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WEAK CONVERGENCE THEOREMS OF ITERATIVE SEQUENCES IN HILBERT SPACES

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Abstract. In this paper, we prove the approximation of common fixed points of new iterative schemes, which extend the iterative sequences introduced by Moudafi, Takahashi and Tamura, respectively, for two nonspreading mappings and an α-nonexpansive mapping in Hilbert spaces.

1. Introduction

Throughout this paper, we denote $\mathbb{R}$ by the set of real numbers. Let C be a nonempty closed convex subset of a real Hilbert space H. We denote the fixed point set of a mapping $T : C \to C$ by $F(T)$.

A mapping $T : C \to C$ is said to be *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|$$

for all $x, y \in C$. A mapping $T : C \to C$ is said to be *quasi-nonexpansive* if the set of fixed points of T is nonempty and

$$\|Tx - y\| \leq \|x - y\|$$

for all $x \in C$ and $y \in F(T)$. If a mapping $T : C \to C$ is nonexpansive and the set of fixed points of T is nonempty, then T is quasi-nonexpansive. Furthermore, a mapping $F : C \to C$ is said to be *firmly nonexpansive* if

$$\|Fx - Fy\|^2 \leq \langle x - y, Fx - Fy\rangle$$

for all $x, y \in C$ (see [3, 5, 6, 18]).

It is known that a mapping $F : C \to C$ is firmly nonexpansive if and only if

$$\|Fx - Fy\|^2 + \|(I - F)x - (I - F)y\|^2 \leq \|x - y\|^2$$

for all $x, y \in C$, where I is the identity mapping on H. Also, it is known that every firmly nonexpansive mapping is nonexpansive (see Remark 2.3) and of the from $F = \frac{1}{2}(I + T)$ with a some nonexpansive mapping T (see, for example, [5, 6]).

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In 2008, Kohsaka and Takahashi [11] studied the existence of fixed points of mappings of firmly nonexpansive type in Banach spaces. They also introduced the class of mappings, which is called the class of nonspreading mappings.

Let E be a real smooth, strictly convex and reflexive Banach space and let j denote the duality mapping of E. Let C be a nonempty closed convex subset of E. A mapping $T : C \to C$ is said to be *nonspreading* if

$$\phi(Tx, Ty) + \phi(Ty, Tx) \leq \phi(Tx, y) + \phi(Ty, x)$$

for all $x, y \in C$, where $\phi(x, y) = \|x\|^2 - 2\langle x, j(y)\rangle + \|y\|^2$ for all $x, y \in E$. In the case when E is a Hilbert space, we know that $\phi(x, y) = \|x - y\|^2$ for all $x, y \in E$. So, a nonspreading mapping $T : C \to C$ in a Hilbert space H is defined as follows:

$$2\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|x - Ty\|^2 \tag{1.1}$$

for all $x, y \in C$. It is shown in [7] that (1.1) is equivalent to the following:

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + 2\langle x - Tx, y - Ty\rangle \tag{1.2}$$

for all $x, y \in C$. It is known that, in a Hilbert space H, every firmly nonexpansive mapping is nonspreading and, if the set of fixed points of a nonspreading mapping is nonempty, then every nonspreading mapping is quasi-nonexpansive [12]. In 2010, Takahashi [23] introduced the class of hybrid mappings in Hilbert spaces, that is, a mapping $T : C \to C$ is said to be *hybrid mappings* if

$$3\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|Tx - y\|^2 + \|x - Ty\|^2$$

for all $x, y \in C$, which contains the class of firmly nonexpansive mappings in Hilbert spaces.

Recently, Aoyama and Kohsaka [1] introduced the new class of α-nonexpansive mappings $T : C \to C$ in a Banach space E which is defined as follows:

$$\|Tx - Ty\|^2 \leq \alpha\|Tx - y\|^2 + \alpha\|Ty - x\|^2 + (1 - 2\alpha)\|x - y\|^2 \tag{1.3}$$

for all $x, y \in C$, where α is real number such that $\alpha < 1$. This class contains the classes of nonexpansive, nonspreading and hybrid mappings, respectively.

Remark 1.1. From [1], we conclude that

(1) If T is a nonexpansive mapping, then T is 0-nonexpansive;
(2) If T is a nonspreading mapping, then T is $\frac{1}{2}$-nonexpansive;
(3) If T is a hybrid mapping, then T is $\frac{1}{3}$-nonexpansive;
(4) If T is an α-nonexpansive mapping and $F(T) \neq \emptyset$, then T is quasi-nonexpansive.

For (4), for any $x \in C$ and $p \in F(T)$, we have

$$\begin{aligned}\|Tx - y\|^2 &= \|Tx - Ty\|^2 \\ &\leq \alpha\|Tx - y\|^2 + \alpha\|Ty - x\|^2 + (1 - 2\alpha)\|x - y\|^2 \\ &= \alpha\|Tx - y\|^2 + (1 - \alpha)\|x - y\|^2,\end{aligned}$$

which implies that $\|Tx - y\| \leq \|x - y\|$, that is, T is a quasi-nonexpansive mapping.

Now, we give some relations among an α-nonexpansive mapping, a nonspreading mapping, a hybrid mapping, a firmly nonexpansive mapping and a nonexpansive mapping.

