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CHARACTERIZATIONS FOR THE SET OF HIGHER ORDER GLOBAL STRICT MINIMIZERS

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ABSTRACT. In this article, a new solution concept of higher order global strict minimizers with respect to a nonlinear function for a scalar optimization problem is introduced. Several characterizations for the solution set via the set equivalence are established to envision the entire solution set, given any one of the solution points. To this aim, we first introduce the notion of higher order strong invexity for differentiable scalar functions. The significance of the new notion of invexity is twofold; it not only allows us to relax the notion of convexity usually associated with characterization of solution sets of optimization problems but also enables us to derive the requisite characterizations for the set of higher order strict minimizers in a lucid manner.

1. INTRODUCTION

Characterization and properties of solution sets are useful for understanding the behavior of solution methods for extremum problems that have multiple optimal solutions. Several authors have characterized the solution sets of optimization problems by imposing varied conditions on the functions involved. Mangasarian [9] presented several characterizations of the solution sets of convex extremum problems and applied them to study monotone linear complementarity problems. Jeyakumar [5] extended Mangasarian's study to derive characterizations for pseudolinear programs. Ansari [1] introduced the notion of η-pseudolinearity and derived several characterizations of solution sets of η-pseudolinear programs. Lalitha and Mehta [7] introduced the notion of h-pseudolinear function and characterized the solution sets of h-pseudolinear programs where h is a bifunction associated with the objective. Yang [14] presented characterization of solution set of a pseudoinvex extremum problem.

On the other hand, complexity of optimization problems has caused the appearance of several solution concepts. In recent years, some attention has been focused on the study of several variants of minimizers, among them the notion of strict minimizer has been widely studied. Auslender [2] introduced the notion of isolated local minima of order 1 and 2 for a scalar optimization problem. Studniarski [11] extended the notion of Auslender for isolated local minima of order m, where m is any positive integer. Ward [12] renamed the notion of isolated minima of order m as strict local minimum of order m for a scalar optimization problem. Jiménez [6] extended the ideas of Ward [12] to define the notion of strict local efficient solution of order m for the vector minimization problem. Recently, Bhatia [3] extended the notion of Ward to define a global strict minimizer of order m for a multiobjective

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optimization problem. It is worth mentioning here that the concept of local minimizer of higher order plays a crucial role in the convergence of iterative schemes and stability analysis for optimization problems, see ([4], [11]) for more details.

The intent of the paper is to characterize the set of global strict minimizers of order m for a scalar optimization problem. The paper is organized as follows. Section 2 commences by introducing a special type of strict minimizer of order m. For the purpose of studying this new solution concept, we found it most appropriate to employ the notion of strong pre-invexity of order m which is a generalization of strong convexity [8]. The section proceeds further by establishing a characterization for strong pre-invexity of order m which in turn leads to an important generalization viz. strong pseudoinvexity of order m. The notion of invariant monotonicity of order m is also introduced. A result relating the strong invexity of order m of a differentiable function to the invariant monotonicity of order m of its gradient is established. Section 3 presents certain characterizations for the set of strict minimizers of order m for higher order strong invex/pseudoinvex scalar optimization problems in terms of set equivalence.

2. Higher order global strict minimizer

In this paper, we study the following scalar minimization problem

$$\textbf{(P)} \quad \min f(x)$$
$$\text{subject to } \; x \in S\,,$$

where $f : X \subseteq R^n \to R$ and $S \subseteq X$. The notion of strict minimizer of order m for **(P)** turned out to be very fruitful in the optimization theory. Let us recall the same from literature. A point $\bar{x} \in S$ is said to be strict local minimizer of order m ($m \geq 1$, an integer) for **(P)** if there exist $c > 0$ and an $\varepsilon > 0$ such that

$$f(x) \geq f(\bar{x}) + c\|x - \bar{x}\|^m, \quad \forall\, x \in S \cap B(\bar{x}, \varepsilon)$$

where $B(\bar{x}, \varepsilon)$ denotes the ball with centre $\bar{x}$ and radius ε. If we replace the ball $B(\bar{x}, \varepsilon)$ by R^n then $\bar{x}$ is called a strict (global) minimizer of order m for **(P)**.

