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FRÉCHET AND PROXIMAL REGULARITIES OF PERTURBED DISTANCE FUNCTIONS AT POINTS IN THE TARGET SET IN BANACH SPACES

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Abstract. This paper is devoted to study the Fréchet and proximal Regularity at points in the target set of perturbed distance functions $d_S^J(\cdot)$ determined by a closed subset S and a Lipschitz function $J(\cdot)$. Also, we provide some important results on the Clarke subdifferential of $d_S^J(\cdot)$ at those points in arbitrary Banach spaces.

1. Introduction

Let X be a real Banach space endowed with a norm $\|\cdot\|$, S a nonempty closed subset of X, and $J : S \to \mathbb{R}$ a lower semicontinuous function. The present paper is concerned with the study of the perturbed distance function $d_S^J : X \to \mathbb{R}$ defined by

$$d_S^J(x) := \inf_{s\in S}\{\|x-s\| + J(s)\} \quad \text{for each } x \in X. \tag{1.1}$$

The perturbed distance function (1.1) can be associated with a perturbed optimization problem, which is denoted by $\min_J(x, S)$ and defined as follows:

$$\min_{w\in S}\{\|x-w\| + J(w)\}, \tag{1.2}$$

that is, the d_S^J coincides with the value function of the perturbed optimization problem $\min_J(\cdot, S)$. This kind of perturbed optimization problems, which has a lot of applications in optimal control problems governed by partial differential equations (cf. [2, 4, 3]), has been studied extensively in [2, 3, 4, 5, 17, 28, 33, 35], where the interests are mainly focused on generic and porousity properties for the well-posedness of problems $\min_J(x, S)$ on X. Clearly, if $J \equiv 0$, then the perturbed distance function $d_S^J(x)$ and the corresponding perturbed optimization problem $\min_J(x, S)$ are reduced to the classical distance function d_S defined by

$$d_S(x) := \inf_{s\in S}\{\|x-s\|\} \quad \text{for each } x \in X$$

and the well-known best approximation problem, respectively. These two important problems have been studied extensively and played key roles in variational analysis; see for example, [9, 10, 12, 29, 30] and [11, 14, 15, 18, 19, 23, 38, 40].

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The regularity of the distance function d_S has been presented and explored in [6, 7]. Under some natural assumptions, it was proved in [6, 7] that the normal regularity (Fréchet and proximal) of a closed set S at a point $\bar{x}$ in S is equivalent to the subdifferential (Fréchet and proximal) regularity of its distance function d_S at $\bar{x}$. Recently, several various subdifferentials, such as the Fréchet, the proximal and limiting (or Mordukhovich) subdifferentials of the perturbed distance function are characterized in [27, 39] and then some known results on subdifferentials of the classical distance function in [6, 7] are extended. Our main aim in this paper is to extend those results to the perturbed distance function $d_S^J(x)$ at points on the target set $S_0 := \{x \in S : d_S^J(x) = J(x)\}$.

The paper is organized as follows. In Section 2 we recall and summarize the needed concepts and existing results. In Sections 3 and 4, we treat the Fréchet and proximal regularity (respectively) of $d_S^J(x)$. The paper is ended with illustrative examples of our abstract results.

2. Preliminaries

Let X be a real Banach space with a norm $\|\cdot\|$ and X^* the dual space of X. We use $\langle\cdot,\cdot\rangle$ to denote the pairing between X^* and X. The closed ball in X (resp. X^*) with radius r and center x is denoted by $\mathbb{B}(x,r)$(resp. $\mathbb{B}^*(x,r)$). In particular, we write $\mathbb{B} = \mathbb{B}(0,1)$ (resp. $\mathbb{B}^* = \mathbb{B}^*(0,1)$). Let S be a nonempty closed subset of X. We use δ_S to denote the indicator function of S, i.e.,

$$\delta_S(x) := \begin{cases} 0 & x \in S \\ +\infty & \text{otherwise.} \end{cases}$$

Consider a function h defined on S. It would be convenient to define the sum $h + \delta_S : X \to \overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$ by

$$(h + \delta_S)(x) := \begin{cases} h(x) & x \in S \\ +\infty & \text{otherwise.} \end{cases}$$

Let $f : X \to \overline{\mathbb{R}}$ be a proper *lower semicontinuous* (l.s.c.) function. As usual, the effective domain of f is denoted by

$$\mathrm{D}(f) := \{x \in X \mid f(x) < +\infty\}.$$

Moreover, we adapt the standard symbols "$x \to^f \bar{x}$" and "$x \to^S \bar{x}$" to represent for $(x, f(x)) \to (\bar{x}, f(\bar{x}))$ and $x \to \bar{x}$ with $x \in S$, respectively. The sequential Painlevé-Kuratowski upper/outer limit $\limsup_{x\to\bar{x}} F(x)$ at $\bar{x}$ for a set-valued mapping $F : X \rightrightarrows X^*$ is defined by

$$\limsup_{x\to\bar{x}} F(x) := \left\{ x^* \in X^* \,\middle|\, \begin{array}{l} \text{there are } \{x_k\} \subset X \text{ and } \{x_k^*\} \subset X^* \text{ with} \\ x_k^* \in F(x_k) \forall k, \text{ such that } x_k \to \bar{x} \text{ and } x_k^* \to^{w^*} x^* \end{array} \right\}.$$

The various notions of subdifferentials and normal cones in the following definitions, and the relative facts provided in the following remarks are well-known (see for example [6, 7, 23, 24]).

Definition 2.1. Let $\bar{x} \in \mathrm{D}(f)$, and $\varepsilon \geq 0$.

(a) The Fréchet ε-subdifferential $\partial^F_\varepsilon f(\bar{x})$ of f at $\bar{x}$ is defined by

$$\partial^F_\varepsilon f(\bar{x}) := \left\{ x^* \in X^* \middle| \liminf_{x \to \bar{x}} \frac{f(x) - f(\bar{x}) - \langle x^*, x - \bar{x} \rangle}{\|x - \bar{x}\|} \geq -\varepsilon \right\}.$$

(b) The Fréchet subdifferential $\partial^F f(\bar{x})$ of f at $\bar{x}$ is defined as the 0-subdifferential of f at the corresponding point, that is

$$\partial^F f(\bar{x}) := \left\{ x^* \in X^* \middle| \liminf_{x \to \bar{x}} \frac{f(x) - f(\bar{x}) - \langle x^*, x - \bar{x} \rangle}{\|x - \bar{x}\|} \geq 0 \right\}.$$

(c) The limiting Fréchet (also called Mordukhovich) subdifferential $\partial^M f(\bar{x})$ of f at $\bar{x}$ is defined by

$$\partial^M f(\bar{x}) := \limsup_{x \to \bar{x}, \varepsilon \downarrow o} \partial^F_\varepsilon f(\bar{x}).$$

