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GENERALIZED QUASIVARIATIONAL INCLUSION PROBLEMS WITH APPLICATIONS

SAN-HUA WANG, NAN-JING HUANG*, AND DONAL O'REGAN

Abstract. In this paper, a class of generalized quasivariational inclusion problems are introduced and studied. By using the famous FKKM theorem and the Kakutani-Fan-Glicksberg fixed point theorem, the existence of solutions and the compactness of the solution sets for the generalized quasivariational inclusion problems are obtained under some suitable assumptions. As applications, some existence results for generalized upper quasivariational inclusion problems and quasioptimization problems are given in Hausdorff topological vector spaces.

1. Introduction

Let X be a nonempty subset of a Hausdorff topological vector space E, and $f : X \times X \to R$ a bifunction such that $f(x, x) \geq 0$ for all $x \in X$. Then the scalar equilibrium problem consists in finding $\bar{x} \in X$ such that

$$f(\bar{x}, y) \geq 0, \quad \forall\, y \in X.$$

It provides a unifying framework for many important problems, such as, optimization problems, variational inequality problems, complementary problems, minimax inequality problems and fixed point problems, and has been widely applied to study problems arising in economics, mechanics, and engineering science (see [3]). In recent years, lots of existence results concerning equilibrium problems and variational inequality problems have been established by many authors in different ways. For details, we refer the reader to [1, 3–5, 9, 11–16, 20–24, 27, 28, 33, 34, 36, 39] and the references therein.

Recently, Di Bella [6] introduced and studied a class of inclusion problems (for short, IP), which is formulated to find $\bar{x} \in X$ such that

$$\text{(IP)} \qquad X \subseteq F(\bar{x}),$$

where $X \subseteq E$ is a nonempty convex subset, and $F : X \to 2^E$ is a multi-valued mapping. By using the Michael continuous selection theorem and the famous Brouwer fixed point theorem, Di Bella [6] obtained a general existence result for (IP):

Theorem 1.1 ([6, Theorem 1]). *Let E be a real Hausdorff topological vector space; V a linear subspace of E; X a convex subset of E, with $ri(X) \neq \emptyset$ (where $ri(X)$ is the interior of X in the affine hull of X); K a compact subset of X; $F : X \to 2^E$*

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*Corresponding author.

a multi-valued mapping. Further, let $\mathcal{F}$ be a directed (by inclusion) family of finite-dimensional linear subspaces of V meeting K, with $V = \cup_{S\in\mathcal{F}} S$ and satisfying the following conditions:

(i) *for every $S \in \mathcal{F}$ and every compact convex set Y, with $K \cap S \subseteq Y \subseteq X \cap S$ and dim(Y)=dim($X \cap S$), one has ri(Y)$\backslash F(x) \neq \emptyset$ for all $x \in Y\backslash K$ and $x \notin$ conv(ri(Y)$\backslash F(x)$) for all $x \in Y$;*

(ii) *for every $S \in \mathcal{F}$ and every $y \in (X-X)\cap S$, the set $\{x \in X\cap S : y \in x-F(x)\}$ is closed in $X \cap S$;*

(iii) *for each $x \in X \cap \overline{V}$ such that (ri(X)$\cap V$)$\backslash F(x) \neq \emptyset$, there exist $y_0 \in$ ri(X), with $x - y_0 \in V$ and a neighborhood U of x such that $z - x + y_0 \notin F(z)$ for all $z \in U \cap K \cap V$.*

Then, there exists $\bar{x} \in K$ such that ri(X) $\cap V \subseteq F(\bar{x})$.

Di Bella [6] also pointed out that the inclusion problem includes the variational inequality problem as a special case. In fact, the equilibrium problem also can be regarded as a special case of the above inclusion problem by defining $F : X \to 2^X$ by

$$F(x) = \{y \in X : f(x, y) \geq 0\}, \ \forall x \in X.$$

Very recently, Fang and Huang [8] generalized the above inclusion problem to the following extended inclusion problem (for short, EIP), which is formulated to find $\bar{x} \in X$ such that

$$\text{(EIP)} \qquad X \subseteq F(\bar{x}, \bar{x}),$$

where X is a nonempty closed convex subset of a real Banach space E, and $F : X \times X \to 2^E$ is a multi-valued mapping. By using the auxiliary technique and the well-known Kakutani-Fan-Glicksberg fixed point theorem, Fang and Huang [8] established an existence result for (EIP):

