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# THE SPLIT FEASIBILITY PROBLEM IN BANACH SPACES

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Abstract. In this paper, we consider the split feasibility problem in Banach spaces. Then using the hybrid method in mathematical programming, we prove a strong convergence theorem for finding a solution of the split feasibility problem in Banach spaces.

## 1. Introduction

Let $H_1$ and $H_2$ be two real Hilbert spaces. Let $D$ and $Q$ be nonempty, closed and convex subsets of $H_1$ and $H_2$, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator. Then the *split feasibility problem* [4] is to find $z \in H_1$ such that $z \in D \cap A^{-1}Q$. Recently, Byrne, Censor, Gibali and Reich [3] also considered the following problem: Given set-valued mappings $A_i : H_1 \to 2^{H_1}$, $1 \leq i \leq m$, and $B_j : H_2 \to 2^{H_2}$, $1 \leq j \leq n$, respectively, and bounded linear operators $T_j : H_1 \to H_2$, $1 \leq j \leq n$, the *split common null point problem* [3] is to find a point $z \in H_1$ such that

$$z \in \Big(\bigcap_{i=1}^{m} A_i^{-1}0\Big) \cap \Big(\bigcap_{j=1}^{n} T_j^{-1}(B_j^{-1}0)\Big),$$

where $A_i^{-1}0$ and $B_j^{-1}0$ are null point sets of $A_i$ and $B_j$, respectively. Defining $U = A^*(I - P_Q)A$ in the split feasibility problem, we have that $U : H_1 \to H_1$ is an inverse strongly monotone operator [1], where $A^*$ is the adjoint operator of $A$ and $P_Q$ is the metric projection of $H_2$ onto $Q$. Furthermore, if $D \cap A^{-1}Q$ is nonempty, then $z \in D \cap A^{-1}Q$ is equivalent to

$$z = P_D(I - \lambda A^*(I - P_Q)A)z, \tag{1.1}$$

where $\lambda > 0$ and $P_D$ is the metric projection of $H_1$ onto $D$. Using such results regarding nonlinear operators and fixed points, many authors have studied the split feasibility problem and the split common null point problem; see, for instance, [1, 3, 5, 6, 13, 14].

On the other hand, Solodov and Svaiter [9] introduced the following hybrid method in mathematical programming; see also Nakajo and Takahashi [7].

Let $H$ be a Hilbert space $H$ and let $T$ be a maximal monotone operator of $H$ into $2^H$ such that $T^{-1}0 = \{z \in H : 0 \in Tz\} \neq \emptyset$. Suppose that $x_1 \in H$ and $\{x_n\}$ is

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given by

$$\begin{cases} y_n = J_{r_n} x_n, \\ C_n = \{z \in H : \langle y_n - z, x_n - y_n \rangle \geq 0\}, \\ Q_n = \{z \in H : \langle x_n - z, x_1 - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_1, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $J_{r_n} = (I + r_n T)^{-1}$ for all $r_n > 0$ and $P_{C_n \cap Q_n}$ is the metric projection from $H$ onto $C_n \cap Q_n$.

They showed that the sequence $\{x_n\}$ converges strongly to $P_{T^{-1}0} x_1$; see Ohsawa and Takahashi [8] for the results in Banach spaces.

In this paper, motivated by these problems and results, we consider the split feasibility problem in Banach spaces. Then using the hybrid method in mathematical programming, we prove a strong convergence theorem for finding a solution of the split feasibility problem in Banach spaces.

## 2. Preliminaries

Let $E$ be a real Banach space with norm $\|\cdot\|$ and let $E^*$ be the dual space of $E$. We denote the value of $y^* \in E^*$ at $x \in E$ by $\langle x, y^* \rangle$. When $\{x_n\}$ is a sequence in $E$, we denote the strong convergence of $\{x_n\}$ to $x \in E$ by $x_n \to x$ and the weak convergence by $x_n \rightharpoonup x$. The modulus $\delta$ of convexity of $E$ is defined by

$$\delta(\epsilon) = \inf \left\{ 1 - \frac{\|x + y\|}{2} : \|x\| \leq 1, \|y\| \leq 1, \|x - y\| \geq \epsilon \right\}$$