(d) The *proximal subdifferential* $\partial^P f(\bar{x})$ of f at $\bar{x}$ is defined by

$$\partial^P f(\bar{x}) := \left\{ x^* \in X^* \middle| \liminf_{x \to \bar{x}} \frac{f(x) - f(\bar{x}) - \langle x^*, x - \bar{x} \rangle}{\|x - \bar{x}\|^2} > -\infty \right\}.$$

(e) The Clarke subdifferential of f at $\bar{x}$ is defined by

$$\partial^C f(\bar{x}) := \{x^* \in X^* : \langle x^*, v \rangle \leq f^\uparrow(\bar{x}; v), \text{ for all } v \in X\}, \tag{2.1}$$

where $f^\uparrow(\bar{x}; \cdot)$ is the Rockafellar directional derivative defined by

$$f^\uparrow(\bar{x}; v) := \limsup_{\substack{x \to_f \bar{x} \\ t \downarrow 0}} \inf_{v' \to v} t^{-1} \big[f(x + tv') - f(x) \big] \quad \text{for each } v \in X.$$

Remark 2.2. Let $\bar{x} \in \mathrm{D}(f)$. Then one has the following assertions:

(a) The following inclusions are clear by definitions:

$$\partial^P f(\bar{x}) \subseteq \partial^F f(\bar{x}) \subseteq \partial^M f(\bar{x}) \subseteq \mathrm{co}(\partial^M f(\bar{x})) \subseteq \partial^C f(\bar{x}). \tag{2.2}$$

(b) If X is Fréchet smooth Banach spaces, then we have

$$\partial^M f(\bar{x}) = \limsup_{x \to \bar{x}} \partial^F f(\bar{x}),$$

and if f is additionally assumed to be locally Lipschitz at $\bar{x}$, then we also have

$$\partial^C f(\bar{x}) = \mathrm{cl}^{w^*} \mathrm{co}(\partial^M f(\bar{x})); \tag{2.3}$$

see, for instance, [29, 37].

Remark 2.3. Let f, $g : X \to \overline{\mathbb{R}}$ be two proper l.s.c. functions and let $\bar{x} \in \mathrm{D}(f) \cap \mathrm{D}(g)$. Then we have that

$$\partial^\Delta f(\bar{x}) + \partial^\Delta g(\bar{x}) \subseteq \partial^\Delta (f + g)(\bar{x}) \quad \text{for any } \Delta \in \{F, P\}. \tag{2.4}$$

This inclusion is not true in general for Mordukhovich and Clarke subdifferentials, see for example [14, 29].

Definition 2.4. Let $\bar{x} \in S$ and $\varepsilon \geq 0$.

(a) The set of Fréchet ε-normals to S at $\bar{x}$ is defined by

$$N_{\varepsilon}^{F}(\bar{x};S) := \left\{ x^* \in X^* : \limsup_{x \to^S \bar{x}} \frac{\langle x^*, x - \bar{x} \rangle}{\|x - \bar{x}\|} \leq \varepsilon \right\}.$$

(b) The set of Fréchet normals to S at $\bar{x}$ is defined as the Fréchet 0-normals to S at the same point, that is,

$$N^{F}(\bar{x};S) := \left\{ x^* \in X^* : \limsup_{x \to^S \bar{x}} \frac{\langle x^*, x - \bar{x} \rangle}{\|x - \bar{x}\|} \leq 0 \right\}.$$

(c) The limiting Fréchet (also called Mordukhovich) normal cone to S at $\bar{x}$ is defined by

$$N^{M}(\bar{x};S) := \limsup_{x \to^S \bar{x}, \varepsilon \downarrow 0} N_{\varepsilon}^{F}(\bar{x};S).$$

(d) The proximal normal cone to S at $\bar{x}$ is defined by

$$N^{P}(\bar{x};S) := \left\{ x^* \in X^* : \limsup_{x \to^S \bar{x}} \frac{\langle x^*, x - \bar{x} \rangle}{\|x - \bar{x}\|^2} < +\infty \right\}.$$

(e) The Clarke normal cone to S at $\bar{x}$ is denoted by $N^C(S;\bar{x})$ and defined as the negative polar of the Clarke tangent cone $T^C(S;\bar{x})$, that is,

$$N^{C}(S;\bar{x}) := \{x^* \in X^* : \langle x^*, v \rangle \leq 0, \forall v \in T^C(S;\bar{x})\},$$

where $T^C(S;\bar{x})$ is the Clarke tangent cone defined by

$$T^{C}(S;\bar{x}) := \{v \in X : \forall t_n \downarrow 0,\, \forall x_n \to^S \bar{x}, \exists v_n \to v \text{ such that } \{x_n + t_n v_n\} \subseteq S\}.$$

(f) For any $\Delta \in \{F, M, P, C\}$, we define $N^{\Delta}(\bar{x};S) = \emptyset$ if $\bar{x} \notin S$.

Remark 2.5. Let $\bar{x} \in S$.

(a) By definition (see also [6, 13, 14, 29]), we have that

$$N^{\Delta}(\bar{x};S) = \partial^{\Delta} \delta_S(\bar{x}) \quad \text{for any } \Delta \in \{F, M, P, C\}.$$

Then by Remark 2.2 we have that

$$N^{P}(\bar{x};S) \subseteq N^{F}(\bar{x};S) \subseteq N^{M}(\bar{x};S) \subseteq \operatorname{co} N^{M}(\bar{x};S) \subseteq N^{C}(\bar{x};S), \tag{2.5}$$

and in the case when X is a Fréchet smooth Banach space (see, for instance, [29, 37]), we have

$$N^{M}(\bar{x};S) = \limsup_{x \to \bar{x}} N^{F}(x;S).$$

(b) In the case when X is a Fréchet smooth Banach space (see, for instance, [29, 37]), the following equality holds

$$N^{C}(\bar{x};S) = cl_{w^*}\operatorname{co}(N^{M}(\bar{x};S)). \tag{2.6}$$

We end this section with the following concepts of regularity for sets and functions introduced and studied in [6, 7].

Definition 2.6. Let $\bar{x} \in D(f)$. We say that f is

(a) *Fréchet subdifferentially regular* at $\bar{x}$ if $\partial^F f(\bar{x}) = \partial^C f(\bar{x})$;
(b) *proximal subdifferentially regular* at $\bar{x}$ if $\partial^P f(\bar{x}) = \partial^C f(\bar{x})$;
(c) *Mordukhovich subdifferentially regular* at $\bar{x}$ if $\partial^F f(\bar{x}) = \partial^M f(\bar{x})$.

Definition 2.7. Let $\bar{x} \in S$. We say that S is

(a) *Fréchet normally regular* at $\bar{x}$ if $N^F(S;\bar{x}) = N^C(S;\bar{x})$;
(b) *proximal normally regular* at $\bar{x}$ if $N^P(S;\bar{x}) = N^C(S;\bar{x}))$;
(c) *Mordukhovich normally regular* at $\bar{x}$ if $N^F(S;\bar{x}) = N^M(S;\bar{x})$.

