Journal of Nonlinear and Convex Analysis
Volume 3, Number 2, 2002, 145–154

Yokohama Publishers
ISSN 1880-5221 ONLINE JOURNAL

ON CONDITIONS FOR EQUALITY OF RELAXATIONS IN THE CALCULUS OF VARIATIONS

KEWEI ZHANG

Abstract. We use the quadratic rank-one convex envelope $qr(f)$ for $f : M^{N\times n} \to \mathbb{R}$ to study conditions for equality of semiconvex envelopes and use the corresponding quadratic rank-one convex hull $qr(K)$ for compact sets $K \subset M^{N\times n}$ to give a condition for equality of semiconvex hulls. We show that $L_c(K) = C(K)$ if and only if $qr(K) = C(K)$. We also establish that for a function f bounded below by certain quadratic functions, $R(f) = C(f)$ if and only if $qr(f) = C(f)$. In particular, when $\min\{N, n\} = 2$, $R(f) = C(f)$ if and only if $P(f) = C(f)$.

In this paper we give a simple new condition for equality of semiconvex hulls which improves upon and unifies the known conditions [Z2] (also see [DKK] for sharpening and simplification of the proofs). We also establish a new condition for equality of semiconvex envelopes in the calculus of variations. We denote by $M^{N\times n}$ the linear space of real $N \times n$ matrices with the standard Euclidean norm of $\mathbb{R}^{Nn}$, and assume that N, $n \geq 2$. Let $f : M^{N\times n} \to \mathbb{R}$ be continuous and bounded below. Let $R(f)$, $Q(f)$, $P(f)$, and $C(f)$ be the rank-one convex, quasiconvex, polyconvex and the convex envelope [Sv2] respectively. It is well-known that $C(f) \leq P(f) \leq Q(f) \leq R(f) \leq f$. In this paper we introduce another semiconvex envelope $qr(f)$ called the quadratic rank-one convex envelope of f (see Definition 2 below) which satisfies

$$C(f) \leq qr(f) \leq Q(f) \leq R(f) \leq f. \tag{1}$$

We state our main results first, followed by relevant notation and definitions.

Theorem 1. *Suppose $f : M^{N\times n} \to \mathbb{R}$ is continuous and satisfies*

$$f(A) \geq c|A|^2 - C_1, \qquad A \in M^{N\times n} \tag{2}$$

for some constants $c > 0$, $C_1 \geq 0$. Then

(i) *$C(f) = qr(f)$ if and only if $C(f) = R(f)$.*
(ii) *When $\min\{N, n\} = 2$, $C(f) = P(f)$ if and only if $C(f) = R(f)$.*

It is well known that rank-one convexity does not imply quasiconvexity for functions [Sv1]. Theorem 1 implies that under (2), if $qr(f)$ is convex, $R(f) = Q(f) = C(f)$. Another condition for equality of envelopes was obtained recently in [Z4] which asserts that if $f : M^{N\times n} \to \mathbb{R}$ is continuous and satisfies $\lim_{|A|\to\infty} f(A)/|A| = +\infty$, then $C(f) = Q(f)$ if and only if $C(f) = R(f)$. Theorem 1 provides a simpler test for equality of envelopes under a stronger assumption (2). However, the second statement of Theorem 1 is new for the polyconvex envelope.

1991 *Mathematics Subject Classification.* 26D15.

Key words and phrases. bilinear form, inner product, Hilbert's inequality, Lagrangian multiplier.

By using semiconvex functions, we may define corresponding semiconvex hulls for compact sets $K \subset M^{N\times n}$ by using cosets. Let $R(K)$, $Q(K)$, $P(K)$, $qr(K)$ and $C(K)$ be the rank-one convex, the quasiconvex, the polyconvex, the quadratic rank-one convex and the convex hull of K respectively. By using rank-one connections of K we may also define the so called closed lamination convex hull $L_c(K)$ of K (see below for definitions). We have $K \subset L_c(K) \subset R(K) \subset Q(K) \subset P(K) \subset C(K)$ and

$$K \subset L_c(K) \subset R(K) \subset Q(K) \subset qr(K) \subset C(K). \tag{3}$$

Theorem 2. *Let $K \subset M^{N\times n}$ be compact, then*

(i) $L_c(K) = C(K)$ *if and only if* $qr(K) = C(K)$.