for every $\epsilon$ with $0 \leq \epsilon \leq 2$. A Banach space $E$ is said to be uniformly convex if $\delta(\epsilon) > 0$ for every $\epsilon > 0$. It is known that a Banach space $E$ is uniformly convex if and only if for any two sequences $\{x_n\}$ and $\{y_n\}$ in $E$ such that

$$\lim_{n\to\infty} \|x_n\| = \lim_{n\to\infty} \|y_n\| = 1 \text{ and } \lim_{n\to\infty} \|x_n + y_n\| = 2,$$

$\lim_{n\to\infty} \|x_n - y_n\| = 0$ holds. A uniformly convex Banach space is strictly convex and reflexive. We also know that a uniformly convex Banach space has the Kadec-Klee property, i.e., $x_n \rightharpoonup u$ and $\|x_n\| \to \|u\|$ imply $x_n \to u$.

The duality mapping $J$ from $E$ into $2^{E^*}$ is defined by

$$Jx = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}$$

for every $x \in E$. Let $U = \{x \in E : \|x\| = 1\}$. The norm of $E$ is said to be Gâteaux differentiable if for each $x, y \in U$, the limit

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.1}$$

exists. In the case, $E$ is called smooth. We know that $E$ is smooth if and only if $J$ is a single-valued mapping of $E$ into $E^*$. We also know that $E$ is reflexive if and only if $J$ is surjective, and $E$ is strictly convex if and only if $J$ is one-to-one. Therefore, if $E$ is a smooth, strictly convex and reflexive Banach space, then $J$ is a single-valued bijection and in this case, the inverse mapping $J^{-1}$ coincides with the duality mapping $J_*$ on $E^*$. For more details, see [10] and [11]. We know the following result.

**Lemma 2.1** ([10])**.** *Let $E$ be a smooth Banach space and let $J$ be the duality mapping on $E$. Then, $\langle x-y, Jx-Jy\rangle \geq 0$ for all $x, y \in E$. Furthermore, if $E$ is strictly convex and $\langle x-y, Jx-Jy\rangle = 0$, then $x = y$.*

Let $C$ be a nonempty, closed and convex subset of a strictly convex and reflexive Banach space $E$. Then we know that for any $x \in E$, there exists a unique element $z \in C$ such that $\|x-z\| \leq \|x-y\|$ for all $y \in C$. Putting $z = P_C x$, we call $P_C$ the metric projection of $E$ onto $C$.

**Lemma 2.2** ([10])**.** *Let $E$ be a smooth, strictly convex and reflexive Banach space. Let $C$ be a nonempty, closed and convex subset of $E$ and let $x_1 \in E$ and $z \in C$. Then, the following conditions are equivalent*

(1) $z = P_C x_1$;

(2) $\langle z-y, J(x_1-z)\rangle \geq 0, \quad \forall y \in C$.

Let $E$ be a smooth, strictly convex and reflexive Banach space and let $C$ be a nonempty, closed and convex subset of $E$. If $P_C$ is the metric projection of $E$ onto $C$, then we have from [2] and [12] that

$$\langle P_C x - P_C y, J(x-P_C x) - J(y - P_C y)\rangle \geq 0, \quad \forall x, y \in C.$$

We also have that if $\{x_n\}$ is a sequence in $E$ such that $x_n \rightharpoonup p$ and $x_n - P_C x_n \to 0$, then $p = P_C p$, i.e., $p \in C$. In fact, assume that $x_n \rightharpoonup p$ and $x_n - P_C x_n \to 0$. It is clear that $P_C x_n \rightharpoonup p$ and $\|J(x_n - P_C x_n)\| = \|x_n - P_C x_n\| \to 0$. Since $P_C$ is the metric projection of $E$ onto $C$, then we have that

$$\langle P_C x_n - P_C p, J(x_n - P_C x_n) - J(p - P_C p)\rangle \geq 0.$$

Therefore, $-\|p-P_C p\|^2 = \langle p - P_C p, -J(p-P_C p)\rangle \geq 0$ and hence $p = P_C p$.