Theorem 1.2 ([8, Theorem 2.3])**.** *Let X be a nonempty, bounded, closed and convex subset of a real reflexive Banach space E and $F : X \times X \to 2^E$ be a multi-valued mapping. Assume that the following conditions hold:*

(i) *$x \in F(y, x)$ for all $x, y \in X$;*

(ii) *for each $z \in X$ and each finite dimensional subspace D of E with $X_D = X \cap D \neq \emptyset$, the multi-valued mapping $F^c(z, \cdot) : X_D \to 2^E$ is lower semicontinuous with convex values;*

(iii) *for each $z \in X$ and each finite dimensional subspace D of E with $X_D \neq \emptyset$, the set $\{x \in X_D : X_D \subseteq F(z, x)\}$ is convex and closed;*

(iv) *if $(x_\alpha, z_\alpha) \in X \times X$, (x_α, z_α) converges to $(x, z) \in X \times X$ weakly, and $X \subseteq F(z_\alpha, x_\alpha)$ for all α, then $X \subseteq F(z, x)$.*

Then there exists $\bar{x} \in X$ such that $X \subseteq F(\bar{x}, \bar{x})$.

In 2004, Tan [38] introduced and studied a class of quasivariational inclusion problems and showed some existence of solutions under suitable conditions. In 2007, Lin, Ansari and Huang [28] considered three types of generalized vector quasi-equilibrium problems (GVQEP)(I), (GVQEP)(II) and (GVQEP)(III), which includes many known vector quasi-equilibrium problems and generalized quasi-variational inequality problems as special cases. They proved some existence results for solutions of these problems under some suitable monotonicity conditions. Some more general existence results concerning inclusion problems have been established by different methods (see, for example, [18, 26, 29–32, 37] and the references therein).

The main objective of this paper is to present some further findings concerning some recent works mentioned above in the area of generalized quasivariational inclusion problems.

Let X, Y be nonempty subsets of Hausdorff topological vector spaces E, Z, respectively. Let $S : X \to 2^X, T : X \to 2^Y$ and $F : X \times Y \times X \to 2^E$ be multi-valued mappings. We consider the following generalized quasi-variational inclusion problems (for short, GQVIP): find $\bar{x} \in X$ and $\bar{y} \in T(\bar{x})$ such that $\bar{x} \in S(\bar{x})$ and

$$\text{(GQVIP)} \qquad\qquad S(\bar{x}) \subseteq F(\bar{x}, \bar{y}, \bar{x}).$$

It is worth mentioning that the problem (GQVIP) considered in this paper is different from the generalized vector quasi-equilibrium problems (GVQEP)(I), (GVQEP)(II) and (GVQEP)(III) consider in Lin, Ansari and Huang [28]. We would like to point out that the method used in this paper is quite different from the one used in [28]. In fact, we use the FKKM theorem and fixed point theorem, while Lin, Ansari and Huang [28] employed maximal element theorem. In addition, no any monotonicity condition is assumed in this paper.

Some special cases of (GQVIP):

(I) If E and Z are real Banach spaces and $E = Z, X = Y$, $T = I$ (identity mapping) and $S(x) = X$, $F(x, y, u) = F(x, u)$ for all $x, u \in X$ and $y \in Y$, then (GQVIP) reduces to (EIP) of Fang and Huang [8].

(II) If $E = Z, X = Y$, $T = I$ and $S(x) = X$, $F(x, y, u) = F(u)$ for any $x, u \in X$ and $y \in Y$, then (GQVIP) reduces to (IP) of Di Bella [6].

By using the famous FKKM theorem and the Kakutani-Fan-Glicksberg fixed point theorem, we show the existence of solutions and the compactness of the solution sets for (GQVIP) under suitable assumptions. As applications, we give some existence results for the generalized upper quasivariational inclusion problem and the quasioptimization problem in Hausdorff topological vector spaces.

2. Preliminaries

In this section, we shall recall some definitions and lemmas used in the sequel.

Definition 2.1 ([2]). Let X and Y be two topological spaces. A multi-valued mapping $T : X \to 2^Y$ is said to be

- (i) upper semi-continuous (for short, *u.s.c.*) at $x \in X$ if, for each open set V in Y with $T(x) \subseteq V$, there exists an open neighborhood $U(x)$ of x such that $T(x') \subseteq V$ for all $x' \in U(x)$;
- (ii) lower semi-continuous (for short, *l.s.c.*) at $x \in X$ if, for each open set V in Y with $T(x) \cap V \neq \emptyset$, there exists an open neighborhood $U(x)$ of x such that $T(x') \cap V \neq \emptyset$ for all $x' \in U(x)$;
- (iii) *u.s.c.* (resp. *l.s.c.*) on X if it is *u.s.c.* (resp. *l.s.c.*) at every point $x \in X$;
- (iv) continuous on X if it is both *u.s.c.* and *l.s.c.* on X;
- (v) closed if the graph of T is closed, i.e., the set $Gr(T) = \{(x, y) \in X \times Y : y \in T(x)\}$ is closed in $X \times Y$.