(ii) *When* $\min\{N, n\} = 2$, $L_c(K) = C(K)$ *if and only if* $P(K) = C(K)$.

In [Z2], some conditions for equality of hulls were established. We have $L_c(K) = C(K)$ if and only if $Q(K) = C(K)$. When $N = n = 2$, $L_c(K) = C(K)$ if and only if $P(K) = C(K)$. This latter result was improved in [DKK] by shown that the above statement for $P(K)$ is true precisely when $\min\{N, n\} = 2$. Theorem 2 shows that the quadratic rank-one convex hull $qr(K)$ is a simpler indicator than $Q(K)$ for equality of semiconvex hulls and is more precise than $P(K)$.

Our approach by using rank-one convex quadratic functions [Se,BFJK] is purely geometrical (see Lemma 1 below) so that we can avoid the use of quasiconvex envelope of distance functions and that of the Young measure [Z2,DKK].

Let $f : M^{N\times n} \to \mathbb{R}$ be a continuous function. The following are some conditions related to weak lower semicontinuity of the integral (c.f. [B1,Mo,D,AFu])

$$I(u) = \int_\Omega f(Du(x))dx$$

(i) f is rank-one convex if for each matrix $A \in M^{N\times n}$ and each rank-one matrix $B = a \otimes b \in M^{N\times n}$, the function $t \to f(A + tB)$ is convex.

(ii) f is quasiconvex at $A \in M^{N\times n}$ on Ω, if for any smooth function $\phi : \Omega \to \mathbb{R}^N$ compactly supported in Ω,

$$\int_\Omega f(A + D\phi(x))dx \geq \int_\Omega f(A)dx$$

holds. f is quasiconvex if it is quasiconvex at every $A \in M^{N\times n}$. The class of quasiconvex functions is independent of the choice of Ω.

(iii) f is polyconvex if $f(A) =$ convex function of minors of the matrix A.

It is well-known that (iii)$\Rightarrow$(ii)$\Rightarrow$(i), while (i)$\not\Rightarrow$(ii)$\not\Rightarrow$(iii) (cf. [B1,Mo,D,Sv1]). However, if f is a quadratic function, (i) is equivalent to (ii).

There are several well-known versions of semiconvex envelopes for functions arising from these notions of semiconvex functions. For a given function $f : M^{N\times n} \to \mathbb{R}$, the rank-one convex envelope $R(f)$, the quasiconvex envelope $Q(f)$ and the polyconvex envelope $P(f)$ are defined by $S(f) = \sup\{g \leq f,\ g \text{ is } S\text{-convex}\}$, where if we take the S-convex functions as quasiconvex, the rank-one convex and the polyconvex functions, we obtain $Q(f)$, $R(f)$ and $P(f)$ [D]. Note that there is a trivial relation [D]: $C(f) \leq P(f) \leq Q(f) \leq R(f) \leq f$.

In [F, Sec.3], N. Firoozye defined the following quasiconvex lower bound for $f : M^{N\times n} \to \mathbb{R}$ motivated by the 'translation method' for bounding effective moduli of composite materials (see, for example [Ta,LC,Mi]).

Definition 1. *Let RCQ be the set of all rank-one convex quadratic forms defined on $M^{N\times n}$. The optimal rank-one quadratic translation bound of f is defined by*

$$T_2(f) = \sup_{q\in RCQ} [C[f(A) - q(A)] + q(A)].$$

In [F] the representation of this bound by using probability measures was considered. A more general bound $T(f)$ of optimal rank-one convex quadratic and polyconvex translation bound is also defined by replacing q by a sum of rank-one quadratic form and a linear combination of minors. The notation $T_2(f)$ was not used in [F]. However, for convenience we call this bound $T_2(f)$.

Now, we define another semiconvex envelope called the quadratic rank-one convex envelope $qr(f)$ for $f : M^{N\times n} \to \mathbb{R}$ as follows which is along the same line of other known semiconvex envelopes. We show that $qr(f)$ is precisely the optimal translation bound $T_2(f)$ defined in Definition 1.

Definition 2. *Let $RCQF$ be the set of all rank-one convex quadratic functions defined on $M^{N\times n}$. Then the quadratic rank-one convex envelope $qr(f)$ of $f : M^{N\times n} \to \mathbb{R}$ is defined by*

$$qr(f)(A) = \sup\{q(A),\, q \le f,\, q \in RCQF\}, \tag{4}$$

Recall that a quadratic rank-one convex function q can be written as $q = q_0 + l$, where q_0 is a rank-one convex quadratic form and l an affine function.