Remark 2.8. By definition, together with (2.2) and (2.5), one has the following implications for a l.s.c function f at each $x \in D(f)$:

proximal subdifferential regularity $\Rightarrow$ Fréchet subdifferential regularity
$\Rightarrow$ Mordukhovich subdifferential regularity,

and for a subset S at each $x \in S$:

proximal normal regularity $\Rightarrow$ Fréchet normal regularity
$\Rightarrow$ Mordukhovich normal regularity.

Moreover, using (2.3) and (2.6), one shows (see [6, 7]), that if X is reflexive and f is locally Lipschitz at x, then the following equivalences hold at x:

Fréchet subdifferential regularity $\Leftrightarrow$ Mordukhovich subdifferential regularity

and

Fréchet normal regularity $\Leftrightarrow$ Mordukhovich normal regularity.

3. Fréchet regularity concepts of d_S^J on S_0

As assumed in the previous sections, let S be a nonempty closed subset of X and J be a lsc real valued function on S. Recall that the perturbed distance function is defined by:

$$\mathrm{d}_S^J(x) = \inf_{s\in S}\{\|x-s\| + J(s)\} \quad \text{for each } x \in X.$$

By [35], we know that d_S^J is nonexpansive:

$$|\mathrm{d}_S^J(y) - \mathrm{d}_S^J(x)| \le \|y - x\| \quad \text{for each } x, y \in X. \tag{3.1}$$

The set of all solutions for problem $\min_J(x,S)$ is denoted by $P_S^J(x)$, i.e.,

$$P_S^J(x) := \{w \in S| \quad \|x-w\| + J(w) = d_S^J(x)\}.$$

As in [27], we use S_0 to denote the subset of S defined by

$$S_0 := \{x \in S | d_S^J(x) = J(x)\}.$$

Throughout the whole paper, we always assume that $S_0 \neq \emptyset$. Note that if J is noexpansive on S then we have by definition that $S_0 = S$. Recall from [28] that a sequence $\{z_n\} \subseteq S$ is a *minimizing sequence* of the problem $\min_J(x,S)$ if

$$\lim_{n\to+\infty}(\|x - z_n\| + J(z_n)) = \inf_{z\in S}(\|x-z\| + J(z)), \tag{3.2}$$

and that the problem $\min_J(x,S)$ is *well-posed* (in the sense of Tykhonov) if $\min_J(x,S)$ has a unique solution and every minimizing sequence of the problem $\min_J(x,S)$ converges to this solution. The following proposition, which is known in [39, Lemmas 3.4 and 3.1], provides a sufficient condition ensuring the well-posedness of the problem $\min_J(x,S)$.

Proposition 3.1. *Let $\bar{x} \in S$. Suppose that the center Lipschitz constant on S w.r.t. $\bar{x}$*

$$C := \sup_{x \in S} \frac{|J(x) - J(\bar{x})|}{x - \bar{x}} < 1. \tag{3.3}$$

Then $\bar{x} \in S_0$ and the problem $\min_J(\bar{x}, S)$ is well-posed.

We give the notions of Lipschitz conditions in the following definition.

Definition 3.2. Let $\bar{x} \in S$. The function J is said to satisfy

(a) the center Lipschitz condition at $\bar{x}$ if the center Lipschitz constant at $\bar{x}$

$$C_{\bar{x}} := \inf_{\rho>0} \sup_{y \in \mathbb{B}(\bar{x},\rho)\cap S} \frac{|J(y) - J(\bar{x})|}{\|y - \bar{x}\|} < \infty;$$

(b) the Lipschitz condition at $\bar{x}$ if the Lipschitz constant at $\bar{x}$

$$L_{\bar{x}} := \inf_{\rho>0} \sup_{x,y \in \mathbb{B}(\bar{x},\rho)\cap S} \frac{|J(y) - J(x)|}{\|y - x\|} < \infty.$$

We now recall from [27] the following two results needed for our study.

Proposition 3.3. *Let $\bar{x} \in S_0$. Then the following assertions hold.*

(i) *We have*

$$\partial^F d_S^J(\bar{x}) \subset \partial^F(J + \delta_S)(\bar{x}) \cap \mathbb{B}^*. \tag{3.4}$$

(ii) *If $\min_J(\bar{x}, S)$ is well-posed and the center Lipschitz constant $C_{\bar{x}} < 1$, then we have*

$$\partial^F(J + \delta_S)(\bar{x}) \cap \mathbb{B}^* = \partial^F d_S^J(\bar{x}). \tag{3.5}$$

Proposition 3.4. *Let $\bar{x} \in S_0$ be such that $\min_J(\bar{x}, S)$ is well-posed. Then the following assertions hold.*

(i) *We have*

$$\partial^M d_S^J(\bar{x}) \subset \partial^M(J + \delta_S)(\bar{x}) \cap \mathbb{B}^*. \tag{3.6}$$

(ii) *If the Lipschitz constant $L_{\bar{x}} = 0$, then we have*

$$\bigcup_{\lambda \geq 0} \lambda[\partial^M(J + \delta_S)(\bar{x}) \cap \mathbb{B}^*] = \bigcup_{\lambda \geq 0} \lambda \partial^M d_S^J(\bar{x}). \tag{3.7}$$

Our first theorem provides some sufficient conditions ensuring the implication that the regularity of the target set S implies the regularity of the perturbed distance function d_S^J.

Theorem 3.5. *Let $\bar{x} \in S_0$ be such that $\min_J(\bar{x}, S)$ is well-posed and the center Lipschitz constant $C_{\bar{x}} < 1$. Suppose that J is Mordukhovich subdifferentially regular at $\bar{x}$. Then we have the following assertions:*

(i) *If S is Mordukhovich normally regular at $\bar{x}$, then d_S^J is Mordukhovich subdifferentially regular at $\bar{x}$.*
(ii) *If the space X is reflexive and S is Fréchet normally regular at $\bar{x}$, then d_S^J is Fréchet subdifferentially regular at $\bar{x}$.*

Proof. By Remark 2.8, one sees that assertion (ii) is a direct consequence of assertion (i). Hence we only need to prove assertion (i). Since J is locally Lipschitz at $\bar{x}$ and δ_S is l.s.c. at $\bar{x}$, it follows from the sum rule for the Mordukhovich subdifferential (see for example [14]) that

$$\partial^M (J+\delta_S)(\bar{x}) \subset \partial^M J(\bar{x}) + \partial^M \delta_S(\bar{x}). \tag{3.8}$$

Now assume that S is Mordukhovich normally regular at $\bar{x}$, that is, $N^M(S;\bar{x}) = N^F(S;\bar{x})$. Moreover, we have $\partial^M J(\bar{x}) = \partial^F J(\bar{x})$ by the assumption that J is Mordukhovich subdifferentially regular at $\bar{x}$. The above two equalities together with (3.8) and (2.4) imply that

$$\partial^M (J+\delta_S)(\bar{x}) \subset \partial^M J(\bar{x}) + \partial^M \delta_S(\bar{x}) = \partial^F J(\bar{x}) + \partial^F \delta_S(\bar{x}) \subset \partial^F (J+\delta_S)(\bar{x}).$$

By assumption, equality (3.5) and inclusion (3.6) hold. Hence we conclude that

$$\partial^M d_S^J(\bar{x}) \subset \partial^M (J+\delta_S)(\bar{x}) \cap \mathbb{B}^* \subseteq \partial^F (J+\delta_S)(\bar{x}) \cap \mathbb{B}^* = \partial^F d_S^J(\bar{x}).$$

This shows that d_S^J is Mordukhovich subdifferentially regular at $\bar{x}$ and the proof is complete. $\square$

To provide the converse implication, we need the following lemmas.