Example 1.2. Let H be a Hilbert space. Set $E = \{x \in H : \|x\| \leq 1\}$, $D = \{x \in H : \|x\| \leq 2\}$ and $C = \{x \in H : \|x\| \leq 3\}$. Defined a mapping $T : C \to C$ as follows:

$$Tx = \begin{cases} x, & \text{if } x \in D, \\ P_E(x), & \text{if } x \in C \backslash D, \end{cases} \tag{1.4}$$

where P_E is the metric projection of H onto E. Then T is a nonspreading mapping, but it is not nonexpansive (see [7]).

Example 1.3. Let E be a Banach space and let $S, T : E \to E$ be firmly nonexpansive mappings such that $S(E)$ and $T(E)$ are contained in rB_E for some positive real number r. Let α and δ be real numbers such that $0 < \alpha \leq \frac{1}{2}$ and $\delta \geq (1 + \frac{2}{\sqrt{\alpha}})r$, where B_E is the closed unit ball centered at the origin of E. Then the mapping $U : E \to E$ defined by

$$Ux = \begin{cases} Sx, & \text{if } x \in \delta B_E, \\ Tx & \text{otherwise}, \end{cases}$$

is α-nonexpansive (see [1]).

Example 1.4. Consider $\mathbb{R}^2$ with the usual norm and let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a function defined by

$$T(x, y) = (-y, x)$$

for all $(x, y) \in \mathbb{R}^2$. Clearly, T is a nonexpansive mapping, but T is not firmly nonexpansive. In fact, if we take $x = (1, 2)$ and $y = (4, 6)$ are element in $\mathbb{R}^2$, then

$$\|Tx - Ty\| = \|T(1, 2) - T(4, 6)\| = 5 = \|x - y\|$$

and

$$\|(x - Tx) - (y - Ty)\| = \|[(1, 2) - T(1, 2)] - [(4, 6) - T(4, 6)]\| = 5\sqrt{2}.$$

Thus we have

$$\|Tx - Ty\|^2 + \|(x - Tx) - (y - Ty)\|^2 = 25 + 50 > 25 = \|x - y\|^2.$$

So, T is a nonexpansive mapping, but T is not firmly nonexpansive.

Example 1.5. Consider $C = [0, 1]$ with the euclidean norm and let $T : C \to C$ be a function defined by

$$Tx = \frac{1 - x}{\sqrt{2}}$$

for all $x, y \in C$. Since

$$\|Tx - Ty\|^2 = \frac{1}{2}\|x - y\|^2$$

for all $x, y \in C$, it follows that

$$\|Tx - Ty\|^2 \leq \frac{1}{4}\|Tx - y\|^2 + \frac{1}{4}\|Ty - x\|^2 + \Big(1 - 2\Big(\frac{1}{4}\Big)\Big)\|x - y\|^2,$$

that is, T is a $\frac{1}{4}-$nonexpansive mapping, but T is not nonspreading and hybrid. In fact, if we take $x = 0$, $y = 1$, we have

$$2\|Tx - Ty\|^2 = 1 > 0.08578 = \|Tx - y\|^2 + \|Ty - x\|^2$$

and

$$3\|Tx-Ty\|^2 = \frac{3}{2} = 1.5 > 1.08578 = 1 + \left\|\frac{1}{\sqrt{2}} - 1\right\|^2$$
$$= \|x-y\|^2 + \|Tx-y\|^2 + \|Ty-x\|^2.$$

Example 1.6. Consider $C = [0,4]$ with the Euclidean norm and let $T : C \to C$ be a function defined by

$$Tx = \begin{cases} 0, & \text{if } x \neq 4, \\ 2, & \text{if } x = 4. \end{cases}$$

Now, we show that T is a $\frac{1}{4}$-nonexpansive mapping. In fact, if $x \neq 4$, $y \neq 4$, $x = 4$, $y = 4$, it is clear. Assume that $x = 4$ and $y \neq 4$. Since

$$\|Tx-Ty\|^2 = 4, \quad \|Tx-y\|^2 = \|2-y\|^2, \quad \|Ty-x\|^2 = 16, \quad \|x-y\|^2 = \|4-y\|^2,$$

we have

$$\|Tx-Ty\|^2 \leq \frac{1}{4}\|Tx-y\|^2 + \frac{1}{4}\|Ty-x\|^2 + \Big(1 - 2\Big(\frac{1}{4}\Big)\Big)\|x-y\|^2.$$

Therefore, T is a $\frac{1}{4}$-nonexpansive mapping, but T is not nonexpansive. In fact, if we take $x = 4$ and $y = 3.5$, then we have

$$\|Tx-Ty\| = 2 > 0.5 = \|x-y\|.$$

On the other hand, recently, Takahashi and Tamura [24] proved some weak convergence theorems for two nonexpansive mappings $T_1, T_2 : C \to C$ in a Hilbert space H by using the following iterative sequence $\{x_n\}$:

$$(1.5) \qquad \begin{cases} x_1 \in C \text{ chosen arbitrary}, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T_1(\beta_n T_2 x_n + (1-\beta_n)x_n) \end{cases}$$

for all $n \geq 1$, where $F(T_1) \cap F(T_2)$ is nonempty. If $T_1 = T_2$ in (1.5), we get the iterative sequence considered by Ishikawa [9] and so Mann [13].

In 2007, Moudafi [15] also considered another iterative sequence $\{x_n\}$ for two nonexpansive mappings $T_1, T_2 : C \to C$ defined as follows:

$$(1.6) \qquad \begin{cases} x_1 \in C \text{ chosen arbitrary}, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n(\beta_n T_1 x_n + (1-\beta_n)T_2 x_n) \end{cases}$$

for all $n \geq 1$, where $F(T_1)$ and $F(T_2)$ are nonempty.

