|\overline{y} - y\|$ is Gâteaux differentiable at $y = f(\overline{x})$ and its derivative at this point equals $\dfrac{\overline{y} - f(\overline{x})}{\|\overline{y} - f(\overline{x})\|}$. Furthermore, for the function $x \to \|x\|$, the closed unit ball $B_n(0,1)$ is a generalized subdifferential at any point. We now apply the sum rule (see [11]) and the chain rule to obtain the following generalized subdifferential of φ at $\overline{x}$:

$$\partial\varphi(\overline{x}) := \left\{ \frac{\overline{y} - f(\overline{x})}{\|\overline{y} - f(\overline{x})\|} M + \frac{2}{\delta}\|\overline{y}\|\xi : M \in Q \right\}$$

where $Q := \partial f(\overline{x}) + (\partial f(\overline{x}))^{\delta}_{\infty}$.

With this subdifferential, inequality (3.14) becomes

$$\sup_{M\in Q, \xi\in B_n(0,1)} \langle \frac{\overline{y} - f(\overline{x})}{\|\overline{y} - f(\overline{x})\|} M + \frac{2}{\delta}\|\overline{y}\|\xi, u\rangle \geq 0, \quad \text{for } u \in C - \overline{x}.$$

This implies

$$\frac{1}{2k} \geq -\sup_{M\in Q} \langle \frac{\overline{y} - f(\overline{x})}{\|\overline{y} - f(\overline{x})\|}, M(u)\rangle \quad \text{for } u \in B_n(0,1) \cap (C - \overline{x}).$$

or equivalently,

$$\begin{aligned}\frac{1}{2k} &\geq \sup_{u\in B_n(0,1)\cap(C-\overline{x})} \left(-\sup_{M\in Q} \langle \frac{\overline{y} - f(\overline{x})}{\|\overline{y} - f(\overline{x})\|},\ M(u)\rangle \right) \\ &\geq -\inf_{u\in B_n(0,1)\cap(C-\overline{x})} \sup_{M\in Q} \langle \frac{\overline{y} - f(\overline{x})}{\|\overline{y} - f(\overline{x})\|},\ M(u)\rangle.\end{aligned}$$

In virtue of Lemma 2.1, the last inequality gives

$$\frac{1}{2k} \geq -\sup_{M\in Q} \inf_{u\in B_n(0,1)\cap(C-\overline{x})} \langle \frac{\overline{y} - f(\overline{x})}{\|\overline{y} - f(\overline{x})\|},\ M(u)\rangle. \tag{3.15}$$

According to Lemma 3.1, for each $M \in Q$, we have the inclusion

$$B_m(0,1) \subseteq kM[B_n(0,1) \cap (C - \overline{x})].$$

In particular, there is $u \in B_n(0,1) \cap (C - \overline{x})$ such that $M(u) = \dfrac{1}{k}\frac{f(\overline{x})-\overline{y}}{\|\overline{y}-f(\overline{x})\|}$. Hence (3.15) implies

$$\frac{1}{2k} \geq \frac{1}{k}$$

which is impossible. This completes the proof. $\square$

Example 3.4. *Let $f(x,y) = g(x) + y^2$ be a continuous map from $\mathbb{R}^2$ to $\mathbb{R}$, where g is a real function which is differentiable at $x \neq 0$ with $\lim_{x\to 0} g'(x) = -\infty$. It can be seen that*

$$\partial f(x,y) = \begin{cases} \{(g'(x), 2y)\} & \text{if } x \neq 0 \\ \{(\alpha, 2y) : \alpha \leq -1\} & \text{if } x = 0 \end{cases}$$

is an approximate Jacobian of f which is upper semicontinuous at $(0,0)$*. At* $(0,0)$ *we have*

$$\partial f(0,0) = \{(\alpha, 0) : \alpha \leq -1\}$$

and

$$(\partial f(0,0))_\infty = \{(\alpha, 0) : \alpha \leq 0\}.$$

Define the convex set $C \subset \mathbb{R}^2$ by

$$C := \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}.$$

Then all the conditions of Theorem 3.3 are satisfied and its conclusion holds.

When f is a locally Lipschitz function, Theorem 3.3 yields Pourciau's convex interior mapping theorem.