Lemma 3.6. *Let $\bar{x} \in S$ be such that $C_{\bar{x}} = 0$. Then we have that*

$$\partial^F (J+\delta_S)(\bar{x}) = N^F(S,\bar{x}).$$

Proof. Let $x^* \in \partial^F (J+\delta_S)(\bar{x})$ and let $\epsilon > 0$. By definition, there exists $\delta > 0$ such that

$$\langle x^*, x-\bar{x}\rangle \le (J+\delta_S)(x) - (J+\delta_S)(\bar{x}) + \frac{\epsilon}{2}\|x-\bar{x}\|,\ \forall x \in \bar{x}+\delta\mathbb{B},$$

that is

$$\langle x^*, x-\bar{x}\rangle \le J(x) - J(\bar{x}) + \frac{\epsilon}{2}\|x-\bar{x}\|,\ \forall x \in (\bar{x}+\delta\mathbb{B}) \cap S.$$

Since $C_{\bar{x}} = 0$ by assumption, we may assume without loss of generality that $\delta > 0$ is small enough such that

$$|J(x) - J(\bar{x})| \le \frac{\epsilon}{2}\|x-\bar{x}\|,\ \forall x \in (\bar{x}+\delta\mathbb{B}) \cap S. \tag{3.9}$$

Combining the two above inequalities we obtain

$$\langle x^*, x-\bar{x}\rangle \le \epsilon\|x-\bar{x}\|, \quad \forall x \in (\bar{x}+\delta\mathbb{B}) \cap S.$$

This shows that $x^* \in N^F(S,\bar{x})$ as $\epsilon > 0$ is arbitrary. Conversely, let $x^* \in N^F(S,\bar{x})$ and let $\epsilon > 0$. Then by definition, there exists $\delta > 0$ such that inequality (3.9) holds and that $\langle x^*, x-\bar{x}\rangle \le \frac{\epsilon}{2}\|x-\bar{x}\|$ for any $x \in (\bar{x}+\delta\mathbb{B}) \cap S$. It follows then that

$$\begin{aligned}\langle x^*, x-\bar{x}\rangle &\le J(x) - J(\bar{x}) + |J(x) - J(\bar{x})| + \tfrac{\epsilon}{2}\|x-\bar{x}\| \\ &\le J(x) - J(\bar{x}) + \epsilon\|x-\bar{x}\|.\end{aligned}$$

This shows that $x^* \in \partial^F (J+\delta_S)(\bar{x})$ as $\epsilon > 0$ is arbitrary, and the proof is complete. $\square$

An analog result for the Mordukhovich subdifferential is needed to prove our next result on the Fréchet regularity concepts.

Lemma 3.7. *Let $\bar{x} \in S$ be such that $L_{\bar{x}} = 0$. Then we have that*

$$\partial^M (J + \delta_S)(\bar{x}) = N^M(S, \bar{x}).$$

Proof. Let $x^* \in \partial^M (J+\delta_S)(\bar{x})$. By definition, there exist sequences $\varepsilon_k \downarrow 0$, $x_k \to \bar{x}$, and $x_k^* \to^{w^*} x^*$ as $k \to \infty$ such that

$$x_k^* \in \partial^F_{\varepsilon_k}(J + \delta_S)(x_k)$$

for all $k \in \mathbb{N}$. Obviously $(x_k) \subset S$. Hence, there exists δ_k such that

$$\langle x_k^*, x - x_k\rangle \le (J + \delta_S)(x) - (J + \delta_S)(x_k) + \frac{\epsilon_k}{2}\|x - x_k\|, \; \forall x \in x_k + \delta_k \mathbb{B},$$

that is,

$$\langle x_k^*, x - x_k\rangle \le J(x) - J(x_k) + \frac{\epsilon_k}{2}\|x - x_k\|, \; \forall x \in (x_k + \delta_k \mathbb{B}) \cap S.$$

Since $x_k \to \bar{x}$ with $(x_k) \subset S$ and $L_{\bar{x}} = 0$ by assumption, we may assume without loss of generality that there exists $\tilde{\delta}_k \in (0, \frac{\delta_k}{2})$ (small enough) such that for k large enough

$$x_k \in (\bar{x} + \tilde{\delta}_k \mathbb{B}) \cap S$$

and

$$|J(z) - J(y)| \le \frac{\epsilon_k}{2}\|z - y\|, \; \forall z, y \in (\bar{x} + \tilde{\delta}_k \mathbb{B}) \cap S.$$

Fix now any $x \in (x_k + \tilde{\delta}_k \mathbb{B}) \cap S$. Then $\|x - \bar{x}\| \le \|x - x_k\| + \|x_k - \bar{x}\| \le \tilde{\delta}_k + \tilde{\delta}_k = 2\tilde{\delta}_k < \delta_k$. Therefore, we can write by using the above inequalities

$$\begin{aligned}
\langle x_k^*, x - x_k\rangle &\le J(x) - J(x_k) + \frac{\epsilon_k}{2}\|x - x_k\| \\
&\le \frac{\epsilon_k}{2}\|x - x_k\| + \frac{\epsilon_k}{2}\|x - x_k\| \\
&= \epsilon_k \|x - x_k\|,
\end{aligned}$$

for all $x \in (x_k + \tilde{\delta}_k \mathbb{B}) \cap S$. This shows that $x_k^* \in N^F_{\epsilon_k}(S, x_k)$ and hence $x^* \in N^M(S, \bar{x})$. Conversely, let $x^* \in N^M(S, \bar{x})$. By definition, there exist sequences $\varepsilon_k \downarrow 0$, $x_k \to^S \bar{x}$, and $x_k^* \to^{w^*} x^*$ as $k \to \infty$ such that

$$x_k^* \in N^F_{\varepsilon_k}(S; x_k),$$

for all $k \in \mathbb{N}$. Hence, there exists δ_k such that

$$\langle x_k^*, x - x_k\rangle \le \frac{\epsilon_k}{2}\|x - x_k\|, \; \forall x \in (x_k + \delta_k \mathbb{B}) \cap S.$$