The following example depicts that in some cases $\bar{x}$ may fail to be strict minimizer in the above sense.

Example 2.1. Let $X = R$, $S = [0, 1]$ and $f(x) = x^3$, then $\bar{x} = 0$ is not a strict minimizer of order 1 since for any $c > 0$ there exists an x, $0 < x < c^{1/2}$ such that

$$f(x) < f(\bar{x}) + c\|x - \bar{x}\|.$$

The above example motivates us to introduce a new notion of strict minimizer of order m for **(P)** with respect to a nonlinear function ψ defined as follows.

Definition 2.2. Let $m \geq 1$ be an integer. A point $\bar{x} \in S$ is said to be strict minimizer of order m for **(P)** with respect to a nonlinear function $\psi : S \times S \to R^n$, if there exists a constant $c > 0$ such that

$$f(x) \geq f(\bar{x}) + c\|\psi(x, \bar{x})\|^m, \quad \forall\, x \in S.$$

Remark 2.3. For the problem considered in Example 2.1, $\bar{x} = 0$ failed to be strict minimizer of order m in the usual sense, however it is important to observe here that $\bar{x} = 0$ is a strict minimizer of order 1 with respect to $\psi(x, \bar{x}) = \sin^3 x - \sin^3 \bar{x}$, for $c = 1$.

Remark 2.4. The integer m plays a vital role in the above definition. For the problem considered in Example 2.1, if $\psi(x, \bar{x}) = \sin x - \sin \bar{x}$ and $c = 1$, then $\bar{x} = 0$ is a strict minimizer of order $m \geq 3$. However, $\bar{x} = 0$ is not a strict minimizer of order 1 as for any $c > 0$ there exists an x, $0 < x^3/\sin x < c$ such that

$$f(x) < f(\bar{x}) + c\|\psi(x, \bar{x})\|.$$

We recall that [13] a set $S \subseteq R^n$ is invex with respect to η if there exists $\eta : S \times S \to R^n$ such that for all $x, y \in S$ and all $\lambda \in [0, 1]$, $y + \lambda\eta(x, y) \in S$. Throughout the paper, we assume the set S to be invex.

The following assumption introduced by Mohan and Neogy [10] and was shown to hold for almost all invex sets, thereby making it a fairly reasonable assumption.

A function $\eta : S \times S \to R^n$ is said to satisfy Assumption C, if for all $x, y \in S$

$$\begin{aligned} &\eta(y, y + \lambda\eta(x, y)) = -\lambda\eta(x, y), \\ &\eta(x, y + \lambda\eta(x, y)) = (1 - \lambda)\eta(x, y), \ \forall\ \lambda \in [0, 1]\,. \end{aligned}$$

If $\eta(x, y) = x - y$, Assumption C is obviously true. Also, if Assumption C holds, then it follows [15] that for all $x, y \in S$ and all $\lambda \in [0, 1]$, $\eta(y + \lambda\eta(x, y), y) = \lambda\eta(x, y)$.

We now present a class of higher order strongly pre-invex functions as follows.

Definition 2.5. Let $S \subseteq X$ be invex with respect to η. A function $f : X \to R$ is said to be strongly pre-invex of order $m \geq 1$ with respect to η, ψ on S if there exists a constant $c > 0$ such that for all $x, y \in S$ and all $\lambda \in [0, 1]$

$$f(y + \lambda\eta(x, y)) \leq \lambda f(x) + (1 - \lambda)f(y) - c\lambda(1 - \lambda)\|\psi(x, y)\|^m.$$

If we take $\psi(x, y) = x - y$ and $\eta(x, y) = x - y$, in the above definition, it reduces to the notion of strong convexity of order m defined by Lin and Fukushima [8]. If $\psi(x, y) = 0$, then the above definition reduces to the usual notion of pre-invexity.