Corollary 3.5. *Suppose that $f : \mathbb{R}^n \to \mathbb{R}^m$ is locally Lipschitz and C is a convex set in $\mathbb{R}^n$. If every matrix of the Clarke generalized Jacobian $\partial^C f(a)$ of f at $a \in \overline{C}$ is surjective on C at a, then $f(a) \in intf(C)$.*

Proof. When f is locally Lipschitz, the Clarke generalized Jacobian map $x \to \partial^C f(x)$ is an upper semicontinuous approximate Jacobian map with bounded and convex values. The conclusion now easily follows from Theorem 3.3. □

4. Interior Mapping Theorems & Partial Approximate Jacobians

In classical calculus a practical way of obtaining the derivative of a vector function is to compute partial derivatives of component functions. As we have seen in Section 2 and Proposition 2.2 this method applies to approximate Jacobians. For applications purposes we shall derive a convex interior mapping theorem involving partial approximate Jacobians.

Lemma 4.1. *Let $F_i : \mathbb{R}^n \rightrightarrows \mathbb{R}^{k_i}$, $i = 1,2$ be set-valued maps with closed values which are upper semicontinuous at $a \in \mathbb{R}^n$. Then for every $\delta \geq 0$, the set-valued map $F^\delta : \mathbb{R}^n \rightrightarrows \mathbb{R}^{k_1} \times \mathbb{R}^{k_2}$ defined by*

$$F^\delta(x) = (F_1(x) + [F_1(x)]^\delta_\infty,\ F_2(x) + [F_2(x)]^\delta_\infty)$$

is upper semicontinuous at a.

Proof. Let $\epsilon > 0$ be given. By the upper semicontinuity, there is some $\delta > 0$ such that for $i = 1,2$,

$$F_i(x) \subseteq F_i(a) + \epsilon B_{k_i}(0,1),$$

whenever $x \in a + B_n(0,1)$. Thus for each $x \in a + B_n(0,1)$,

$$[F_i(x)]_\infty \subseteq [F_i(a)]_\infty.$$

Consequently,

$$\begin{aligned} F^\delta(x) &\subseteq (F_1(a) + [F_1(a)]^\delta_\infty + \epsilon B_{k_1}(0,1),\ F_2(a) + [F_2(a)]^\delta_\infty + \epsilon B_{k_2}(0,1)) \\ &\subseteq F^\delta(a) + \epsilon[B_{k_1}(0,1) \times B_{k_2}(0,1)] \end{aligned}$$

which shows that F^δ is upper semicontinuous at a. □

Lemma 4.2. *Let $C \subseteq \mathbb{R}^{n_1+n_2}$ be a nonempty convex set with $0 \in \overline{C}$. Let $F_i : \mathbb{R}^{n_1+n_2} \rightrightarrows L(\mathbb{R}^{n_i}, \mathbb{R}^m)$ $i = 1, 2$ be closed set-valued maps which are upper semicontinuous at 0. If for each pair of matrices $M \in \overline{co}F_1(0) \cup co[(F_1(0))_\infty \backslash \{0\}]$ and $N \in \overline{co}F_2(0) \cup co[(F_2(0))_\infty \backslash \{0\}]$, the matrix (MN) is surjective on C at 0, then there exist some $k > 0$ and $\delta > 0$ such that*

$$B_m(0,1) \subseteq k(MN)[B_n(0,1) \cap (C - x)],$$

for every $x \in B_n(0,\delta) \cap \overline{C}$ and for every

$$(MN) \in \bigcup_{y \in B_n(0,\delta)} (\overline{co}[F_1(y) + (F_1(y))^\delta_\infty],\ \overline{co}[F_2(y) + (F_2(y))^\delta_\infty]),$$