Proposition 1. *For a continuous function $f : M^{N\times n} \to \mathbb{R}$ satisfying (2), we have*

$$qr(f)(A) = T_2(f)(A), \qquad A \in M^{N\times n}.$$

Proof. For a fixed $q_0 \in RCQ$, we have [ET,R] that

$$C[f(A) - q_0(A)] + q_0(A) = \sup\{l(A) + q_0(A),\ l + q_0 \le f,\ l \,\text{is affine}\}.$$

Since $l + q_0 \in RCQF$, one has that

$$T_2 f(A) \le \sup\{q(A),\, q \le f,\, q \in RCQF\} = qr(f)(A).$$

For any $q \in RCQF$, we may write $q = l + q_0$, where $q_0 \in RCQ$ and l affine. Therefore

$$q(A) = q_0(A) + l(A) \le \sup_{p\in RCQ} \left[\sup_{r\le f-p\ r\ \text{affine}} [p(A) + r(A)] \right] = T_2 f(A).$$

□

Remark 1. Clearly, if the function f is of sub-quadratic growth, both $T_2(f)$ and $qr(f)$ result in $C(f)$. Notice that every rank-one convex quadratic function $q(A) = q_0(A) + l(A)$ is polyconvex if and only if $\min\{N, n\} = 2$ [Te,Se,B2]. Therefore in general, $P(f)$ and $qr(f)$ are not directly related while $qr(f) = P(f)$ whenever $\min\{N, n\} = 2$. However, it is easy to see in this case that (1) holds and

$$C(f) \leq qr(f) = P(f) \leq Q(f) \leq R(f) \leq f. \tag{5}$$

Remark 2. A direct consequence of Theorem 1 is that when (2) holds, $R(f) = C(f)$ if and only if $Q(f) = C(f)$. Theorem 1 gives a theoretically simple condition for equality of semiconvex envelopes by saying in (1) that if the smallest semiconvex relaxation $qr(f)$ is convex, all other semiconvex envelopes are convex.

We do not require any upper bound for f such as $f(A) \leq C_1|A|^2 + C_2$, however, the lower bound (2) is necessary for Theorem 1 to hold because if f is of sub-quadratic growth in the sense that $f(A) \leq C_1|A|^p + C_2$ with $1 \leq p < 2$, any rank-one convex quadratic function less than f must be an affine function hence $qr(f) = C(f)$.

In the study of material microstructure, the following concepts of semiconvex hulls for a compact set $K \subset M^{N\times n}$ are naturally introduced.

$K \subset M^{N\times n}$ is called stable under lamination (or lamination convex) [MS] if A, $B \in K$ is rank-one connected, that is, $\mathrm{rank}(A - B) = 1$, then for all $\lambda \in (0, 1)$, one has $(1 - \lambda)A + \lambda B \in K$. For a given $K \subset M^{N\times n}$, The *lamination convex hull* $L(K)$ is defined as the smallest lamination convex set that contains K [MS]. We also define the *closed lamination convex hull* $L_c(K)$ as the smallest closed lamination convex set that contains K [Z2].

We also have the *rank-one convex hull* $R(K)$ [Sv2]

$$R(K) = \{X \in M^{N\times n},\ f(X) \leq \sup_{Y\in K} f(Y), \text{ for every rank-one convex } f : M^{N\times n} \to \mathbb{R}\};$$

the quasiconvex hull $Q(K)$ [Sv2]

$$Q(K) = \{X \in M^{N\times n},\ f(X) \leq \sup_{Y\in K} f(Y), \text{ for every quasiconvex } f : M^{N\times n} \to \mathbb{R}\};$$

and the *polyconvex hull* $P(K)$ [Sv2]

$$P(K) = \{X \in M^{N\times n},\ f(X) \leq \sup_{Y\in K} f(Y), \text{ for every polyconvex } f : M^{N\times n} \to \mathbb{R}\}.$$

Clearly, if K is closed,

$$K \subset L_c(K) \subset R(K) \subset Q(K) \subset P(K) \subset C(K).$$

If $L_c(K)$ is convex, obviously, all other 'semiconvex' hulls are identical to $C(K)$.