Lemma 2.2 ([2]). *Let X and Y be two topological spaces, $F : X \to 2^Y$ a multi-valued mapping.*

(i) *If F is u.s.c. and closed-valued, then F is closed;*
(ii) *If F is closed and Y is compact, then F is u.s.c.;*
(iii) *If F is compact-valued, then F is u.s.c. at $x \in X$ if and only if for any net $\{x_\alpha\} \subseteq X$ with $x_\alpha \to x$ and for any net $\{y_\alpha\} \subseteq Y$ with $y_\alpha \in F(x_\alpha)$ for all α, there exist $y \in F(x)$ and a subnet $\{y_\beta\}$ of $\{y_\alpha\}$ such that $y_\beta \to y$;*
(iv) *F is l.s.c. at $x \in X$ if and only if for any $y \in F(x)$ and for any net $\{x_\alpha\}$ with $x_\alpha \to x$, there is a net $\{y_\alpha\}$ such that $y_\alpha \in F(x_\alpha)$ for all α and $y_\alpha \to y$.*

Definition 2.3 ([35, 38]). Let E and Z be Hausdorff topological vector spaces, $X \subseteq E$ a nonempty convex subset and $C \subseteq Z$ a cone. A multi-valued mapping $F : X \to 2^Z$ is said be

(i) upper [lower] C-continuous at $x \in X$ if, for any neighborhood V of the origin in Z, there exists a neighborhood U of x such that, for every $y \in U \cap X$, $F(y) \subseteq F(x) + V + C$ $[F(x) \subseteq F(y) + V - C]$;
(ii) upper [lower] C-continuous on X if F is upper [lower] C-continuous at every point of X;
(iii) upper properly C-quasiconvex on X if, for any $x, y \in X$, $\lambda \in [0, 1]$,
either $F(x) \subseteq F(\lambda x + (1-\lambda)y) + C$;
or $F(y) \subseteq F(\lambda x + (1-\lambda)y) + C$.
(iv) lower properly C-quasiconvex on X if, for any $x, y \in X$, $\lambda \in [0, 1]$,
either $F(\lambda x + (1-\lambda)y) \subseteq F(x) - C$;
or $F(\lambda x + (1-\lambda)y) \subseteq F(y) - C$.

Remark 2.4. (i) By induction, it is easy to prove that F is upper [lower] properly C-quasiconvex on X if and only if for any finite subset $\{x_1, x_2, \ldots, x_n\} \subseteq X$ and for any $x \in co\{x_1, x_2, \ldots, x_n\}$ (where $co(A)$ denotes the convex hull of A), there exists some $i \in \{1, 2, \ldots, n\}$ such that $F(x_i) \subseteq F(x) + C$ [$F(x) \subseteq F(x_i) - C$]; (ii) The above upper [lower] proper C-quasiconvexity for multi-valued mapping is a generalization of proper C-quasiconvexity for single-valued mappings in [10]. The concept of C-quasiconvexity plays an important role in the study of minimax theorems, equilibrium problems and some related problems (see, for example, [10, 11, 15, 16] and the references therein).

Definition 2.5. Let E_1, E_2 and Z be Hausdorff topological vector spaces, $X \subseteq E_1$ and $Y \subseteq E_2$ be nonempty subsets. Let $C : X \times Y \to 2^Z$ be a multi-valued mapping such that, for any $(x, y) \in X \times Y$, $C(x, y)$ is a cone with apex at the origin of Z. Let $F : X \times Y \times X \to 2^Z$ be a multi-valued mapping. For any fixed $(x, y) \in X \times Y$, F is called upper [lower] $C(x, y)$-continuous if, for any $z \in X$ and any neighborhood V of the origin in Z, there exist neighborhoods U_x, U_y and U_z of x, y and z, respectively, such that

$$F(\tilde{x}, \tilde{y}, \tilde{z}) \subseteq F(x, y, z) + V + C(x, y),\ \forall\, (\tilde{x}, \tilde{y}, \tilde{z}) \in (U_x \cap X) \times (U_y \cap Y) \times (U_z \cap X).$$

$$[F(x, y, z) \subseteq F(\tilde{x}, \tilde{y}, \tilde{z}) + V - C(x, y),\ \forall\, (\tilde{x}, \tilde{y}, \tilde{z}) \in (U_x \cap X) \times (U_y \cap Y) \times (U_z \cap X).]$$