The following result provides a characterization for higher order differentiable strong pre-invex functions which will play a central role in deriving the main results in the subsequent section.

Theorem 2.6. *Let $X \subseteq R^n$ be an open set, $S \subseteq X$ be invex with respect to η and η, ψ satisfy Assumption C. Then a differentiable function $f : X \to R$ is strongly pre-invex of order m with respect to η, ψ on S if and only if there exists a constant $c > 0$ such that for all $x, y \in S$*

$$f(x) - f(y) \geq \nabla f(y)^t\eta(x, y) + c\|\psi(x, y)\|^m. \tag{2.1}$$

Proof. Suppose that f is strongly pre-invex of order m on S. Then there exists a constant $c > 0$ such that for $x, y \in S$ and $\lambda \in (0, 1)$

$$f(y + \lambda\eta(x, y)) \leq \lambda f(x) + (1 - \lambda)f(y) - c\lambda(1 - \lambda)\|\psi(x, y)\|^m.$$

Dividing by λ and taking limit as $\lambda \downarrow 0$, we have

$$\nabla f(y)^t\eta(x, y) \leq f(x) - f(y) - c\|\psi(x, y)\|^m.$$

Conversely, suppose that (2.1) holds for some $c > 0$. Since S is an invex set then for all $x, y \in S$, $z = y + \lambda\eta(x, y) \in S$, for all $\lambda \in (0, 1)$ and the following inequalities hold

$$f(x) - f(z) \geq \nabla f(z)^t \eta(x, z) + c\|\psi(x, z)\|^m,$$
$$f(y) - f(z) \geq \nabla f(z)^t \eta(y, z) + c\|\psi(y, z)\|^m.$$

Using Assumption C, these inequalities can be rewritten as

$$f(x) - f(z) \geq (1 - \lambda)\nabla f(z)^t \eta(x, y) + c(1 - \lambda)^m \|\psi(x, y)\|^m,$$
$$f(y) - f(z) \geq -\lambda\nabla f(z)^t \eta(x, y) + c\lambda^m \|\psi(x, y)\|^m.$$

Multiplying the above two inequalities by λ and $(1 - \lambda)$ respectively and adding, we have

(2.2)
$$f(y + \lambda\eta(x, y)) \leq \lambda f(x) + (1 - \lambda)f(y) - c\lambda(1 - \lambda)[(1 - \lambda)^{m-1} + \lambda^{m-1}]\|\psi(x, y)\|^m.$$

If $0 < m \leq 2$, then $(1 - \lambda)^{m-1} + \lambda^{m-1} \geq (1 - \lambda) + \lambda = 1$. On the other hand if $m > 2$, the real function $\varphi(\lambda) = \lambda^{m-1}$ is convex on $(0, 1)$, therefore

$$(1 - \lambda)^{m-1} + \lambda^{m-1} \geq (1/2)^{m-1}.$$

It follows from (2.2) that there exists $c' > 0$ independent of x, y, λ such that

$$f(y + \lambda\eta(x, y)) \leq \lambda f(x) + (1 - \lambda)f(y) - c'\lambda(1 - \lambda)\|\psi(x, y)\|^m.$$

Hence, f is strongly pre-invex of order m on S. □

The above theorem leads us to introduce the notion of higher order strong invexity as follows.

Definition 2.7. Let $X \subseteq R^n$ be an open set and $S \subseteq X$ be invex with respect to η. A differentiable function $f : X \to R$ is said to be strongly invex of order $m \geq 1$ with respect to η, ψ on S if there exists a constant $c > 0$ such that for all $x, y \in S$

$$f(x) - f(y) \geq \nabla f(y)^t \eta(x, y) + c\|\psi(x, y)\|^m.$$

In general, every differentiable higher order strongly pre-invex function is higher order invex. If $\psi(x, y) = 0$, then the above notion reduces to the notion of invexity.