Proof. We proceed the proof in the similar way as in Lemma 3.1. By supposing to the contrary, we find

$$x_k \in B_n\left(0, \frac{1}{k}\right) \cap \overline{C}, \quad v_k \in B_m(0,1),\ y_k \in B_n\left(0, \frac{1}{k}\right)$$
$$M_k \in \overline{co}[F_1(y_k) + (F_1(y_k))^{1/k}_\infty],\ N_k \in \overline{co}[F_2(y_k) + (F_2(y_k))^{1/k}_\infty],$$

such that

$$\lim_{k\to\infty} v_k = v_0 \in B_m(0,1) \tag{4.1}$$

$$v_k \notin k(M_k N_k)[B_n(0,1) \cap (C - x_k)]. \tag{4.2}$$

For $\{M_k\}$ and $\{N_k\}$ we have two possible cases (by using a subsequence of necessary)

$$\lim_{k\to\infty} M_k = M_0 \in \overline{co}F_1(0)$$
$$\lim_{k\to\infty} t_k M_k = M_* \in co[(F_1(0))_\infty \backslash \{0\}],$$

where $\{t_k\}$ is some sequence of positive numbers converging to 0, and similar relations for $\{N_k\}$.

Then we have

$$v_0 + B_m(0,\epsilon) \subseteq P[B_n(0,1) \cap C]$$

for some $\epsilon > 0$, where P is one of the four matrices $(M_0 N_0), (M_0 N_*), (M_* N_0)$, and $(M_* N_*)$.

Since P is surjective by hypothesis, for k sufficiently large

$$v_0 + B_m(0,\epsilon/2) \subseteq k_0 P[B_n(0,1) \cap (C - x_k)]$$

and this implies

$$v_0 + B_m(0,\epsilon/2) \subseteq k_0 P_k[B_n(0,1) \cap (C - x_k)] \tag{4.3}$$

for k large, where P_k is either $(M_k N_k)$, $(M_k(s_k N_k))$, $((t_k M_k) N_k)$, $((t_k M_k)(s_k N_k))$ with $\lim t_k M_k = M_*$, $\lim s_k N_k = N_*$. Since for k large, $0 < t_k \leq 1$, relation (4.3) yields

$$v_0 + B_m(0,\epsilon/2) \subseteq k_0(M_k N_k)[B_n(0,1) \cap (C - x_k)],$$

which contradicts (4.2). □

Theorem 4.3. *Let $C \subseteq \mathbb{R}^n = \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ be a nonempty convex set and let $f : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}^m$ be continuous. Assume that*

i) $\partial_x f$ *and* $\partial_y f$ *are partial approximate Jacobian maps of* f *with respect to* x *and* y *respectively and are upper semicontinuous at* $a \in \overline{C}$.

ii) *Every matrix* (MN) *where* $M \in \overline{co}\partial_x f(a) \cup co[(\partial_x f(a))_\infty \backslash \{0\}]$ *and* $N \in \overline{co}\partial_y f(a) \cup co[(\partial_y f(a))_\infty \backslash \{0\}]$ *is surjective on* C *at* a.

Then $f(a) \in int f(C)$.

Proof. We proceed in a similar way as that of Theorem 3.3. In view of Proposition 2.2, the set

$$Q := (\partial_x f(a) + (\partial_x f(a))^\delta_\infty,\ \partial_y f(a) + (\partial_y f(a))^\delta_\infty)$$

is an approximate Jacobian of f at a. Now Lemma 4.2 yields

$$B_m(0,1) \subseteq k(MN)[B_n(0,1) \cap (C-a)]$$

for every $(MN) \in Q$. By this the same contradiction is obtained. □

5. Generalized Lagrange Multiplier Rule

In this section we shall apply the convex interior mapping theorem to derive a multiplier rule for a constrained optimization problem with continuous data.

We consider the following problem (P):

$$\min f(x)$$

$$g_i(x) \leq 0, \qquad i = 1, \dots, p \tag{5.1}$$

$$h_j(x) = 0, \qquad j = 1, \dots, q \tag{5.2}$$

We denote $g = (g_1, \dots, g_p)$, $h = (h_1, \dots, h_q)$ and $F = (f, g, h)$.