Definition 2.6 ([7]). Let X be a nonempty subset of a vector space E. A multi-valued mapping $G : X \to 2^E$ is said be a KKM mapping if, for any finite subset $\{x_1, x_2, \ldots, x_n\} \subseteq X$, one has $co\{x_1, x_2, \ldots, x_n\} \subseteq \cup_{i=1}^n G(x_i)$.

The following lemmas are very important to establish our main results.

Lemma 2.7 ([7, FKKM Theorem]). *Let X be a nonempty subset of a Hausdorff topological vector space E, and $G : X \to 2^E$ a KKM mapping. If for any $x \in X$, $G(x)$ is closed and for at least one point $x \in X$, $G(x)$ is compact, then $\cap_{x\in X} G(x) \neq \emptyset$.*

Lemma 2.8 ([19, Kakutani-Fan-Glicksberg Fixed Point Theorem]). *Let X be a nonempty compact convex subset of a locally convex Hausdorff topological vector space E. If $S : X \to 2^X$ is u.s.c. with nonempty closed convex values, then there exists an $\bar{x} \in X$ such that $\bar{x} \in S(\bar{x})$.*

3. Main results

In this section, we shall present some existence theorems of solution for (GQVIP) under suitable assumptions by using the famous FKKM theorem and the Kakutani-Fan-Glicksberg fixed point theorem.

Theorem 3.1. *Let E and Z be locally convex Hausdorff topological vector spaces, $X \subseteq E$ and $Y \subseteq Z$ be nonempty compact convex subsets. Let $S : X \to 2^X$ and $T : X \to 2^Y$ be u.s.c. with nonempty closed convex values. Let $F : X \times Y \times X \to 2^E$ be a multi-valued mapping satisfying the following conditions:*

(i) *for any $x \in X$, $y \in Y$, and for any finite subset $\{u_1, u_2, \ldots, u_n\} \subseteq X$ and any $u \in co\{u_1, u_2, \ldots, u_n\}$, there exists some i, such that $u_i \in F(x, y, u)$;*

(ii) *for any $x \in X$, $y \in Y$, and for any $v \in S(x)$, the set $\{u \in X : v \in F(x, y, u)\}$ is closed in X;*

(iii) *for any $x \in X$, $y \in Y$, the set $\{u \in X : S(x) \subseteq F(x, y, u)\}$ is empty or convex;*

(iv) *if $(x_\alpha, y_\alpha, u_\alpha) \in X \times Y \times X$, $(x_\alpha, y_\alpha, u_\alpha)$ converges to $(x, y, u) \in X \times Y \times X$, and $S(x_\alpha) \subseteq F(x_\alpha, y_\alpha, u_\alpha)$ for all α, then $S(x) \subseteq F(x, y, u)$.*

Then (GQVIP) is solvable, i.e., there exist $\bar{x} \in X$ and $\bar{y} \in T(\bar{x})$ such that $\bar{x} \in S(\bar{x})$ and $S(\bar{x}) \subseteq F(\bar{x}, \bar{y}, \bar{x})$. Moreover, the solution set of (GQVIP) is a compact subset of X.

Proof. Define a multi-valued mapping $A : X \times Y \to 2^X$ by

$$A(x, y) = \{u \in S(x) : S(x) \subseteq F(x, y, u)\}, \ \forall\, (x, y) \in X \times Y.$$

The proof is divided into the following steps.

(I) For any $(x, y) \in X \times Y$, $A(x, y)$ is nonempty and closed.