Remark 2.8. Every strongly invex function of order m is invex. However, the converse may not be true. For example, let $X = R$, $S = [0, \pi/2)$, $f(x) = 1 + \sin x$ and $\eta(x, y) = \dfrac{(\sin x - \sin y)}{\cos y}$ then f is invex with respect to η on S. However, it is evident that f is not strongly invex of any order m with respect to η as for every $x, y \in S$, the inequality (2.1) fails to hold for any $c > 0$ and for any function $\psi(x, y)$.

We now present the following generalization of higher order strong invexity.

Definition 2.9. Let $X \subseteq R^n$ be an open set and $S \subseteq X$ be invex with respect to η. A differentiable function $f : X \to R$ is said to be strongly pseudoinvex of order m with respect to η, ψ on S if there exists a constant $c > 0$ such that for all $x, y \in S$

$$\nabla f(y)^t \eta(x, y) \geq 0 \quad \Rightarrow \quad f(x) - f(y) \geq c\|\psi(x, y)\|^m.$$

Remark 2.10. It is evident that strong invexity of order m implies strong pseudoinvexity of order m. However, converse is not true in general. For the example considered in Remark 2.8, if $\psi(x,y) = \sin x - \sin y$ and $c = 1/2^{m-1}$ then f is strongly pseudoinvex of order $m \geq 1$ with respect to η, ψ on S but is not strongly invex of any order m with respect to η.

We now introduce higher order strongly invariant monotonicity as a generalization of invariant monotonicity (Yang, Yang and Teo [15]).

Definition 2.11. A vector valued function $F : S \to R^n$ is strongly invariant monotone of order m with respect to η, ψ on S, if there exists $c > 0$ such that for all $x, y \in S$

$$\eta(x,y)^t F(y) + \eta(y,x)^t F(x) + c(\|\psi(x,y)\|^m + \|\psi(y,x)\|^m) \leq 0.$$

The following theorem associates the higher order strong invexity of a function to the higher order strong invariant monotonicity of its gradient.

Theorem 2.12. *Let $X \subseteq R^n$ be an open set and $S \subseteq X$ be invex with respect to η. If the differentiable function $f : X \to R$ is strongly invex of order m with respect to η and ψ on S, then ∇f is strongly invariant monotone of order m on S with respect to same η and ψ.*

Proof. As f is strongly invex of order m with respect to η and ψ on S, therefore

$$f(y) - f(x) \geq \nabla f(x)^t \eta(y,x) + c\|\psi(y,x)\|^m$$

and

$$f(x) - f(y) \geq \nabla f(y)^t \eta(x,y) + c\|\psi(x,y)\|^m.$$

Hence, it follows that

$$0 \geq \eta(x,y)^t \nabla f(y) + \eta(y,x)^t \nabla f(x) + c(\|\psi(x,y)\|^m + \|\psi(y,x)\|^m).$$

Therefore, ∇f is strongly invariant monotone of order m with respect to same η and ψ on S. □

3. Characterizations for solutions set

Let $\bar{S} = \{\bar{x} \in S : f(x) \geq f(\bar{x}) + c\|\psi(x,\bar{x})\|^m, \forall\, x \in S\}$ denote the set of all strict minimizers of order m with respect to a nonlinear function ψ for **(P)**. Throughout this section, we shall assume f to be real valued differentiable function defined on an open subset X of R^n, containing an invex set S, the solution set $\bar{S}$ to be nonempty and ψ satisfies Assumption C.

Lemma 3.1. *If f is strongly invex of order m and $\bar{x}, \bar{y} \in \bar{S}$ then*

$$\nabla f(\bar{x})^t \eta(\bar{y},\bar{x}) \;=\; \nabla f(\bar{y})^t \eta(\bar{x},\bar{y}) = 0.$$

Proof. For $\bar{x} \in \bar{S}$, we have for all $z \in S$

$$f(z) \geq f(\bar{x}) + c\|\psi(z,\bar{x})\|^m. \tag{3.1}$$

Since S is an invex set, $z = \bar{x} + \lambda\eta(\bar{y},\bar{x}) \in S$, for $\bar{x}, \bar{y} \in S$ and $\lambda \in (0,1)$. Therefore (3.1) can be rewritten as

$$f(\bar{x} + \lambda\eta(\bar{y},\bar{x})) \geq f(\bar{x}) + c\|\psi(\bar{x} + \lambda\eta(\bar{y},\bar{x}),\bar{x})\|^m.$$