In 2009, Iemoto and Takahashi [7] extended the results of [15] in a Hilbert space H by using Moudafi's iterative sequence $\{x_n\}$ defined as follows:

$$(1.7) \qquad \begin{cases} x_1 \in C \text{ chosen arbitrary}, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n(\beta_n S x_n + (1-\beta_n)T x_n) \end{cases}$$

for all $n \geq 1$, where S is a nonspreading mapping, T is a nonexpansive mapping and $F(S) \cap F(S)$ is nonempty.

The aim of this paper is to study the approximation of common fixed points for two nonspreading mappings and an α-nonexpansive mapping in Hilbert spaces by using the new iterative sequence $\{x_n\}$, which is an extension of the iterative sequences (1.6) and (1.7), defined as follow:

$$\begin{cases} x_1 \in C \text{ chosen arbitrary}, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n y_n, \\ y_n = (1-\beta_n)Tx_n + \beta_n S z_n, \\ z_n = (1-\gamma_n)x_n + \gamma_n U x_n \end{cases} \tag{1.8}$$

for all $n \geq 1$, where $\{\alpha_n\}$, $\{\beta_n\}$ are the sequences in $(0,1]$ and $\{\gamma_n\}$ is a sequence in $[0,1]$ with some conditions.

If $\gamma_n = 0$ for all $n \geq 1$ in (1.8), then $z_n = x_n$ and the iterative sequence (1.8) reduces to (1.6).

If $\beta_n = 1$ for all $n \geq 1$ (1.8), then $y_n = Sz_n$ and the iterative sequence (1.8) reduces to (1.5).

In 2002, Xu and Noor [28] introduced a three-step scheme to approximate fixed points of asymptotically nonexpansive mappings in a Banach space and Glowinski and Le Tallec [4] used tree-step iterative schemes to find approximate solutions of the elastoviscoplasticity problems, liquid crystal theory and eigenvalue computation. Also, it has been shown in [4] that the three-step scheme gives better numerical results than the two-step and one-step approximate schemes. In [8], Haubruge et al. studied the converegnce analysis of three-step schemes of Glowinski and Le Tallec [4] and applied these schemes to obtain new spltting-type algorithms for solving variational inequalities, separable convex programming and minimization of a sum of convex conditions. Thus we know that the three-step schemes play an important and significant part in solving various problems. For further details on three-step schemes, see [2, 16, 19] and [27].

2. Preliminaries

Let H be a real Hilbert space with an inner product $\langle \cdot, \cdot \rangle$ and a norm $\| \cdot \|$, respectively. First, we start with a brief recollection of basic concepts and facts in a Hilbert space H, which are very important for our main results in next sections.

In a Hilbert space H, it is known that

$$\|x+y\|^2 = \|x\|^2 + \|y\|^2 + 2\langle x, y\rangle \tag{2.1}$$

and

$$\|\alpha x + (1-\alpha)y\|^2 = \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\|x-y\|^2 \tag{2.2}$$

for all $x, y \in H$ and $\alpha \in \mathbb{R}$ (see, for instance, [21]). Further, if $\{x_n\}$ is a sequence in a Hilbert space H which converges weakly to a point $z \in H$, then we have

$$\limsup_{n\to\infty} \|x_n - y\|^2 = \limsup_{n\to\infty} \|x_n - z\|^2 + \|z-y\|^2 \tag{2.3}$$

for all $y \in H$. It is known that a Hilbert space H satisfies Opial's condition ([17]), that is, for any sequence $\{x_n\}$ in H such that $x_n \rightharpoonup x$,

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{i\to\infty} \|x_n - y\| \tag{2.4}$$

for all $y \in H$ with $y \neq x$ where "$\rightharpoonup$" stands for the weak convergence of the sequence $\{x_n\}$.

Lemma 2.1 ([26]). *Suppose that $\{s_n\}$ and $\{e_n\}$ are the sequences of nonnegative real numbers such that $s_{n+1} \leq s_n + e_n$ for all $n \geq 1$. If $\sum_1^\infty e_n < \infty$, then $\lim_{n\to\infty} s_n$ exists.*

Lemma 2.2 ([25]). *Let C be a nonempty closed convex subset of a Hilbert space H and let $\{x_n\}$ be a sequence in H. Suppose that, for all $y \in C$,*

$$\|x_{n+1} - y\| \leq \|x_n - y\|$$

for all $n \geq 1$. Then the sequence $\{P_C x_n\}$ converges strongly to a point $z \in C$, where P_C is the metric projection of H onto C.

Lemma 2.3 ([25]). *Let C be a nonempty closed convex subset of Hilbert space H. Then, for any $x \in H$ and $y \in C$, $y = P_C x$ if and only if $\langle x - y, y - z\rangle \geq 0$ for all $z \in C$.*

Lemma 2.4 ([7]). *Let C be a nonempty closed convex subset of a Hilbert space H. Let S be a nonseading mapping of C into itself such that $F(S) \neq \emptyset$. Then S is demicosed, i.e., $x_n \rightharpoonup u$ and $x_n - Sx_n \to 0$ imply $u \in F(S)$.*

Theorem 2.5 ([22]). *Let H be a Hilbert space and let $\{x_n\}$ be a bounded sequence in H. Then the sequence $\{x_n\}$ is weakly convergent if and only if each weakly convergent subsequence of $\{x_n\}$ has the same weak limit, that is, for any $x \in H$,*

$$x_n \rightharpoonup x \quad \Longleftrightarrow \quad (x_{n_i} \rightharpoonup y \Longrightarrow x = y).$$

3. Weak convergence theorems

In this section, we prove the approximation of common fixed points of two nonspreading mappings and an α-nonexpansive mapping in Hilbert spaces by using the iterative sequence $\{x_n\}$ defined by (1.8). First, we give a lemma which is very important for our main results.