Since $x_k \to^S \bar{x}$ and $L_{\bar{x}} = 0$ by assumption, we may assume without loss of generality that there exists $\tilde{\delta}_k \in (0, \delta_k)$ (small enough) such that for k large enough

$$x_k \in (\bar{x} + \tilde{\delta}_k \mathbb{B}) \cap S$$

and

$$|J(z) - J(y)| \le \frac{\epsilon_k}{2}\|z - y\|, \; \forall z, y \in (\bar{x} + \tilde{\delta}_k \mathbb{B}) \cap S.$$

Fix now any $x \in (x_k + \tilde{\delta}_k \mathbb{B}) \cap S$. Then $x \in (\bar{x} + \delta_k \mathbb{B}) \cap S$. Therefore,

$$\langle x_k^*, x - x_k\rangle \le \frac{\epsilon_k}{2}\|x - x_k\|$$

$$\begin{aligned} &\leq J(x) - J(x_k) + |J(x) - J(x_k)| + \frac{\epsilon_k}{2}\|x - x_k\| \\ &\leq J(x) - J(x_k) + \frac{\epsilon_k}{2}\|x - x_k\| + \frac{\epsilon_k}{2}\|x - x_k\| \\ &= J(x) - J(x_k) + \epsilon_k\|x - x_k\|, \end{aligned}$$

for all $x \in (x_k + \tilde{\delta}_k\mathbb{B}) \cap S$. This shows that $x_k^* \in \partial_{\epsilon_k}^F(J + \delta_S)(x_k)$ and hence $x^* \in \partial^M(J + \delta_S)(\bar{x})$. Thus completing the proof. □

The following proposition extends the corresponding result in [11] for the special case when $J = 0$ due to Burke *et al.* [11].

Proposition 3.8. *Let $\bar{x} \in S_0$. Assume that $\min_J(\bar{x}, S)$ is well-posed and the Lipschitz constant $L_{\bar{x}} = 0$. Then we have the following assertions:*

$$\left(d_S^J\right)^{\uparrow}(\bar{x}; v) \leq d_{T^C(S;\bar{x})}(v) \quad \textit{for all } v \in X; \tag{3.10}$$

and

$$\partial^C d_S^J(\bar{x}) \subset N^C(S; \bar{x}) \cap \mathbb{B}^*. \tag{3.11}$$

Proof. By definition, one checks that inclusion (3.11) is a direct consequence of inequality (3.10). Thus it suffices to verify inequality (3.10). To do this, fix any $v \in X$ and let $\epsilon > 0$. Then there exists some $\bar{v} \in T^C(S; \bar{x})$ such that

$$\|v - \bar{v}\| \leq d_{T^C(S;\bar{x})}(v) + \epsilon. \tag{3.12}$$

Since the distance function d_S^J is Lipschitz, we have that

$$\left(d_S^J\right)^{\uparrow}(\bar{x}; v) = \limsup_{t\downarrow 0, x\to\bar{x}} \frac{d_S^J(x + tv) - d_S^J(x)}{t}.$$

Thus there exists a sequence (t_n, x_n) in $(0, +\infty) \times X$ converging to $(0, \bar{x})$ such that

$$\left(d_S^J\right)^{\uparrow}(\bar{x}; v) = \lim_n \frac{d_S^J(x_n + t_n v) - d_S^J(x_n)}{t_n}. \tag{3.13}$$

By the definition of d_S^J, we may choose a sequence $\{y_n\} \subset S$ such that

$$\|y_n - x_n\| + J(y_n) \leq d_S^J(x_n) + t_n^2 \quad \text{for each } n. \tag{3.14}$$

Then

$$\|y_n - \bar{x}\| + J(y_n) \leq d_S^J(x_n) + \|x_n - \bar{x}\| + t_n^2 \quad \text{for each } n.$$

This, together with the fact that $\|x_n - \bar{x}\| \to 0$ and $t_n^2 \to 0$, implies that $\{y_n\}$ is a minimizing sequence for problem $\min_J(\bar{x}, S)$. Then, by the assumed well-posedness, we get that $y_n \to^S \bar{x}$. Using now the sequential characterization of the Clarke tangent cone $T^C(S; \bar{x})$, we obtain a sequence $\{v_n\}$ with $v_n \to \bar{v}$ such that $\{y_n + t_n v_n\} \subseteq S$. Let $n \in \mathbb{N}$. Then, by relation (3.14), we have

$$\begin{aligned} [d_S^J(x_n + t_n v) - d_S^J(x_n)] &\leq [d_S^J(y_n + t_n v) + \|y_n - x_n\| - d_S^J(x_n)] \\ &\leq [d_S^J(y_n + t_n v) - J(y_n) + t_n^2] \\ &\leq [J(y_n + t_n v_n) - J(y_n) + t_n\|v_n - v\| + t_n^2], \end{aligned}$$

where the last inequality holds because $y_n + t_n v_n \in S$. Thus

$$\frac{d_S^J(x_n + t_n v) - d_S^J(x_n)}{t_n} \le \frac{J(y_n + t_n v_n) - J(y_n)}{t_n} + \|v_n - v\| + t_n. \tag{3.15}$$

Using now the assumption $L_{\bar{x}} = 0$ and the fact that $y_n \to \bar{x}$, we obtain that

$$\lim_n \frac{J(y_n + t_n v_n) - J(y_n)}{\|t_n v_n\|} = 0.$$

This together with (3.12) and (3.15) implies that

$$(d_S^J)^{\uparrow}(\bar{x}; v) \le \|v - \bar{v}\| \le d_{T^C(S;\bar{x})}(v) + \epsilon.$$

Hence (3.10) is established as $\epsilon > 0$ is arbitrary and the proof is complete. □

Now, we can state our second theorem on the relationship between the subdifferential regularity of d_S^J and the normal regularity of S.

Theorem 3.9. *Let $\bar{x} \in S_0$ be such that $\min_J(\bar{x}, S)$ is well-posed and the Lipschitz constant $L_{\bar{x}} = 0$. Then the following assertions hold:*

(i) *S is Mordukhovich normally regular at $\bar{x}$ if and only if d_S^J is Mordukhovich subdifferentially regular at $\bar{x}$.*

(ii) *If S is Fréchet normally regular at $\bar{x}$, then d_S^J is Fréchet subdifferentially regular at $\bar{x}$.*

(iii) *If the space X is reflexive then S is Fréchet normally regular at $\bar{x}$ if and only if d_S^J is Fréchet subdifferentially regular at $\bar{x}$.*

Proof. By assumptions, Lemmas 3.6-3.7 and Propositions 3.3-3.4 are applicable. Hence we have that

$$\partial^M (J + \delta_S)(\bar{x}) = N^M(S;\bar{x}) = \bigcup_{\lambda>0} \lambda \partial^M d_S^J(\bar{x}) \tag{3.16}$$

$$\partial^F d_S^J(\bar{x}) = N^F(S;\bar{x}) \cap \mathbb{B}^* \quad \text{and} \quad \partial^F (J + \delta_S)(\bar{x}) = N^F(S;\bar{x}). \tag{3.17}$$