## 3. Main result

In this section, using the hybrid method in mathematical programming, we prove a strong convergence theorem for finding a solution of the split feasibility problem in Banach spaces.

**Lemma 3.1.** *Let $E$ and $F$ be strictly convex, reflexive and smooth Banach spaces and let $J_E$ and $J_F$ be the duality mappings on $E$ and $F$, respectively. Let $C$ and $D$ be nonempty, closed and convex subsets of $E$ and $F$ and let $P_C$ and $P_D$ be the metric projections of $E$ onto $C$ and $F$ onto $D$, respectively. Let $A : E \to F$ be a bounded linear operator such that $A \neq 0$ and let $A^*$ be the adjoint operator of $A$. Suppose that $C \cap A^{-1}D \neq \emptyset$. Let $r > 0$ and $z \in E$. Then the following are equivalent:*

(i) $z = P_C\big(z - rJ_E^{-1}A^*J_F(Az - P_D Az)\big)$;

(ii) $z \in C \cap A^{-1}D$.

*Proof.* Since $C \cap A^{-1}D \neq \emptyset$, there exists $z_0 \in C$ such that $Az_0 \in D$.

(i) $\Rightarrow$ (ii). Assuming $z = P_C\big(z - rJ_E^{-1}A^*J_F(Az - P_D Az)\big)$, we have from properties of $P_C$ that

$$\langle J_E\big(z - rJ_E^{-1}A^*J_F(I_F - P_D)Az - z\big), z - y\rangle \geq 0, \quad \forall y \in C.$$

This implies that

$$\langle J_E\big( - rJ_E^{-1}A^*J_F(Az - P_D Az)\big), z - y\rangle \geq 0.$$

Thus we have that

$$\langle -rA^*J_F(Az - P_DAz), z - y\rangle \geq 0$$

and hence

$$\langle A^*J_F(Az - P_DAz), z - y\rangle \leq 0.$$

Since $A^*$ is the adjoint operator of $A$, we have that

$$\langle J_F(Az - P_DAz), Az - Ay\rangle \leq 0.$$

From $z_0 \in C$ we have that

$$\langle J_F(Az - P_DAz), Az - Az_0\rangle \leq 0. \tag{3.1}$$

On the other hand, since $P_D$ is the metric projection of $F$ onto $D$, we have that

$$\langle J_F(Az - P_DAz), P_DAz - v\rangle \geq 0, \quad \forall v \in D.$$

From $Az_0 \in D$ we have that

$$\langle J_F(Az - P_DAz), P_DAz - Az_0\rangle \geq 0. \tag{3.2}$$

Using (3.1) and (3.2), we have that

$$\langle J_F(Az - P_DAz), Az - P_DAz\rangle \leq 0$$

and hence

$$\|Az - P_DAz\|^2 \leq 0.$$

This implies that $Az = P_DAz$. Using this and $z = P_C\big(z - rJ_E^{-1}A^*J_F(Az - P_DAz)\big)$, we have that $z = P_Cz$. Therefore $z \in C \cap A^{-1}D$.

(ii) $\Rightarrow$ (i). Since $z \in C \cap A^{-1}D$, we have that $Az \in D$ and $z \in C$. It follows that $Az = P_DAz$ and $z = P_Cz$. Thus we have that

$$P_C\big(z - rJ_E^{-1}A^*J_F(Az - P_DAz)\big) = P_Cz = z.$$

The proof is complete. □

**Theorem 3.2.** *Let $E$ and $F$ be uniformly convex and smooth Banach spaces and let $J_E$ and $J_F$ be the duality mappings on $E$ and $F$, respectively. Let $C$ and $D$ be nonempty, closed and convex subsets of $E$ and $F$, respectively. Let $P_D$ be the metric projection of $F$ onto $D$. Let $A : E \to F$ be a bounded linear operator such that $A \neq 0$ and let $A^*$ be the adjoint operator of $A$. Suppose that $C \cap A^{-1}D \neq \emptyset$. Let $x_1 \in E$ and let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} z_n = x_n - rJ_E^{-1}A^*J_F(Ax_n - P_DAx_n), \\ C_n = \{z \in C : \langle z_n - z, J_E(x_n - z_n)\rangle \geq 0\}, \\ Q_n = \{z \in C : \langle x_n - z, J_E(x_1 - x_n)\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n}x_1, \quad \forall n \in \mathbb{N}, \end{cases}$$