When K is compact, the quasiconvex hull $Q(K)$ can be defined by a single quasiconvex function [Z1] as $Q(K) = \{A \in M^{N\times n},\ Q\,\mathrm{dist}^p(A, K) = 0\}$ for any $1 \leq p < \infty$. Now we have

Definition 3. *For a compact set $K \subset M^{N\times n}$, the quadratic rank-one convex hull $qr(K)$ is defined by*

$$qr(K) = \{A \in M^{N\times n},\ q(A) \leq \sup_{B\in K} q(B),\ q \in RCQF\}. \tag{6}$$

If $qr(K) = K$, we call K a quadratic rank-one convex set. Clearly, $qr(K)$ satisfies (3).

Remark 3. Before we proceed to prove our main results, we make a remark about the quadratic rank-one convex envelope $qr(f)$ or equivalently, the optimal rank-one convex quadratic translation bound $T_2(f)$. It is known that many explicit quasiconvex envelopes can be obtained by calculating $T_2(f)$ and by showing that $T_2(f) = R(f)$ hence $Q(f)$ can be found [K,F,Z3,Z5]. Let us consider the generalized Kohn-Strang function [KS,AL,AF]

$$f(A) = H(|A|) = \begin{cases} \lambda + |A|^2, & A \in M^{N\times n}\ A \neq 0, \\ 0, & A = 0, \end{cases}$$

where $\lambda > 0$ is a constant. When $n = 2$, $N \geq 2$, f is the original Kohn-Strang function arising from the study of an optimal design problem. The explicit quasiconvex envelope of the original Kohn-Strang function $Q(f)$ was obtained in [KS] by calculating $P(f)$ and by showing that $R(f) = P(f)$ and the explicit formula for $Q(f)$ follows. If we notice that $qr(f) = P(f)$ in this case, we see that $Q(f)$ is obtained by calculating $qr(f)$, or equivalently $T_2(f)$.

For the general Kohn-Strang function when $N \geq 2, n \geq 2$, $Q(f)$ was calculated first in [AL,AFr] by using homogenization theory and the G-closure. Later it was established in [Z5] that $T_2(f) = Q(f) = R(f)$ and $T_2(f)$ can be explicitly calculated.

In some simple cases, such as the squared distance function to a two point set $\{A, B\} \subset M^{N\times n}$,

$$f(X) = \min\{|X - A|^2, |X - B|^2\},$$

it can be shown that $T_2(f) = Q(f) = R(f)$ [K,Z3] and $T_2(f)$ can be calculated by using a single translation $q \in RCQ$. The formula for $Q(f)$ in this case was originally calculated in [K] by using Fourier series method and the translation method is also mentioned.

Although the translation method has been used successfully in certain cases, it is known that quadratic rank-one convex relaxation cannot recover quasiconvex hull or quasiconvex envelope in general. If we denote by $\mathrm{diag}(x_1, x_2)$ a 2×2 diagonal matrix, and let

$$K = \{0, \mathrm{diag}(a, 1/a), \mathrm{diag}(1/a, a)\} \subset M^{2\times 2}$$

with $0 < a < 1$, then $Q(K) = K$ [Sv3] while $P(K)$ contains an extra curve which is part of a hyperbola joining $\mathrm{diag}(a, 1/a)$ and $\mathrm{diag}(1/a, a)$ in the diagonal plane [Sv4].

We prove Theorem 2 first and use it as a tool to establish Theorem 1.

Let $E \subset M^{N\times n}$ be a linear subspace without rank-one matrices, and $E^\perp$ being its orthogonal complement. Let

$$q(A) = |P_{E^\perp}(A)|^2 - \lambda_E |P_E(A)|^2, \tag{7}$$

where $P_{E^\perp}$ and P_E are orthogonal projections to $E^\perp$ and E respectively, and $\lambda_E > 0$ is a positive number such that the quadratic form q is rank-one convex (so is quasiconvex [B]). The existence of such $\lambda_E > 0$ was established in [BFJK]. If E_1 is a plane parallel to E and $X \in E_1$, then

$$q_X(A) = |P_{E^\perp}(A - X)|^2 - \lambda_E |P_E(A - X)|^2 \tag{8}$$

is a quadratic rank-one convex function reaching its strict maximum at X in E_1 with $q_X(X) = 0$ and $q_X(A) < 0$ for $A \in E_1 \setminus \{X\}$. We have