(i). This follows from the following chain of equivalences/implications:

$$\begin{aligned} N^M(S;\bar{x}) = N^F(S;\bar{x}) &\Leftrightarrow N^M(S;\bar{x}) \cap \mathbb{B}^* = N^F(S;\bar{x}) \cap \mathbb{B}^* \\ &\Leftrightarrow \partial^M (J + \delta_S)(\bar{x}) \cap \mathbb{B}^* = \partial^F (J + \delta_S)(\bar{x}) \cap \mathbb{B}^* \\ &\Rightarrow \partial^M d_S^J(\bar{x}) = \partial^F d_S^J(\bar{x}) \\ &\Rightarrow \textstyle\bigcup_{\lambda>0} \lambda \partial^M d_S^J(\bar{x}) = \bigcup_{\lambda>0} \lambda \partial^F d_S^J(\bar{x}) \\ &\Rightarrow N^M(S;\bar{x}) = N^F(S;\bar{x}), \end{aligned}$$

where the first equivalence holds by definition, the second equivalence and the last implication hold by (3.16) and (3.17), while the first implication holds because by Propositions 3.3-3.4 we have

$$\partial^F d_S^J(\bar{x}) \subset \partial^M d_S^J(\bar{x}) \subset \partial^M (J + \delta_S)(\bar{x}) \cap \mathbb{B}^* = \partial^F (J + \delta_S)(\bar{x}) \cap \mathbb{B}^* = \partial^F d_S^J(\bar{x}).$$

(ii). Assume that S is Fréchet normally regular at $\bar{x}$. Then $N^C(S;\bar{x}) = N^F(S;\bar{x})$. Using (3.11) and (3.17) we have

$$\partial^C d_S^J(\bar{x}) \subset N^C(S;\bar{x}) \cap \mathbb{B}^* = N^F(S;\bar{x}) \cap \mathbb{B}^* = \partial^F d_S^J(\bar{x}).$$

This shows the Fréchet subdifferential regularity of d_S^J at $\bar{x}$ as the inverse inclusion always holds.

(iii). This follows directly from assertion (i) and the equivalence between the Mordukhovich regularity and the Fréchet regularity in reflexive Banach spaces. □

The following corollary of the above theorem adds, some further characterization of the Fréchet normal regularity of sets, to the list stated in in Theorems 2.5-2.6 in [6].

Corollary 3.10. *Let $\bar{x} \in S_0$. Assume that $L_{\bar{x}} = 0$, that $\min_J(\bar{x}, S)$ is well-posed, and that the space X is reflexive. Then the following assertions are equivalent:*

i) *S is Fréchet normally regular at $\bar{x}$;*
ii) *S is Mordukhovich normally regular at $\bar{x}$;*
iii) *d_S is Fréchet subdifferentially regular at $\bar{x}$;*
iv) *d_S is Mordukhovich subdifferentially regular at $\bar{x}$;*
v) *d_S^J is Fréchet normally regular at $\bar{x}$.*
vi) *d_S^J is Mordukhovich normally regular at $\bar{x}$.*

Proof. The equivalence between (i)-(iv) are proved in Theorems 2.5-2.6 in [6, 7]. The equivalence between (i) and (v) follows from the previous theorem. Finally the equivalence between (v) and (vi) follows from the reflexivity of the space, the 1-Lipschitz continuity of d_S^J, and the the equality $\partial^C d_S^J(\bar{x}) = cl_{w^*} co \left[\partial^M d_S^J(\bar{x})\right]$. □

4. Proximal regularity concepts of d_S^J on S_0

It is well known (see for example [6]) that Fréchet subdifferential regularity of functions is not equivalent to proximal subdifferential regularity, even in finite dimensional spaces. So, the present section is devoted to study some properties of proximal subdifferential regularity of d_S^J, essentially we will give conditions under which this subdifferential regularity can be characterized in terms of the proximal subdifferential regularity of J and the proximal normal regularity of S.

We recall first the following result on the relationship between the proximal subdifferential of d_S^J and the proximal subdifferential of $J + \delta_S$, which is needed in the proof of the main result in this section. It is due to [39].

Proposition 4.1. *Let $\bar{x} \in S_0$. Then the following assertions hold.*

(i) *We have*

$$\partial^P d_S^J(\bar{x}) \subset \partial^P (J + \delta_S)(\bar{x}) \cap \mathbb{B}^*. \tag{4.1}$$

(ii) *If $\min_J(\bar{x}, S)$ is well-posed and the center Lipschitz constant $C_{\bar{x}} < 1$, then we have*

$$\partial^P (J + \delta_S)(\bar{x}) \cap \mathbb{B}^* = \partial^P d_S^J(\bar{x}). \tag{4.2}$$

To study the proximal subdifferential regularity of d_S^J, we introduce the following notions of the second Lipschhitz conditions.

Definition 4.2. Let $\bar{x} \in S$. The function $J : S \to \mathbb{R}$ is said to satisfy

(a) the center second Lipschitz condition at $\bar{x}$ if the center second Lipschitz constant at $\bar{x}$

$$C_{\bar{x}}^{\mathrm{Sec}} := \inf_{\rho>0} \sup_{y\in\mathbb{B}(\bar{x},\rho)\cap S} \frac{|J(y)-J(\bar{x})|}{\|y-\bar{x}\|^2} < \infty;$$

(b) the second Lipschitz condition at $\bar{x}$ if the second Lipschitz constant at $\bar{x}$

$$L_{\bar{x}}^{\mathrm{Sec}} := \inf_{\rho>0} \sup_{x,y\in\mathbb{B}(\bar{x},\rho)\cap S} \frac{|J(y)-J(x)|}{\|y-x\|^2} < \infty.$$

Clearly, one has by definition that $C_{\bar{x}}^{\mathrm{Sec}} \le L_{\bar{x}}^{\mathrm{Sec}}$. Furthermore, we have the following implications:

$$C_{\bar{x}}^{\mathrm{Sec}} < \infty \Longrightarrow C_{\bar{x}} = 0 \quad \text{and} \quad L_{\bar{x}}^{\mathrm{Sec}} < \infty \Longrightarrow L_{\bar{x}} = 0.$$

However, the converse implications are not true in general; see Example 4.6 below.