Indeed, for every $v \in S(x)$, let

$$G(v) = \{u \in S(x) : v \in F(x, y, u)\}.$$

By assumption (i), we have $v \in G(v)$ and so $G(v) \neq \emptyset$. Note that $G(v) = S(x) \cap \{u \in X : v \in F(x, y, u)\}$. Then, by assumption (ii), it is easy to see that $G(v)$ is closed. Further, by noting that X is compact, we get that $G(v)$ is compact. Now we shall show that G is a KKM mapping. Suppose that it is not the case, then there exist $u_1, u_2, \ldots, u_n \in S(x)$ and $\lambda_i \geq 0, i = 1, 2, \ldots, n$ with $\sum_{i=1}^n \lambda_i = 1$ such that $u = \sum_{i=1}^n \lambda_i u_i \notin \cup_{i=1}^n G(u_i)$. Since $S(x)$ is convex, we know that $u \in S(x)$ and so

$$u_i \notin F(x, y, u), \ i = 1, 2, \ldots, n,$$

which contradicts assumption (i). Thus, G is a KKM mapping and it follows from Lemma 2.7 that $\cap_{v\in S(x)}G(v)\neq\emptyset$. Note that $A(x,y)=\cap_{v\in S(x)}G(v)$. We have $A(x,y)\neq\emptyset$. Moreover, it is easy to see that $A(x,y)$ is closed.

(II) For any $(x,y)\in X\times Y$, $A(x,y)$ is convex.

In fact, let $u_1,u_2\in A(x,y)$, $\lambda\in[0,1]$, $u_\lambda=\lambda u_1+(1-\lambda)u_2$. Since $u_1,u_2\in S(x)$ and $S(x)$ is convex, we have $u_\lambda\in S(x)$. In addition,

$$S(x)\subseteq F(x,y,u_i),\quad i=1,2$$

and so $u_i\in\{u\in X: S(x)\subseteq F(x,y,u)\}$ for $i=1,2$. It follows from assumption (iii) that $u_\lambda\in\{u\in X: S(x)\subseteq F(x,y,u)\}$, i.e.,

$$S(x)\subseteq F(x,y,u_\lambda).$$

Hence $u_\lambda\in A(x,y)$ and so $A(x,y)$ is convex.

(III) A is *u.s.c.*.

Since $A: X\times Y\to 2^X$ and X is compact, by Lemma 2.2(ii), we need only to show that A is closed. Let $\{(x_\alpha,y_\alpha,u_\alpha)\}\subseteq Gr(A)$ be an arbitrary net such that $(x_\alpha,y_\alpha,u_\alpha)\to(x,y,u)$. We shall show that $(x,y,u)\in Gr(A)$, i.e., $u\in A(x,y)$. Since S is *u.s.c.* and closed-valued, it follows from Lemma 2.2(i) that S is closed. Moreover, for each α, $u_\alpha\in S(x_\alpha)$. Thus, we have $u\in S(x)$. In addition,

$$S(x_\alpha)\subseteq F(x_\alpha,y_\alpha,u_\alpha),\ \text{for all } \alpha.$$

It follows from assumption (iv) that $S(x)\subseteq F(x,y,u)$. Thus $u\in A(x,y)$, and so A is closed.

(IV) Define a multi-valued mapping $M: X\times Y\to 2^{X\times Y}$ as follows:

$$M(x,y)=(A(x,y),T(x)),\ \forall\,(x,y)\in X\times Y.$$

Then, for every $(x,y)\in X\times Y$, $M(x,y)$ is a nonempty closed convex subset of $X\times Y$, and M is *u.s.c.* on $X\times Y$. By Lemma 2.8, there exists a point $(\bar{x},\bar{y})\in X\times Y$ such that $(\bar{x},\bar{y})\in M(\bar{x},\bar{y})$. Thus, $\bar{x}\in S(\bar{x})$, $\bar{y}\in T(\bar{x})$ and $S(\bar{x})\subseteq F(\bar{x},\bar{y},\bar{x})$. As a result $\bar{x}$ is a solution of (GQVIP).

(V) The solution set V of (GQVIP) is a compact subset of X.

Indeed, by noting that X is compact and $V\subseteq X$, we need only to show that V is closed in X. Let $\{x_\alpha\}\subseteq V$ be an arbitrary net such that $x_\alpha\to x\in X$. Then, for every α, $x_\alpha\in S(x_\alpha)$ and there exists $y_\alpha\in T(x_\alpha)$ such that

$$S(x_\alpha)\subseteq F(x_\alpha,y_\alpha,x_\alpha).$$

Since S is *u.s.c.* and closed-valued, it follows from Lemma 2.2(i) that S is closed. Thus, we have $x\in S(x)$. Since Y is compact and T is *u.s.c.* with closed values, it follows from Lemma 2.2(iii) that there exist $y\in T(x)$ and a subnet $\{y_\beta\}$ of $\{y_\alpha\}$ such that $y_\beta\to y$. So $(x_\beta,y_\beta,x_\beta)\to(x,y,x)$. Note that $S(x_\beta)\subseteq F(x_\beta,y_\beta,x_\beta)$ for all β. It follows from assumption (iv) that

$$S(x)\subseteq F(x,y,x).$$

Thus $x\in V$, and so V is closed. This completes the proof. □

Now we give the following example to illustrate Theorem 3.1.