Since the map ψ satisfies Assumption C, we obtain

$$f(\bar{x} + \lambda\eta(\bar{y}, \bar{x})) - f(\bar{x}) \geq c\lambda^m \|\psi(\bar{y}, \bar{x})\|^n .$$

Dividing by λ and taking limit as $\lambda \downarrow 0$, we have

$$\nabla f(\bar{x})^t \eta(\bar{y}, \bar{x}) \geq 0 . \tag{3.2}$$

Similarly, we obtain

$$\nabla f(\bar{y})^t \eta(\bar{x}, \bar{y}) \geq 0 . \tag{3.3}$$

As f is strongly invex of order m, it follows that ∇f is strongly invariant monotone of order m

$$\nabla f(\bar{y})^t \eta(\bar{x}, \bar{y}) + \nabla f(\bar{x})^t \eta(\bar{y}, \bar{x}) + c(\|\psi(\bar{x}, \bar{y})\|^m + \|\psi(\bar{y}, \bar{x})\|^m) \leq 0$$

which implies

$$\nabla f(\bar{y})^t \eta(\bar{x}, \bar{y}) + \nabla f(\bar{x})^t \eta(\bar{y}, \bar{x}) \leq 0 ,$$

which may further be written as

$$\nabla f(\bar{y})^t \eta(\bar{x}, \bar{y}) \leq -\nabla f(\bar{x})^t \eta(\bar{y}, \bar{x}) .$$

Using (3.3) we have

$$-\nabla f(\bar{x})^t \eta(\bar{y}, \bar{x}) \geq 0 . \tag{3.4}$$

Combining (3.2) and (3.4), we have

$$\nabla f(\bar{x})^t \eta(\bar{y}, \bar{x}) = 0.$$

Proceeding on the similar lines, we have

$$\nabla f(\bar{y})^t \eta(\bar{x}, \bar{y}) = 0.$$

□

The following results establish simple characterizations for the set of higher order strict minimizers, in terms of any one of its solution points.

Theorem 3.2. *If f be strongly invex of order m and $\bar{x} \in \bar{S}$, then $\bar{S} = S' = S''$, where*

$$S' = \{x \in S : \nabla f(x)^t \eta(\bar{x}, x) = 0\},$$
$$S'' = \{x \in S : \nabla f(x)^t \eta(\bar{x}, x) \geq 0\}.$$

Proof. If $x \in \bar{S}$, as $\bar{x} \in \bar{S}$ then by Lemma 3.1, we have

$$\nabla f(x)^t \eta(\bar{x}, x) = 0 .$$

Hence, $x \in S'$. Thus, $\bar{S} \subseteq S'$. Conversely if $x \in S'$,then

$$\nabla f(x)^t \eta(\bar{x}, x) = 0.$$

As $c > 0$, it yields that

$$\nabla f(x)^t \eta(\bar{x}, x) + c\|\psi(\bar{x}, x)\|^m \geq 0.$$

Using strong invexity of order m for f, we have

$$f(\bar{x}) \geq f(x).$$

By virtue of $\bar{x} \in \bar{S}$, we have

$$f(x) \geq f(\bar{x}) + c\|\psi(x, \bar{x})\|^m,$$

$$f(x) \geq f(\bar{x}).$$

Therefore, $f(\bar{x}) = f(x)$. Hence, $x \in \bar{S}$. Consequently, $S' = \bar{S}$. It is obvious that $\bar{S} \subseteq S''$. Further, assume that $x \in S''$. Therefore,