Lemma 1. *Suppose $E \subset M^{N\times n}$ is a linear subspace without rank-one matrices and E_1 is a plane parallel to E. Then any closed subset $K \subset E_1$ is a quadratic rank-one convex set, that is, $qr(K) = K$.*

Proof. If $K \neq E_1$, then for any $X \in E_1 \setminus K$, we consider q_X defined by (8), then $q_X \in RCQF$ and $q_X(X) = 0 > \sup_{A\in K} q_X(A)$. Therefore $X \notin qr(K)$. The proof is then finished. □

If one examines both proofs in [Z2] and [DKK] closely, the following two key requirements might be found for any semiconvex hull $S(K)$ containing $L_c(K)$ to have the equality of hulls property: $S(K) = C(K)$ implies $L_c(K) = C(K)$.

(i) For every supporting plane E_1 of $C(K)$, one must have the dimension reduction property for $S(K)$, that is $S(K\cap E_1) = S(K)\cap E_1$. This requirement is satisfied if in the coset definition of the semiconvex hull $S(K)$, the S-convex functions used are continuous and the class contains all affine functions.

(ii) If $E \subset M^{N\times n}$ is a plane which does not have rank-one connections and $K \subset E$, then the semiconvex hull separates points, that is, $S(K) = K$. Function $q_X(\cdot)$ defined by (8) is among the simplest quasiconvex functions which can serve us for this purpose.

When $\min\{N, n\} > 2$, polyconvex functions no longer separate points on subspaces without rank-one matrices. Therefore (ii) fails. In case $\min\{N, n\} = 2$, all quadratic rank-one convex functions are polyconvex [Te,Se] and separate points, hence in our definition of $qr(K)$, we take into account these minimal requirements.

Proof of Theorem 2 (i). We first show that if E_1 is a supporting plane [R] of $C(K)$ then

$$qr(K) \cap E_1 = qr(K \cap E_1). \tag{9}$$

Let E be the plane in $M^{N\times n}$ containing $C(K)$ with the same dimension as $C(K)$ [R]. Obviously, $qr(K \cap E_1) \subset qr(K) \cap E_1$. Let $X \in qr(K) \cap E_1$. There is an affine function l defined on $M^{N\times n}$ such that $l < 0$ on the open half space in E containing $C(K) \setminus E_1$, $l = 0$ on E_1 and $l > 0$ on the opposite half space to $C(K)$ in E. We also define $E_1(\epsilon) = \{A \in E, \operatorname{dist}(A, E_1) \le \epsilon, l(A) \le 0\}$ which is a set on the same side as $C(K)$ in E, where $\operatorname{dist}(A, E_1)$ is the Euclidean distance from A to E_1. For any fixed $q \in RCQF$ we consider, for every integer $n > 0$ the quadratic function $q(\cdot)+nl(\cdot) \in RCQF$. Since for any $A \in E_1$, $l(A) = 0$, we have, for every fixed point $X \in qr(K) \cap E_1$,

$$q(X) = q(X) + nl(X) \le \sup_{A\in K}[q(A) + nl(A)].$$

Since $q+nl$ is continuous and K compact, the maximum is attained at some $A_n \in K$, that is, $\sup_{A\in K}[q(A) + nl(A)] = q(A_n) + nl(A_n)$, so that $q(X) \le q(A_n) + nl(A_n)$. Since K is compact there is a subsequence $A_{n_k} \to A_0 \in K$ as $k \to \infty$. Notice that

$l(A_n) \leq 0$ for all n. If we let $k \to \infty$ we see that $\delta_k := \mathrm{dist}(A_{n_k}, E_1) \to 0$. Otherwise $q(X)$ cannot be finite. Now we have

$$q(X) \leq q(A_{n_k}) + n_k l(A_{n_k}) \leq \sup\{q(A),\, A \in K \cap E_1(\delta_k)\}. \tag{10}$$

Again the 'sup' in (10) can be reached by, say $B_k \in K \cap E_1(\delta_k)$, and up to a subsequence $B_k \to B_0 \in K \cap E_1$ as $k \to \infty$.