Lemma 3.1. *Let C be a nonempty closed convex subset of a Hilbert space H and let $T : C \to C$ be an α-nonexpansive mapping. If a sequence $\{x_n\}$ in C with $x_n \rightharpoonup x^*$ and $\|x_n - Tx_n\| \to 0$, then $x^* = Tx^*$.*

Proof. Since $x_n \rightharpoonup x^*$, we conclude that $\{x_n\}$ is bounded. For any $x \in H$, define a mapping $f : H \to [0, \infty)$ by

$$f(x) = \limsup_{n\to\infty} \|x_n - x\|^2.$$

By (2.3), we obtain

$$f(x) = \limsup_{n\to\infty} \|x_n - x^*\|^2 + \|x^* - x\|^2$$

for all $x \in H$. Thus

$$f(x) = f(x^*) + \|x^* - x\|^2$$

for all $x \in H$ and

$$f(Tx^*) = f(x^*) + \|x^* - Tx^*\|^2. \tag{3.1}$$

Observe that

$$\begin{aligned} f(Tx^*) &= \limsup_{n\to\infty}\|x_n - Tx^*\|^2 \\ &= \limsup_{n\to\infty}\|x_n - Tx_n + Tx_n - Tx^*\|^2 \\ &= \limsup_{n\to\infty}\|Tx_n - Tx^*\|^2. \end{aligned} \tag{3.2}$$

Using (2.3), (3.2) and the $\alpha-$nonexpansive of T, we get

$$\begin{aligned} f(Tx^*) &= \limsup_{n\to\infty}\|Tx_n - Tx^*\|^2 \\ &\leq \limsup_{n\to\infty}\big[\alpha\|Tx_n - x^*\|^2 + \alpha\|Tx^* - x_n\|^2 + (1-2\alpha)\|x_n - x^*\|^2\big] \\ &= \limsup_{n\to\infty}\big[\alpha\|Tx_n - x_n + x_n - x^*\|^2 + \alpha\|Tx^* - x^* + x^* - x_n\|^2 \\ &\qquad +(1-2\alpha)\|x_n - x^*\|^2\big] \\ &\leq \limsup_{n\to\infty}\|x_n - x^*\|^2 + \alpha\|Tx^* - x^*\|^2 \\ &= f(x^*) + \alpha\|Tx^* - x^*\|^2. \end{aligned} \tag{3.3}$$

It follow from (3.1) and (3.3) that $(1-\alpha)\|Tx^* - x^*\|^2 = 0$. Therefore, $x^* = Tx^*$. This completes the proof. □

Lemma 3.2. *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let $T : C \to C$ be an α-nonexpansive mapping such that $F(T) \neq \emptyset$. Then $F(T)$ is closed and convex.*

Proof. First, we show that $F(T)$ is closed. Let $\{x_n\}$ be a sequence in $F(T)$ which converges to a point $x \in C$. Since T is α-nonexpansive, we have

$$\begin{aligned} \|x_n - Tx\|^2 &= \|Tx_n - Tx\|^2 \\ &\leq \alpha\|Tx_n - x\|^2 + \alpha\|Tx - x_n\|^2 + (1-2\alpha)\|x_n - x\|^2 \\ &= \alpha\|Tx - x_n\|^2 + (1-\alpha)\|x_n - x\|^2 \end{aligned}$$

for all $n \geq 1$. Taking $n \to \infty$ in the above inequality, it follows that

$$\|x - Tx\| \leq \alpha\|Tx - x\|$$

and hence $(1-\alpha)\|Tx - x\| = 0$. Therefore, $x \in F(T)$.

Next, we show that $F(T)$ is convex. For any $u, v \in F(T)$ and $t \in [0,1]$, let $z = tu + (1-t)v$. Then we have

$$u - z = (1-t)(u-v), \quad v - z = t(v-u). \tag{3.4}$$

Since T is α-nonexpansive, by (2.2) and (3.4), we get

$$\begin{aligned} \|z - Tz\|^2 &= \|tu + (1-t)v - Tz\|^2 \\ &= \|t(u - Tz) + (1-t)(v - Tz)\|^2 \\ &= t\|u - Tz\|^2 + (1-t)\|v - Tz\|^2 - t(1-t)\|u - v\|^2 \\ &\leq t[\alpha\|u - z\|^2 + \alpha\|Tz - u\|^2 + (1-2\alpha)\|u - z\|^2] \\ &\quad +(1-t)[\alpha\|v - z\|^2 + \alpha\|Tz - v\|^2 + (1-2\alpha)\|v - z\|^2] \end{aligned}$$

$$
\begin{aligned}
&\quad -t(1-t)\|u-v\|^2\\
&= t\alpha\|Tz-u\|^2+t(1-\alpha)\|u-z\|^2+(1-t)\alpha\|Tz-v\|^2\\
&\quad +(1-t)(1-\alpha)\|v-z\|^2-t(1-t)\|u-v\|^2\\
&= (1-\alpha)[t\|u-z\|^2+(1-t)\|v-z\|^2-t(1-t)\|u-v\|^2]\\
&\quad +\alpha[t\|Tz-u\|^2+(1-t)\|Tz-v\|^2-t(1-t)\|u-v\|^2]\\
&= (1-\alpha)\|t(u-z)+(1-t)(v-z)\|^2\\
&\quad +\alpha\|t(Tz-u)+(1-t)(Tz-v)\|^2\\
&= (1-\alpha)\|t(1-t)(u-v)+(1-t)t(v-u)\|^2\\
&\quad +\alpha\|Tz-(tu+(1-t)v)\|^2\\
&= \alpha\|Tz-z\|^2,
\end{aligned}
$$

which implies that $(1-\alpha)\|Tz-z\|=0$ and so $z\in F(T)$. Therefore, $F(T)$ is convex. This completes the proof. □