*where $0 < r\|A\|^2 < 1$. Then $\{x_n\}$ converges strongly to a point $z_0 \in C \cap A^{-1}D$, where $z_0 = P_{C \cap A^{-1}D}x_1$.*

*Proof.* It is obvious that $C_n \cap Q_n$ is closed and convex for all $n \in \mathbb{N}$. To show that $C \cap A^{-1}D \subset C_n$ for all $n \in \mathbb{N}$, let us show that $\langle z_n - z, J_E(x_n - z_n)\rangle \geq 0$ for all $z \in A^{-1}D$ and $n \in \mathbb{N}$. In fact, we have that for all $z \in A^{-1}D$ and $n \in \mathbb{N}$,

$$\begin{aligned}
\langle z_n - z, J_E(x_n - z_n)\rangle &= \langle z_n - x_n + x_n - z, J_E(x_n - z_n)\rangle \\
&= \langle -rJ_E^{-1}A^*J_F(Ax_n - P_DAx_n) \\
&\quad + x_n - z, J_E(rJ_E^{-1}A^*J_F(Ax_n - P_DAx_n))\rangle \\
&= \langle -rJ_E^{-1}A^*J_F(Ax_n - P_DAx_n) + x_n - z, rA^*J_F(Ax_n - P_DAx_n)\rangle \\
&= -r^2\|A^*J_F(Ax_n - P_DAx_n)\|^2 + \langle x_n - z, rA^*J_F(Ax_n - P_DAx_n)\rangle \\
&= -r^2\|A^*J_F(Ax_n - P_DAx_n)\|^2 + r\langle Ax_n - Az, J_F(Ax_n - P_DAx_n)\rangle \\
&= -r^2\|A^*J_F(Ax_n - P_DAx_n)\|^2 \\
&\quad + r\langle Ax_n - P_DAx_n + P_DAx_n - Az, J_F(Ax_n - P_DAx_n)\rangle \\
&= -r^2\|A^*J_F(Ax_n - P_DAx_n)\|^2 \\
&\quad + r\|Ax_n - P_DAx_n\|^2 + r\langle P_DAx_n - Az, J_F(Ax_n - P_DAx_n)\rangle \\
&\geq -r^2\|A\|^2\|Ax_n - P_DAx_n\|^2 + r\|Ax_n - P_DAx_n\|^2 \\
&= r(1 - r\|A\|^2)\|Ax_n - P_DAx_n\|^2 \\
&\geq 0.
\end{aligned} \tag{3.3}$$

Then we have that $C \cap A^{-1}D \subset C_n$ for all $n \in \mathbb{N}$. We show that $C \cap A^{-1}D \subset Q_n$ for all $n \in \mathbb{N}$. Since $Q_1 = \{z \in C : \langle x_1 - z, J_E(x_1 - x_1)\rangle \geq 0$, it is obvious that $C \cap A^{-1}D \subset Q_1$. Suppose that $C \cap A^{-1}D \subset Q_k$ for some $k \in \mathbb{N}$. Then $C \cap A^{-1}D \subset C_k \cap Q_k$. From $x_{k+1} = P_{C_k \cap Q_k}x_1$, we have that

$$\langle x_{k+1} - z, J_E(x_1 - x_{k+1})\rangle \geq 0, \quad \forall z \in C_k \cap Q_k$$

and hence

$$\langle x_{k+1} - z, J_E(x_1 - x_{k+1})\rangle \geq 0, \quad \forall z \in C \cap A^{-1}D.$$

Then, $C \cap A^{-1}D \subset Q_{k+1}$. We have by mathematical induction that $C \cap A^{-1}D \subset Q_n$ for all $n \in \mathbb{N}$. Thus, we have that $C \cap A^{-1}D \subset C_n \cap Q_n$ for all $n \in \mathbb{N}$. This implies that $\{x_n\}$ is well defined.