Lemma 4.3. *Let $\bar{x} \in S$ be such that $C_{\bar{x}}^{\mathrm{sec}} < \infty$. Then we have*

$$\partial^P (J+\delta_S)(\bar{x}) = N^P(S,\bar{x}).$$

Proof. Let $x^* \in \partial^P(J+\delta_S)(\bar{x})$. By definition, there exists $\sigma, \delta > 0$ such that

$$\langle x^*, x-\bar{x}\rangle \le (J+\delta_S)(x) - (J+\delta_S)(\bar{x}) + \sigma\|x-\bar{x}\|^2,\ \forall x \in \bar{x}+\delta\mathbb{B},$$

that is,

$$\langle x^*, x-\bar{x}\rangle \le J(x) - J(\bar{x}) + \sigma\|x-\bar{x}\|^2,\ \forall x \in (\bar{x}+\delta\mathbb{B})\cap S.$$

Since $C_{\bar{x}}^{\mathrm{sec}} < \infty$ by assumption, there exist constants $\alpha, \delta_1 > 0$ such that

$$|J(x) - J(\bar{x})| \le \alpha\|x-\bar{x}\|^2,\ \forall x \in (\bar{x}+\delta_1\mathbb{B})\cap S. \tag{4.3}$$

Without loss of generality we assume that $\delta_1 \le \delta$. Combining the above two inequalities we get

$$\langle x^*, x-\bar{x}\rangle \le [\alpha+\sigma]\|x-\bar{x}\|^2, \quad \forall x \in (\bar{x}+\delta_1\mathbb{B})\cap S.$$

This shows that $x^* \in N^P(S,\bar{x})$. Conversely, let $x^* \in N^P(S,\bar{x})$. Then by definition, there exist $\sigma, \delta_1 > 0$ such that inequality (4.3) holds and that $\langle x^*, x-\bar{x}\rangle \le \sigma\|x-\bar{x}\|^2$ for any $x \in (\bar{x}+\delta_1\mathbb{B})\cap S$. It then follows that

$$\begin{aligned}\langle x^*, x-\bar{x}\rangle &\le J(x) - J(\bar{x}) + |J(x)-J(\bar{x})| + \sigma\|x-\bar{x}\|^2 \\ &\le J(x) - J(\bar{x}) + [\sigma+\alpha]\|x-\bar{x}\|^2, \quad \forall x \in (\bar{x}+\delta_1\mathbb{B})\cap S.\end{aligned}$$

This ensures

$$\langle x^*, x-\bar{x}\rangle \le (J+\delta_S)(x) - (J+\delta_S)(\bar{x}) + [\sigma+\alpha]\|x-\bar{x}\|^2, \quad \forall x \in \bar{x}+\delta_1\mathbb{B}.$$

This shows that $x^* \in \partial^P(J+\delta_S)(\bar{x})$ and the proof is complete. □

Now, we are ready to prove the following theorem.

Theorem 4.4. *Let $\bar{x} \in S_0$ be such that $\min_J(\bar{x},S)$ is well-posed. Suppose that the Lipschitz constant $L_{\bar{x}} = 0$ and the center second Lipschitz constant $C_{\bar{x}}^{\mathrm{sec}} < \infty$. If S is proximal normally regular at $\bar{x}$, then d_S^J is proximal subdifferentially regular at $\bar{x}$. The converse is true provided that X is reflexive.*

Proof. By the assumption that $C_{\bar{x}}^{\sec} < \infty$, we can apply Lemma 4.3 and Proposition 4.1 to get that

$$\partial^P d_S^J(\bar{x}) = N^P(S;\bar{x}) \cap \mathbb{B}^*. \tag{4.4}$$

Now assume that S is proximal normally regular at $\bar{x}$, i.e., $N^C(S;\bar{x}) = N^P(S;\bar{x})$. Since $L_{\bar{x}} = 0$, it follows from (4.4) and Proposition 3.8 that

$$\partial^C d_S^J(\bar{x}) \subset N^C(S;\bar{x}) \cap \mathbb{B}^* = N^P(S;\bar{x}) \cap \mathbb{B}^* = \partial^P d_S^J(\bar{x}).$$

Hence $\partial^C d_S^J(\bar{x}) = \partial^P d_S^J(\bar{x})$ as the inverse inclusion always holds, and so d_S^J is proximal subdifferentially regular at $\bar{x}$.

Conversely, assume that d_S^J is proximal subdifferentially regular at $\bar{x}$. Then

$$\partial^C d_S^J(\bar{x}) = \partial^M d_S^J(\bar{x}) = \partial^F d_S^J(\bar{x}) = \partial^P d_S^J(\bar{x}). \tag{4.5}$$

Since $L_{\bar{x}} = 0$, it follows from Proposition 3.3 and Lemma 3.6 that

$$N^F(S;\bar{x}) = \bigcup_{\lambda \geq 0} \lambda \partial^F d_S^J(\bar{x}) \tag{4.6}$$

This together with (4.4) and (4.5) imply

$$N^F(S;\bar{x}) = \bigcup_{\lambda \geq 0} \lambda \partial^F d_S^J(\bar{x}) = \bigcup_{\lambda \geq 0} \lambda \partial^P d_S^J(\bar{x}) = N^P(S;\bar{x}). \tag{4.7}$$

Furthermore, (4.5) ensures that d_S^J is Fréchet subdifferentially regular at $\bar{x}$. Now assume that X is reflexive. Then S is Fréchet normally regular at $\bar{x}$ by assertion (iii) of Theorem 3.9, that is $N^F(S;\bar{x}) = N^C(S;\bar{x})$. This together with (4.7) yield that $N^P(S;\bar{x}) = N^C(S;\bar{x})$, which ensures the proximal normal regularity of S at $\bar{x}$. The proof of the theorem is then complete. □

The following corollary on the proximal regularity concepts is direct from the previous theorem because the condition $L_{\bar{x}}^{\sec} < \infty$ implies both $L_{\bar{x}} = 0$ and $C_{\bar{x}}^{\sec} < \infty$.

Theorem 4.5. *Let X be reflexive and let $\bar{x} \in S_0$. Assume that $\min_J(\bar{x}, S)$ is well-posed and that $L_{\bar{x}}^{\sec} < \infty$. Then the following assertions are equivalent:*

i) *S is proximal normally regular at $\bar{x}$.*
ii) *d_S is proximal subdifferentially regular at $\bar{x}$.*
iii) *d_S^J is proximal subdifferentially regular at $\bar{x}$.*

We end this paper with an example to illustrate the results obtained in the present paper.

Example 4.6. Let $X := \mathbb{R}^2$, the 2-dimensional Euclidean space, and let S^1, S^2 and S^3 be the closed sets defined respectively by

$$S^1 := \{(t,s) \in \mathbb{R}^2 : s \geq 0,\ |\sin 2t|^3 - s \geq 0,\ t \in [-\pi,\pi]\},$$

$$S^2 := \{(t,s) \in \mathbb{R}^2 : s \geq 0,\ |\sin 2t|^{\frac{3}{2}} - s \geq 0,\ t \in [-\pi,\pi]\}$$

and

$$S^3 := \{(t,s) \in \mathbb{R}^2 : s > 0,\ \mathrm{sign}(\sin 2t)(\sin 2t - s) \geq 0,\ t \in (-\pi,\pi]\} \cup \{(-\pi,0)\}.$$

The figures of the three sets are shown in the Figures 1,2, and 3.