Theorem 3.3. *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let $S,U:C\to C$ be two nonspreading mappings and $T:C\to C$ be an α-nonexpansive mapping such that $F(S)\cap F(T)\cap F(U)\neq\emptyset$. Let $\{x_n\}$ be the sequence defined by* (1.8). *If $\liminf_{n\to\infty}(1-\alpha_n)\alpha_n>0$, $\liminf_{n\to\infty}(1-\beta_n)\beta_n>0$ and $\sum_{n=1}^{\infty}\gamma_n<\infty$, then the sequence $\{x_n\}$ converges weakly to a point $v\in F(T)\cap F(S)$.*

Proof. First, we show that $\{x_n\}$ is bounded. Since a nonspreading mapping with the nonempty fixed point set is quasi-nonexpansive, for any $p\in F(S)\cap F(T)\cap F(U)$ and $x\in C$, we have

$$
\begin{aligned}
\|x_{n+1}-p\| &= \|(1-\alpha_n)x_n+\alpha_n y_n-p\|\\
&\le (1-\alpha_n)\|x_n-p\|+\alpha_n\|y_n-p\|\\
&= (1-\alpha_n)\|x_n-p\|+\alpha_n\|(1-\beta_n)Tx_n+\beta_n Sz_n-p\|\\
&\le (1-\alpha_n)\|x_n-p\|+\alpha_n[(1-\beta_n)\|Tx_n-p\|+\beta_n\|Sz_n-p\|]\\
&\le (1-\alpha_n)\|x_n-p\|\\
&\quad +\alpha_n\big[(1-\beta_n)\|x_n-p\|+\beta_n\big[(1-\gamma_n)\|x_n-p\|+\gamma_n\|Ux_n-p\|\big]\big]\\
&\le (1-\alpha_n)\|x_n-p\|\\
&\quad +\alpha_n\big[(1-\beta_n)\|x_n-p\|+\beta_n[(1-\gamma_n)\|x_n-p\|+\gamma_n\|x_n-p\|]\big]\\
&= \|x_n-p\|
\end{aligned}
\tag{3.5}
$$

for all $n\ge 1$. Thus $\lim_{n\to\infty}\|x_n-p\|$ exists and hence $\{x_n\}$ is bounded, say $c=\lim_{n\to\infty}\|x_n-p\|$. As in (3.5), similarly, we can show that

$$
\|y_n-p\|\le\|x_n-p\|,\quad \|z_n-p\|\le\|x_n-p\| \tag{3.6}
$$

for all $n\ge 1$. Observe that, for any $p\in F(S)\cap F(T)\cap F(U)$,

$$
\begin{aligned}
\|x_{n+1}-p\| &= \|(1-\alpha_n)x_n+\alpha_n y_n-p\|\\
&\le (1-\alpha_n)\|x_n-p\|+\alpha_n\|y_n-p\|,
\end{aligned}
$$

which implies that

$$\frac{\|x_{n+1}-p\|-\|x_n-p\|}{\alpha_n}+\|x_n-p\| \quad\leq\quad \|y_n-p\| \tag{3.7}$$

for all $n \geq 1$. From (3.6), (3.7) and $c=\lim_{n\to\infty}\|x_n-p\|$, it follows that

$$\begin{aligned} c \quad &\leq \quad \liminf_{n\to\infty}\|y_n-p\| \leq \limsup_{n\to\infty}\|y_n-p\| \\ &\leq \quad \limsup_{n\to\infty}\|x_n-p\| = c \end{aligned} \tag{3.8}$$

and hence

$$\lim_{n\to\infty}\|y_n-p\| = c = \lim_{n\to\infty}\|x_n-p\|. \tag{3.9}$$

Since we have

$$\begin{aligned} \|Sz_n-p\| \quad &\leq \quad \|z_n-p\| \\ &\leq \quad \|(1-\gamma_n)x_n+\gamma_n Ux_n-p\| \\ &= \quad (1-\gamma_n)\|x_n-p\|+\gamma_n\|Ux_n-p\| \\ &\leq \quad (1-\gamma_n)\|x_n-p\|+\gamma_n\|x_n-p\| \\ &= \quad \|x_n-p\|, \end{aligned} \tag{3.10}$$

it follows from (2.2) and (3.10) that

$$\begin{aligned} \beta_n(1-\beta_n)\|Sz_n-Tx_n\|^2 \quad &= \quad \beta_n\|Sz_n-p\|^2+(1-\beta_n)\|Tx_n-p\|^2 \\ &\quad\quad -\|\beta_n(Sz_n-p)+(1-\beta_n)(Tx_n-p)\|^2 \\ &\leq \quad \beta_n\|x_n-p\|^2+(1-\beta_n)\|x_n-p\|^2 \\ &\quad\quad -\|\beta_n(Sz_n-p)+(1-\beta_n)(Tx_n-p)\|^2 \\ &= \quad \|x_n-p\|^2-\|y_n-p\|^2. \end{aligned} \tag{3.11}$$