Passing to the limit $k \to \infty$ on both side of the inequality $q(X) \leq q(B_k)$ and noticing that $B_0 \in K \cap E_1$, we have $q(X) \leq q(B_0) \leq \sup_{A \in K \cap E_1} q(A)$, hence $X \in qr(K \cap E_1)$, (9) is proved. Notice also that $C(K) \cap E_1 = C(K \cap E_1)$.

Now we follow the logic argument of [DKK]. Suppose $K \subset M^{N\times n}$ is compact while $L_c(K) \neq C(K)$, but $qr(K) = C(K)$. We may assume that K is a closed laminated convex set. Then among all these K's there is one for which the affine dimension $\dim C(K) \geq 1$ of $C(K)$ is the smallest. For such K we claim that the plane E spanned by $C(K)$ does not have rank-one connections. Otherwise it is easy to see [Z2] that there is a supporting plane E_1 of $C(K)$ such that $E_1 \cap K$ is not convex and is still a closed laminated convex set while $qr(K \cap E_1) = qr(K) \cap E_1 = C(K) \cap E_1$ is convex. This contradicts to the fact that the dimension $\dim C(K)$ is the smallest. Now since $C(K) \subset E$ and E does not have rank-one connection, there is some $X \in C(K) \neq K$. If we define q_X as in Lemma 1, then there is $\delta > 0$, such that $q_X(X) = 0 > -\delta = \sup_{A \in K \subset E_1} q_X(A)$. Hence $X \notin qr(K)$ and $qr(K) \neq C(K)$, a contradiction. □

Proof of Theorem 2 (ii). When $\min\{N, n\} = 2$, every $q \in RCQF$ is a sum of a null Lagrangian and an affine function, so $P(K) \subset qr(K)$. However, in this case every polyconvex function f can be defined as $f(A) = \sup\{l(M(A)),\, l(M(\cdot)) \leq f(\cdot)\}$, where $M(A)$ are the 2×2 minors of A and l is an affine function of M. Thus $l(M(A)) \in RCQF$. Hence $P(K) = qr(K)$. The conclusion then follows from Part (i). □

For a given function $g : M^{N\times n} \to \mathbb{R}$, let $\mathrm{epi}(g)$ be the epi-graph of g ([R]) defined by

$$\mathrm{epi}(g) = \{(A, t) \in M^{N\times n} \times \mathbb{R},\ t \geq g(A)\},$$

we see that if g satisfies (2), then g is convex if and only if for every supporting plane E of the convex hull $C(\mathrm{epi}(g))$, $\mathrm{epi}(g) \cap E$ is convex (see [R]).

Proof of Theorem 1 (i). Since f satisfies (2), and the function $c|A|^2 - C_1$ is convex, we have $R(f)(A) \geq c|A|^2 - C_1$ and $R(f)$ is continuous. We assume that $R(f)$ is not convex and prove that $qr(f)$ is not convex.

We claim that there is a supporting plane E of $\mathrm{epi}(C(f))$ such that $K = \mathrm{epi}(R(f)) \cap E$ is not convex while $C(K) = \mathrm{epi}(Cf) \cap E$. If this is not true, we can easily see that $R(f) = C(f)$ on $P_{M^{N\times n}}(K) = P_{M^{N\times n}}(C(K))$, where $P_{M^{N\times n}} : \mathbb{R} \times M^{N\times n} \to M^{N\times n}$ is the orthogonal projection, so that $R(f) = C(f)$ (see [R]) and we reach a contradiction.

Now we use the supporting plane E to construct a non-negative rank-one convex function F vanishing exactly on K and is bounded below by a similar bound to (2).

Since plane E is the graph of a real-valued affine function $L(\cdot)$ defined on $M^{N\times n}$, we see that $R(f)(\cdot) - L(\cdot) \geq 0$ and $R(f)(A) - L(A) = 0$ if and only if $A \in K$. We also see that K is compact because $R(f)$ still satisfies (2). Let us consider $F(A) = R(f)(A) - L(A)$, for $A \in M^{N\times n}$. Then F is rank-one convex, $F \geq 0$, and $F(A) = 0$ if and only if $A \in K$. Furthermore

$$F(A) \geq \frac{c}{2}|A|^2 - C_2 \tag{11}$$

for some $C_2 > 0$.

We show that $qr(f)(X) > Cf(X)$ for a certain matrix X, if $qr(F)(X) > 0$ holds.