$$\nabla f(x)^t \eta(\bar{x}, x) \geq 0.$$

Again as $c > 0$, it follows that

$$\nabla f(x)^t \eta(\bar{x}, x) + c\|\psi(\bar{x}, x)\|^m \geq 0.$$

Using the strong invexity of order m for f, we have

$$f(\bar{x}) \geq f(x).$$

From $\bar{x} \in \bar{S}$ and above inequality, we have

$$f(\bar{x}) = f(x).$$

Therefore, $S'' = \bar{S}$. □

Remark 3.3. It is worth noting that in the above theorem the set equivalence $\bar{S} = S'$ implies that any feasible point x wherein the gradient of the objective is orthogonal to $\eta(\bar{x}, x)$ is also a solution, here and otherwise we refer by solution as the strict minimizer of order m for **(P)**, whereas the second equality $\bar{S} = S''$ allows any feasible point x wherein the gradient of the objective makes an acute angle with $\eta(\bar{x}, x)$ to be an entrant into the solution set.

Theorem 3.4. *If f is strongly invex of order m and $\bar{x} \in \bar{S}$, then $\bar{S} = S_1' = S_1''$, where*

$$S_1' = \{x \in S : \nabla f(\bar{x})^t \eta(x, \bar{x}) = \nabla f(x)^t \eta(\bar{x}, x)\},$$
$$S_1'' = \{x \in S : \nabla f(\bar{x})^t \eta(x, \bar{x}) \leq \nabla f(x)^t \eta(\bar{x}, x)\}.$$

Proof. If $x \in \bar{S}$, then from $\bar{x} \in \bar{S}$ and Lemma 3.1, we have

$$\nabla f(\bar{x})^t \eta(x, \bar{x}) \;=\; \nabla f(x)^t \eta(\bar{x}, x) = 0.$$

Hence, $x \in S_1'$. Therefore,

$$\bar{S} \subseteq S_1' . \tag{3.5}$$

It is obvious that

$$S_1' \subseteq S_1'' . \tag{3.6}$$

Further, assume that $x \in S_1''$, then

$$\nabla f(\bar{x})^t \eta(x, \bar{x}) \leq \nabla f(x)^t \eta(\bar{x}, x) .$$

Since $\bar{x} \in \bar{S}$, $\nabla f(\bar{x})^t \eta(x, \bar{x}) \geq 0$. Therefore, $\nabla f(x)^t \eta(\bar{x}, x) \geq 0$. As $c > 0$, we have

$$\nabla f(x)^t \eta(\bar{x}, x) + c\|\psi(\bar{x}, x)\|^m \geq 0.$$

Using the strong invexity of order m for f, we have

$$f(\bar{x}) \geq f(x).$$

Thus, from $\bar{x} \in \bar{S}$ and above inequality, it follows that $f(\bar{x}) = f(x)$. Hence, $x \in \bar{S}$. Consequently, $S_1'' \subseteq \bar{S}$. Further on combining this with (3.5) and (3.6) we conclude that $\bar{S} = S_1' = S_1''$. □

Remark 3.5. Let $d = \nabla f(x) - \nabla f(\bar{x})$ and if η is skew, that is, $\eta(x, \bar{x}) = -\eta(\bar{x}, x)$, then the set equivalence $\bar{S} = S_1'$ in the above theorem enables any feasible point x to be a solution if d is orthogonal to $\eta(\bar{x}, x)$, while the equality $\bar{S} = S_1''$ ensures that any feasible point x wherein d makes an acute angle with $\eta(\bar{x}, x)$ is a candidate for solution.

Remark 3.6. It is important to observe that Lemma 3.1, Theorem 3.2 and Theorem 3.4 also hold if the function f is strongly pseudoinvex of order m.

Conclusions

In this paper, we characterize the set of strict minimizers of order m defined with respect to a nonlinear function for a scalar optimization problem, when one such solution is known. These characterizations endeavor to provide the decision maker with an easy way to generate the entire solution set, in the sense of a strict minimizer of order m for strong invex optimization problem. Subsequently, enable the decision maker to choose the solution that suits him the most. However, in view of the precondition for the existence of at least one solution, it would be interesting to develop sufficient conditions for the well-posedness of the problem under consideration, which will ensure that any suitable algorithm terminates with a solution point.

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