From (3.9), (3.11) and $\liminf_{n\to\infty}(1-\beta_n)\beta_n > 0$, we obtain

$$\lim_{n\to\infty}\|Sz_n-Tx_n\| = 0. \tag{3.12}$$

Let $t_{n+1}=(1-\alpha_n)x_n+\alpha_n Sz_n$. Then we get

$$\begin{aligned} \|x_{n+1}-t_{n+1}\| \quad &= \quad \|(1-\alpha_n)x_n+\alpha_n)y_n-(1-\alpha_n)x_n-\alpha_n Sz_n\| \\ &= \quad \alpha_n\|(1-\beta_n)Tx_n+\beta_n Sz_n-Sz_n\| \\ &= \quad \alpha_n(1-\beta_n)\|Tx_n-Sz_n\|. \end{aligned} \tag{3.13}$$

Thus it follows from (3.12) and (3.13) that

$$\lim_{n\to\infty}\|x_n-t_n\| = 0 \tag{3.14}$$

and hence

$$\lim_{n\to\infty}\|t_n-p\| = \lim_{n\to\infty}\|x_n-p\| = c. \tag{3.15}$$

Since

$$\begin{aligned} \|t_{n+1}-p\|^2 \quad &= \quad \|(1-\alpha_n)x_n+\alpha_n Sz_n-p\|^2 \\ &= \quad (1-\alpha_n)\|x_n-p\|^2+\alpha_n\|Sz_n-p\|^2-\alpha_n(1-\alpha_n)\|x_n-Sz_n\|^2 \\ &\leq \quad (1-\alpha_n)\|x_n-p\|^2+\alpha_n\|x_n-p\|^2-\alpha_n(1-\alpha_n)\|x_n-Sz_n\|^2 \\ &\leq \quad \|x_n-p\|^2-\alpha_n(1-\alpha_n)\|x_n-Sz_n\|^2, \end{aligned} \tag{3.16}$$

it follows that

$$(1-\alpha_n)\alpha_n\|x_n - Sz_n\|^2 \le \|x_n - p\|^2 - \|t_{n+1} - p\|^2,$$

which implies that

$$\lim_{n\to\infty}(1-\alpha_n)\alpha_n\|x_n - Sz_n\|^2 = 0.$$

Since $\liminf_{n\to\infty}(1-\alpha_n)\alpha_n > 0$, we conclude that

$$\lim_{n\to\infty}\|x_n - Sz_n\| = 0. \tag{3.17}$$

Thus, using (1.2) and (2.1), we have

$$\begin{aligned}
\|Sx_n - x_n\|^2 &= \|Sx_n - Sz_n\|^2 + \|Sz_n - x_n\|^2 + 2\langle Sx_n - Sz_n, Sz_n - x_n\rangle \\
&\le \|x_n - z_n\|^2 + 2\langle x_n - Sx_n, z_n - Sz_n\rangle + \|Sz_n - x_n\|^2 \\
&\quad +2\langle Sx_n - Sz_n, Sz_n - x_n\rangle \\
&\le \gamma_n^2\|x_n - Ux_n\|^2 + 2\|x_n - Sx_n\|\|Sz_n - z_n\| \\
&\quad +\|Sz_n - x_n\|^2 + 2\|Sx_n - Sz_n\|\|Sz_n - x_n\| \\
&\le \gamma_n^2\|x_n - Ux_n\|^2 + 2\|x_n - Sx_n\|[\|Sz_n - x_n\| + \|x_n - z_n\|] \\
&\quad +\|Sz_n - x_n\|^2 + 2\|Sx_n - Sz_n\|\|Sz_n - x_n\| \\
&\le \gamma_n^2\|x_n - Ux_n\|^2 + 2\|x_n - Sx_n\|[\|Sz_n - x_n\| + \gamma_n\|x_n - Ux_n\|] \\
&\quad +\|Sz_n - x_n\|^2 + 2\|Sx_n - Sz_n\|\|Sz_n - x_n\|.
\end{aligned} \tag{3.18}$$

From $\sum_{n=1}^{\infty}(\gamma_n) < \infty$, (3.17) and (3.18), it follows that

$$\lim_{n\to\infty}\|x_n - Sx_n\| = 0. \tag{3.19}$$

Since $\{x_n\}$ is bounded, there exist a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ and a point $v \in C$ such that $x_{n_i} \rightharpoonup v$. By Lemma 2.4, we obtain $v \in F(S)$.