Since $C \cap A^{-1}D$ is nonempty, closed and convex, there exists $z_1 \in C \cap A^{-1}D$ such that $z_1 = P_{C \cap A^{-1}D}x_1$. From $x_{n+1} = P_{C_n \cap Q_n}x_1$, we have that

$$\|x_1 - x_{n+1}\| \leq \|x_1 - y\|$$

for all $y \in C_n \cap Q_n$. Since $z_1 \in C \cap A^{-1}D \subset C_n \cap Q_n$, we have that

$$\|x_1 - x_{n+1}\| \leq \|x_1 - z_1\|. \tag{3.4}$$

This means that $\{x_n\}$ is bounded.

Next we show that $\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0$. From the definition of $Q_n$, we have that $x_n = P_{Q_n}x_1$. From $x_{n+1} = P_{C_n \cap Q_n}x_1$ we have that $x_{n+1} \in Q_n$. Thus

$$\|x_n - x_1\| \leq \|x_{n+1} - x_1\|$$

for all $n \in \mathbb{N}$. This implies that $\{\|x_1 - x_n\|\}$ is bounded and nondecreasing. Then there exists the limit of $\{\|x_1 - x_n\|\}$. Put $\lim_{n\to\infty} \|x_n - x_1\| = c$. If $c = 0$, then

$\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0$. Assume that $c > 0$. Since $x_n = P_{Q_n}x_1$, $x_{n+1} \in Q_n$ and $\frac{x_n+x_{n+1}}{2} \in Q_n$, we have that

$$\|x_1 - x_n\| \le \left\|x_1 - \frac{x_n + x_{n+1}}{2}\right\| \le \frac{1}{2}(\|x_1 - x_n\| + \|x_1 - x_{n+1}\|)$$

and hence

$$\lim_{n\to\infty} \left\|x_1 - \frac{x_n + x_{n+1}}{2}\right\| = c.$$

Since $E$ is uniformly convex, we get that $\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0$.

From the definition of $C_n$, we also have that $z_n = P_{C_n}x_n$. From $x_{n+1} \in C_n$ we have that

$$\|x_n - z_n\| \le \|x_n - x_{n+1}\|.$$

From $\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0$ we have that $\lim_{n\to\infty} \|x_n - z_n\| = 0$. On the other hand, we know that

$$\|x_n - z_n\| = \|J_E(x_n - z_n)\| = \|rA^*J_F(Ax_n - P_DAx_n)\|.$$

Using $0 < r\|A\|^2 < 1$ and $\|x_n - z_n\| \to 0$, we have that $A^*J_F(Ax_n - P_DAx_n) \to 0$. Then we get from (3.3) that

$$\lim_{n\to\infty} \|Ax_n - P_DAx_n\| = 0.$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ converging weakly to $w$. Note that $w \in C$. Since $A$ is bounded and linear, we also have that $\{Ax_{n_i}\}$ converges weakly to $Aw$. Using this and $\lim_{n\to\infty} \|Ax_n - P_DAx_n\| = 0$, we have from the property of the metric projection $P_D$ that $Aw = P_DAw$ and hence $Aw \in D$.

From $z_1 = P_{C\cap A^{-1}D}x_1$, $w \in C \cap A^{-1}D$ and (3.4), we have that

$$\|x_1 - z_1\| \le \|x_1 - w\| \le \liminf_{i\to\infty} \|x_1 - x_{n_i}\| \le \limsup_{i\to\infty} \|x_1 - x_{n_i}\| \le \|x_1 - z_1\|.$$

Then we get that

$$\lim_{i\to\infty} \|x_1 - x_{n_i}\| = \|x_1 - w\| = \|x_1 - z_1\|.$$

From the Kadec-Klee property of $E$, we have that $x_1 - x_{n_i} \to x_1 - w$ and hence

$$x_{n_i} \to w = z_1.$$

Therefore, we have $x_n \to w = z_1$. This completes the proof. □

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