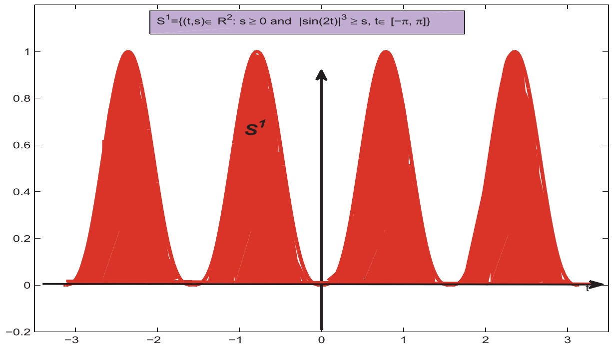

FIGURE 1. The set S^1 in Example 4.1

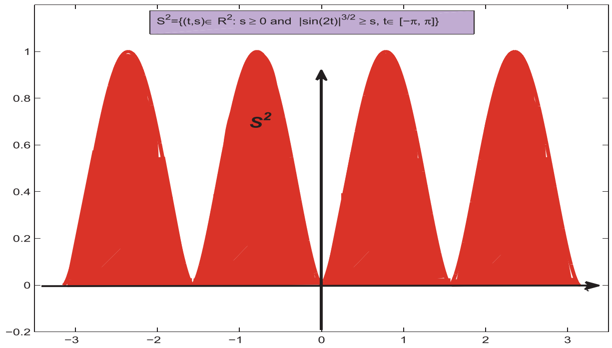

FIGURE 2. The set S^2 in Example 4.1

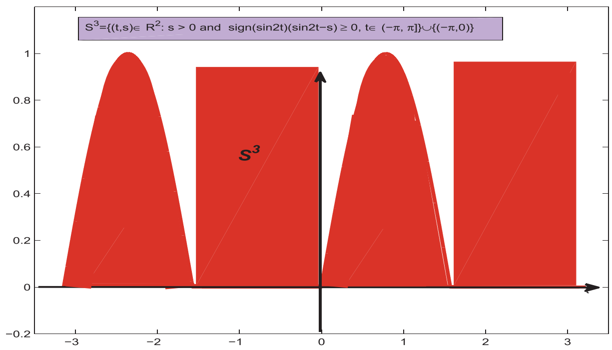

FIGURE 3. The set S^3 in Example 4.1

We first discuss the regularity of the above sets. To do this, we write

$$D_0 := \left\{(t,0) : t = -\frac{\pi}{2}, 0, \frac{\pi}{2}\right\}.$$

Note that, for each $i \in \{1,2,3\}$ and each $\bar{x} \in S^i \setminus D_0$, S^i is locally convex at $\bar{x}$, that is, there exists an open set U containing $\bar{x}$ such that $S^i \cap U$ is convex; hence S^i is proximal (so both Fréchet and Mordukhovich by Remark 2.8) normally regular at each $\bar{x} \in S^i$. Considering the point $\bar{x} \in D_0$, we have the following regularity results:

(a) S^1 is proximal normally regular at $\bar{x}$.
(b) S^2 is Fréchet normally regular but not proximal normally regular at $\bar{x}$.
(c) S^3 is not (proximal, Fréchet, Mordukhovich) normally regular at $\bar{x}$.

To see this, we calculate by definition that

$$N^P(S^1, \bar{x}) = \mathbb{R}_+\{(0,1),(0,-1)\} = N^C(S^1, \bar{x})$$

and

$$N^P(S^2, \bar{x}) = \mathbb{R}_+\{(0,-1)\}.$$

Moreover, by [6, P. 37, Remark 2.1],

$$N^F(S^2, \bar{x}) = \mathbb{R}_+\{(0,1),(0,-1)\} = N^C(S^2, \bar{x}).$$

Thus, we get the assertions (a) and (b). Finally assertion (c) holds because for $\bar{x} = (\pm\frac{\pi}{2}, 0)$ we have

$$N^F(S^3, \bar{x}) = \mathbb{R}_+\{(0,-1)\} \quad \text{and} \quad N^M(S^3, \bar{x}) = \mathbb{R}_+\{(0,-1);(-1,0);(2,1)\}.$$

Also, for $\bar{x} = (0,0)$, we have

$$N^F(S^3, \bar{x}) = \mathbb{R}_+\{(0,-1)\} \quad \text{and} \quad N^M(S^3, \bar{x}) = \mathbb{R}_+\{(0,-1);(-1,0);(-2,1)\}.$$

Thus, all the assertions (a), (b), and (c) are fulfilled.

Now fix $i = 1,2,3$ and fix $S := S^i$. Let $p \in (1,2]$ and consider the function J on S defined by

$$J(t,s) := \begin{cases} \frac{1}{6}|\sin s|^p & t \in [-\pi, 0], \\ \frac{1}{6}(|\sin s|^p + |\sin t|^p) & t \in [0, \pi], \end{cases}$$

for each $(t,s) \in S$. Then one checks that J satisfies the following Lipschitz condition:

$$|J(t,s) - J(\bar{t},\bar{s})| \le \frac{2}{3}\|(t,s) - (\bar{t},\bar{s})\| \quad \text{for any } (t,s),\ (\bar{t},\bar{s}) \in S.$$

This implies that $S_0 = S$ and that, for each $\bar{x} \in S$, $C_{\bar{x}} < 1$ and the problem $\min_J(\bar{x}, S)$ is well posed. Moreover, it is easy to see that J is Mordukhovich subdifferentially regular at each point $\bar{x} \in S$. Thus Theorem 3.5 is applicable for any point $\bar{x} \in S$. Consequently, thanks to assertions (a) and (b) above, the perturbed distance function d_S^J is Fréchet subdifferentially regular at any points $\bar{x} \in S$ in the case when $S = S^1$ or $S = S^2$.

To illustrate the applicability of Theorems 3.9 and 4.4, we consider the point $\bar{x} \in D := \{(t,0) \in S : -\pi \le t \le 0\}$. Then

$$\begin{array}{ll} L_{\bar{x}} = 0 & \text{if } p > 1, \\ L_{\bar{x}}^{\sec} = C_{\bar{x}}^{\sec} = \infty & \text{if } 1 < p < 2, \\ L_{\bar{x}}^{\sec} = C_{\bar{x}}^{\sec} < \infty & \text{if } p = 2. \end{array}$$

Thus Theorem 3.9 is applicable and then we have from assertions (a)-(b) above that the perturbed distance function d_S^J is Fréchet normally regular at each point $\bar{x} \in D$ if $S = S^1$ or $S = S^2$, while, if $S = S^3$, d_S^J is Fréchet normally regular at each $\bar{x} \in D \setminus D_0$ but not at point $\bar{x} \in D_0$.

Similarly, in the special case when $p = 2$, Theorem 4.4 is applicable to concluding that, if $S = S^2$, d_S^J is proximal subdifferentially regular at each point $\bar{x} \in D$; while, if $S = S^1$ or $S = S^3$, d_S^J is proximal subdifferentially regular at each point $\bar{x} \in D \setminus D_0$ but not at point $\bar{x} \in D_0$.

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