Next, we show that $v \in F(T)$. Let

$$s_{n+1} = (1-\alpha_n)x_n + \alpha_n Tx_n. \tag{3.20}$$

Similarly, as n (3.13) with (3.12), (3.17) and triangle property, we can show that $\|x_n - s_n\| \to 0$ as $n \to \infty$ and

$$\lim_{n\to\infty}\|s_n - p\| = \lim_{n\to\infty}\|x_n - p\| = c.$$

From

$$\begin{aligned}
\|s_{n+1} - p\|^2 &= \|(1-\alpha_n)x_n + \alpha_n Tx_n - p\|^2 \\
&= (1-\alpha_n)\|x_n - p\|^2 + \alpha_n\|Tx_n - p\|^2 - \alpha_n(1-\alpha_n)\|x_n - Tx_n\|^2 \\
&\le (1-\alpha_n)\|x_n - p\|^2 + \alpha_n\|x_n - p\|^2 - \alpha_n(1-\alpha_n)\|x_n - Tx_n\|^2 \\
&= \|x_n - p\|^2 - \alpha_n(1-\alpha_n)\|x_n - Tx_n\|^2,
\end{aligned} \tag{3.21}$$

it follows that

$$(1-\alpha_n)\alpha_n\|x_n - Tx_n\|^2 \le \|x_n - p\|^2 - \|s_{n+1} - p\|^2,$$

which implies that

$$\lim_{n\to\infty}(1-\alpha_n)\alpha_n\|x_n - Tx_n\| = 0. \tag{3.22}$$

Since $\liminf_{n\to\infty}(1-\alpha_n)\alpha_n > 0$, we conclude that

$$\lim_{n\to\infty}\|x_n - Tx_n\| = 0. \tag{3.23}$$

Since the subsequence $\{x_{n_i}\}$ converges weakly to the point v, we have $v \in F(T)$, that is, $v \in F(S)\cap F(T)$.

Let $\{x_{n_j}\}$ be another subsequence of $\{x_n\}$ which converges weakly to a point $v^* \in C$. Now, we show that $v = v^*$. For this, first we prove that $\lim_{n\to\infty}\|x_n - z\|$ exists for all $z \in F(S)\cap F(T)$. Similarly, as the inequality (3.16), we can show that

$$\|t_{n+1} - z\| \le \|x_n - z\|$$

and so

$$\|t_{n+1} - z\| \le \|x_n - z\| \le \|t_n - z\| + \|x_n - t_n\|.$$

Thus, from Lemma 2.1, we can conclude that $\lim_{n\to\infty}\|t_n - z\|$ exists. Consequently, $\lim_{n\to\infty}\|x_n - z\|$ exists for all $z \in F(S)\cap F(T)$. Suppose that $v \ne v^*$. Then, by Opial's condition, we have

$$\begin{aligned}\liminf_{i\to\infty}\|x_{n_i} - v\| &< \liminf_{i\to\infty}\|x_{n_i} - v^*\| = \lim_{n\to\infty}\|x_n - v^*\| \\ &= \liminf_{j\to\infty}\|x_{n_j} - v^*\| < \liminf_{j\to\infty}\|x_{n_j} - v\| \\ &= \lim_{n\to\infty}\|x_n - v\| = \liminf_{i\to\infty}\|x_{n_i} - v\|,\end{aligned}$$

which is a contradiction. Therefore, $v = v^*$, by Theorem 2.5, the sequence $\{x_n\}$ converges weakly to $v \in F(T)\cap F(S)$. This completes the proof. □

Since the class of α-nonexpansive mappings contains the classes of nonexpansive, nonspreading and hybrid mappings, respectively, from Theorem 3.3, we have the following:

Corollary 3.4. *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let $S, U : C \to C$ be two nonspreading mappings and let $T : C \to C$ be a mapping satisfying one of the following conditions:*

(i) *T is a nonexpansive mapping;*
(ii) *T is a nonspreading mappings;*
(iii) *T is a hybrid mapping*

such that $F(S)\cap F(T)\cap F(U) \ne \emptyset$. Let $\{x_n\}$ be the sequence defined by (1.8). If $\liminf_{n\to\infty}(1-\alpha_n)\alpha_n > 0$, $\liminf_{n\to\infty}(1-\beta_n)\beta_n > 0$ and $\sum_{n=1}^{\infty}\gamma_n < \infty$, then the sequence $\{x_n\}$ converges weakly to a point $v \in F(T)\cap F(S)$.

Corollary 3.5 ([7])**.** *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let $S : C \to C$ be a nonspreading mapping and let $T : C \to C$ be a nonexpansive mapping such that $F(S)\cap F(T) \ne \emptyset$. Define the sequence $\{x_n\}$ in C as follows:*

$$\begin{cases} x_1 \in C \text{ chosen arbitrary}, \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)(\beta_n Sx_n + (1-\beta_n)Tx_n) \end{cases} \tag{3.24}$$

for all $n \ge 1$. If $\liminf_{n\to\infty}(1-\alpha_n)\alpha_n > 0$ and $\liminf_{n\to\infty}(1-\beta_n)\beta_n > 0$, then the sequence $\{x_n\}$ converges weakly to a point $v \in F(T)\cap F(S)$.

Proof. Putting $\gamma_n = 0$ for all $n \ge 1$ in Corollary 3.4, we obtain Corollary 3.5. □

Corollary 3.6. ([14]) *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let $S : C \to C$ be a nonspreading mapping of C into itself such that $F(S) \neq \emptyset$. Define the sequence $\{x_n\}$ in C as follows:*

$$\begin{cases} x_1 \in C \text{ chosen arbitrary}, \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n) S x_n \end{cases} \tag{3.25}$$

for all $n \geq 1$. If $\liminf_{n\to\infty}(1-\alpha_n)\alpha_n > 0$, then the sequence $\{x_n\}$ converges weakly to a point $v \in F(S)$.