Theorem 5.1. *Assume that* F *is continuous and admits an approximate Jacobian map* ∂F *which is upper semicontinuous at* $\overline{x} \in \mathbb{R}^n$. *If* $\overline{x}$ *is a local optimal solution of* (P), *then there exists a nonzero vector* $(\alpha, \beta, \gamma) \in \mathbb{R} \times \mathbb{R}^p \times \mathbb{R}^q$ *with* $\alpha \geq 0$, $\beta = (\beta_1, \dots, \beta_p)$ *with* $\beta_i \geq 0$ *and* $\beta_i g_i(\overline{x}) = 0, i = 1, ..., p$, *such that*

$$0 \in (\alpha, \beta, \gamma)(\overline{co}\partial F(\overline{x}) \cup co[(\partial F(\overline{x}))_\infty \backslash \{0\}]).$$

Proof. Let $\epsilon > 0$ be given so that $f(x) \geq f(\overline{x})$ for every feasible $x \in \overline{x} + B_n(0, \epsilon)$. Without loss of generality we may assume $\overline{x} = 0$ and $F(\overline{x}) = 0$. Let us denote

$$W = \{(t, a, 0) \in \mathbb{R} \times \mathbb{R}^p \times \mathbb{R}^q : t < 0, a = (a_1, \dots, a_p) \quad \text{with } a_i < 0, i = 1, \dots, p\},$$

$$C = B_n(0, \epsilon) \times W \subseteq \mathbb{R}^n \times \mathbb{R}^{1+p+q}.$$

Let us also define a vector function $\phi : \mathbb{R}^n \times \mathbb{R}^{1+p+q} \to \mathbb{R}^{1+p+q}$ by

$$\phi(x, w) = F(x) - w.$$

By denoting by I the identity $(1+p+q) \times (1+p+q)$-matrix, we see that

$$(x, w) \to \partial_x \phi(x, w) = \partial F(x)$$
$$(x, w) \to \partial_y \phi(x, w) = \{I\}$$

are partial approximate Jacobian maps of ϕ which are upper semicontinuous at $(0, 0)$. Moreover,

$$[\partial_x \phi(x, w)]_\infty = [\partial F(x)]_\infty,\ [\partial_w \phi(x, w)] = \{0\}.$$

We observe further that

$$\phi(0,0) \notin \phi[B_n(0,\epsilon) \times W],$$

otherwise we can find some $x \in B_n(0,\epsilon)$ and $w \in W$ such that

$$0 = \phi(0,0) = F(x) - w$$

which shows that x is feasible for (P) and $f(x) < f(\overline{x})$ and contradicts the hypothesis. It follows that

$$\phi(0,0) \notin \mathrm{int}\phi[B_n(0,\epsilon) \times W].$$

In view of our convex interior mapping theorem, there must exist a matrix from the set

$$(\overline{co}\partial F(0) \cup co[(\partial F(0))_\infty \backslash \{0\}], -I),$$

say of the form $(M(-I))$ such that

$$(M(-I))(0,0) \notin \mathrm{int}(M(-I))[B_n(0,\epsilon) \times W].$$

Now we apply the separation theorem to find a nonzero vector $(\alpha, \beta, \gamma) \in \mathbb{R}^{1+p+q}$ such that

$$((\alpha,\beta,\gamma),(M(-I))(x,w) \geq 0 \quad \text{for all } (x,w) \in B_n(0,\epsilon) \times W.$$

This is equivalent to

$$\langle(\alpha,\beta,\gamma), Mx\rangle \geq \langle(\alpha,\beta,\gamma), w\rangle \quad \text{for all } x \in \mathbb{R}^n, w \in W.$$

¿¿From this inequality we see that $\alpha \geq 0, \beta_i \geq 0, \beta_i g_i(\overline{x}) = 0$ (because $g_i(\overline{x}) = 0$), $i = \ell, \ldots, p$ and $(\alpha,\beta,\gamma)M = 0$ where $M \in \overline{co}\partial F(\overline{x})$. □

When f, g and h are locally Lipschitz, Theorem 5.1 gives the classical multiplier rule for Lipschitz problems (see [1, 5, 14]).

Corollary 5.2. *Assume that F is locally Lipschitz and $\overline{x}$ is a local optimal solution of (P). Then there exists a nonzero vector $(\alpha,\beta,\gamma) \in \mathbb{R}^{1+p+q}$ with $\alpha \geq 0, \beta_i \geq 0$ and $\beta_i g_i(\overline{x}) = 0, i = 1, \ldots, p$ such that*

$$0 \in (\alpha,\beta,\gamma)\partial^c F(\overline{x}).$$

Proof. We use the Clarke generalized Jacobian $\partial^c F$ as an upper semicontinuous approximate Jacobian of F and apply Theorem 5.1 to produce the desired result. □

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