From (11) and the fact that L is affine, we see that

$$qr(F)(\cdot) = qr[R(f)(\cdot) - L(\cdot)] = qr(f)(\cdot) - L(\cdot),$$

because L is affine. Therefore, we only need to prove that $qr(F)$ is not convex. Since $F \geq 0$, it suffices to show that there is some $X \in C(K) \setminus K$ and some $q \in RCQF$, such that $q(A) \leq 0$ in K, $q \leq F$ and $q(X) > 0$.

Since the non-convex set K is the zero set of the non-negative rank-one convex function F, so it is a closed lamination convex set. Theorem 2 implies that $qr(K)$ is not convex. We fix $X \in C(K) \setminus qr(K)$.

From the definition of $qr(K)$, we see that there is some $q_1 \in RCQF$ such that $q_1(X) > \sup_{A\in K} q_1(A)$. Let $\operatorname{dist}(B, K)$ be the Euclidean distance from a point $B \in M^{N\times n}$ to K, we see that when $\epsilon > 0$ is small enough, $X \notin K_\epsilon$, where $\bar{K}_\epsilon = \{B \in M^{N\times n}, \operatorname{dist}(B, K) \leq \epsilon\}$ is the closed ϵ-neighborhood of K. Let K_ϵ be the open ϵ-neighborhood of K given by $K_\epsilon = \{B \in M^{N\times n}, \operatorname{dist}(B, K) < \epsilon\}$. We may also claim that for sufficiently small $\epsilon > 0$, $q_1(X) > \sup_{A\in \bar{K}_\epsilon} q_1(A) := \alpha$.

Let $B(0, R)$ and $\bar{B}(0, R)$ be the open and closed balls in $M^{N\times n}$ centered at 0 with radius $R > 0$ respectively. Since $F \geq 0$ is continuous and satisfies (11), we have

$$\min_{Y\in M^{N\times n}\setminus K_\epsilon} F(Y) = \beta > 0,$$

and also for $R > 0$ sufficiently large, $F(Y) \geq \frac{1}{4}c|Y|^2$ when $Y \in M^{N\times n} \setminus B(0, R)$. Since $q_1 - \alpha$ is at most of quadratic growth, we may assume that $q_1(Y) - \alpha \leq M|Y|^2$ when $Y \in M^{N\times n} \setminus B(0, R)$, where $M > 0$ is a positive constant.

Now we define $q(\cdot) = \eta(q_1(\cdot) - \alpha)$ for some $\eta > 0$ to be determined. We may choose $\eta < c/(4M)$ so that $q(Y) < F(Y)$ whenever $|Y| \geq R$. Let $N = \sup_{Y\in\bar{B}(0,R)}(q_1(Y) - \alpha)$, we may further require $\eta < \beta/N$ so that on $\bar{B}(0, R) \setminus K_\epsilon$, $q(Y) < \beta \leq F(Y)$. We also have $q(Y) \leq 0 \leq F(Y)$ on $\bar{K}_\epsilon$ because $q_1(Y) \leq \alpha$ on that set. Thus we have $q(Y) \leq F(Y)$ for all $Y \in M^{N\times n}$ when $\eta > 0$ is small enough. However, we also observe that $q(X) > 0$, hence $qr(F)(X) > 0$. Consequently, $qr(F)$ and further, $qr(f)$ are not convex. The proof is complete. □

Theorem 1 (ii) follows from (i) and the fact that $P(f) = qr(f)$ whenever $\min\{N, n\} = 2$. However, we remark that in this case the convexity of $P(K)$ and $P(f)$ can be test purely by using quadratic polyconvex functions even if f satisfies $f(A) \geq c|A|^p - C_1$ with $p > 2$.

Remark 4. We conclude that the class of quadratic rank-one convex functions is enough to detect whether the semiconvex hulls and quasiconvex envelopes are convex. This class can be obtained from the Taylor's expansion of smooth quasiconvex (rank-one convex) functions up to the second order. Comparing this with convex analysis, where one needs only affine functions to obtain the convex hull and convex envelope, we see that we just need to expand the function by one extra term to obtain information about the convexity of most of the semiconvex hulls (envelopes).

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Manuscript received February 11, 2002

Kewei Zhang
School of Mathematical Sciences, University of Sussex, Brighton, BN1 9QH, UK
E-mail address: k.zhang@sussex.ac.uk