Proof. Setting $\beta = 1$ for all $n \geq 1$ in Corollary 3.5, we obtain Corollary 3.6. □

Theorem 3.7. *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let $S, U : C \to C$ be two nonspreading mappings and $T : C \to C$ be an $\alpha-$nonexpansive mapping such that $F(S) \cap F(T) \cap F(U) \neq \emptyset$. Let $\{x_n\}$ be the sequence in C defined by (1.8). If $\liminf_{n\to\infty}(1-\alpha_n)\alpha_n > 0$, $\liminf_{n\to\infty}(1-\beta_n)\beta_n > 0$ and $\liminf_{n\to\infty}(1-\gamma_n)\gamma_n > 0$, then the sequence $\{x_n\}$ converges weakly to a point $v \in F(T) \cap F(S) \cap F(U)$. Moreover, $v = \lim_{n\to\infty} P_{F(T)\cap F(S)\cap F(U)} x_n$.*

Proof. As in the proof of Theorem 3.3 (see (3.23)), we have

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0.$$

and a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ converges weakly to the point $v \in F(T)$.

Next, we show that

$$\lim_{n\to\infty} \|x_n - Sx_n\| = 0$$

and

$$\lim_{n\to\infty} \|x_n - Ux_n\| = 0.$$

For any $p \in F(S) \cap F(T) \cap F(U)$, we have

$$\begin{aligned} \gamma_n(1-\gamma_n)\|Ux_n - x_n\|^2 &= (1-\gamma_n)\|x_n - p\|^2 + \gamma_n\|Ux_n - p\|^2 \\ &\quad - \|(1-\gamma_n)(x_n - p) + \gamma_n(Ux_n - p)\|^2 \\ &\leq (1-\gamma_n)\|x_n - p\|^2 + \gamma_n\|x_n - p\|^2 \\ &\quad - \|(1-\gamma_n)(x_n - p) + \gamma_n(Ux_n - p)\|^2 \\ &= \|x_n - p\|^2 - \|z_n - p\|^2. \end{aligned} \tag{3.26}$$

Similarly, as in (3.7) and (3.8), we can show that

$$\frac{\|y_n - p\| - \|x_n - p\|}{\beta_n} + \|x_n - p\| \leq \|z_n - p\|$$

and also

$$\lim_{n\to\infty} \|z_n - p\| = c = \lim_{n\to\infty} \|x_n - p\|. \tag{3.27}$$

From (3.26), (3.27) and $\liminf_{n\to\infty}(1-\gamma_n)\gamma_n > 0$, it follows that

$$\lim_{n\to\infty} \|x_n - Ux_n\| = 0. \tag{3.28}$$

Since $\{x_{n_i}\}$ converges weakly to the point v, we have $v \in F(U)$, that is, $v \in F(T) \cap F(U)$. Again, as in the proof of Theorem 3.3 (see (3.17)), we get

$$\lim_{n\to\infty} \|x_n - Sz_n\| = 0. \tag{3.29}$$

and (see (3.18))

$$
\begin{aligned}
\|Sx_n - x_n\|^2 \le\ & \gamma_n^2\|x_n - Ux_n\|^2 + 2\|x_n \\
& -Sx_n\|[\|Sz_n - x_n\| + \gamma_n\|x_n - Ux_n\|] \\
& +\|Sz_n - x_n\|^2 + 2\|Sx_n - Sz_n\|\|Sz_n - x_n\|.
\end{aligned}
\tag{3.30}
$$

From (3.28), (3.29) and (3.30), it follows that

$$\lim_{n\to\infty}\|x_n - Sx_n\| = 0.$$

Thus we have $v \in F(S) \cap F(T) \cap F(U)$. Let $\{x_{n_k}\}$ be another subsequence of $\{x_n\}$ which converges weakly to a point $v^* \in C$. Following the proof of Theorem 3.3, we can conclude that $v = v^*$ and so the sequence $\{x_n\}$ converges weakly to $v \in F(S) \cap F(T) \cap F(U)$.

Next, we show that $v = \lim_{n\to\infty} P_{F(T)\cap F(S)\cap F(U)}x_n$. Let $v_n = P_{F(T)\cap F(S)\cap F(U)}x_n$. Then it follows from Lemma 2.3 that $\langle v - v_n, v_n - x_n\rangle \ge 0$ for all $n \ge 1$ since $v \in F(T) \cap F(S) \cap F(U)$. Therefore, it follows from (3.5) that, for any $p \in F(T) \cap F(S) \cap F(U)$, $\|x_{n+1} - p\| \le \|x_n - p\|$. Then, by Lemma 2.2, we get $v_n \to z$ for some $z \in F(T) \cap F(S) \cap F(U)$ and so $\langle v - z, z - v\rangle \ge 0$, which implies that $v = z$. This completes the proof. □

Corollary 3.8. *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let $S, U : C \to C$ be two nonspreading mappings and $T : C \to C$ be a mapping of C satisfying one of the following conditions:*

(i) *T is a nonexpansive mapping;*
(ii) *T is a nonspreading mappings;*
(iii) *T is a hybrid mapping*

such that $F(S) \cap F(T) \cap F(U) \ne \emptyset$. Let $\{x_n\}$ be the sequence in C defined by (1.8)*. If* $\liminf_{n\to\infty}(1-\alpha_n)\alpha_n > 0$, $\liminf_{n\to\infty}(1-\beta_n)\beta_n > 0$ *and* $\liminf_{n\to\infty}(1-\gamma_n)\gamma_n > 0$*, then the sequence $\{x_n\}$ converges weakly to a point $v \in F(T) \cap F(S) \cap F(U)$. Moreover, $v = \lim_{n\to\infty} P_{F(T)\cap F(S)\cap F(U)}x_n$.*

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