Example 3.2. Let $E = Z = R$ and $X = Y = [0, 1]$. Let $S : X \to 2^X$, $T : X \to 2^Y$ and $F : X \times Y \times X \to 2^E$ be defined, respectively, by $S(x) = [x, 1]$, $T(x) = [0, x]$ and $F(x, y, u) = [u - y - 1, u + x]$ for all $x, u \in X,\ y \in Y$. Then, S and T are *u.s.c.* with nonempty closed convex values. Moreover,

(i) for any $x \in X$, $y \in Y$, and for any finite subset $\{u_1, u_2, \ldots, u_n\} \subseteq X$ and any $u \in co\{u_1, u_2, \ldots, u_n\}$, there exists some i such that $u_i \le u$, and thus $u_i \le u + x$. In addition, since $u_i \in X$, we have $u_i \ge 0$. Note that $y \ge 0$ and $u \le 1$. It follows that $u - y - 1 \le u - 1 \le 0$. So $u_i \ge u - y - 1$. Hence, $u_i \in [u - y - 1, u + x] = F(x, y, u)$.

(ii) for any $x \in X$, $y \in Y$, and for any $v \in S(x) = [x, 1]$, the set $M = \{u \in X : v \in F(x, y, u)\}$ is closed in X. Indeed, for any $u \in X$, $u - y - 1 \le u - 1 \le 0$. Then, by noting that $v \ge x \ge 0$, we have

$$\begin{aligned} M &= \{u \in X : v \in F(x, y, u)\} \\ &= \{u \in [0, 1] : v \in [u - y - 1, u + x]\} \\ &= \{u \in [0, 1] : v \le u + x\} \\ &= \{u \in [0, 1] : u \ge v - x\} \\ &= [v - x, 1]. \end{aligned}$$

Thus, M is closed in X.

(iii) for any $x \in X$, $y \in Y$, the set $N = \{u \in X : S(x) \subseteq F(x, y, u)\}$ is convex. In fact, for any $u \in X = [0, 1]$, $u - y - 1 \le u - 1 \le 0$. Then, we have

$$\begin{aligned} N &= \{u \in X : S(x) \subseteq F(x, y, u)\} \\ &= \{u \in [0, 1] : [x, 1] \subseteq [u - y - 1, u + x]\} \\ &= \{u \in [0, 1] : u + x \ge 1\} \\ &= \{u \in [0, 1] : u \ge 1 - x\} \\ &= [1 - x, 1]. \end{aligned}$$

Thus, N is convex.

(iv) if $(x_\alpha, y_\alpha, u_\alpha) \in X \times Y \times X$, $(x_\alpha, y_\alpha, u_\alpha) \to (x, y, u) \in X \times Y \times X$, and $S(x_\alpha) \subseteq F(x_\alpha, y_\alpha, u_\alpha)$, then $S(x) \subseteq F(x, y, u)$. Indeed, for each α, $S(x_\alpha) \subseteq F(x_\alpha, y_\alpha, u_\alpha)$, i.e., $[x_\alpha, 1] \subseteq [u_\alpha - y_\alpha - 1, u_\alpha + x_\alpha]$. It follows that $u_\alpha + x_\alpha \ge 1$. Since $(x_\alpha, u_\alpha) \to (x, u)$, we have

$$u + x = \lim_\alpha (u_\alpha + x_\alpha) \ge 1.$$

By noting that $y \ge 0$ and $u \le 1$, we have $u - y - 1 \le u - 1 \le 0$. Thus, $[x, 1] \subseteq [0, 1] \subseteq [u - y - 1, u + x]$, i.e., $S(x) \subseteq F(x, y, u)$.

Thus, all the conditions of Theorem 3.1 are satisfied and so Theorem 3.1 implies that there exist $\bar{x} \in X$ and $\bar{y} \in T(\bar{x})$ such that $\bar{x} \in S(\bar{x})$ and $S(\bar{x}) \subseteq F(\bar{x}, \bar{y}, \bar{x})$. Indeed, we can see that $\bar{x} = \bar{y} = 1$, the solution set $V = [\frac{1}{2}, 1]$, which is a compact subset of X.

Corollary 3.3. *Let E, Z, X, Y, S, T and F be as in Theorem 3.1. Assume that the conditions (i) and (iii) of Theorem 3.1, and the following condition hold:*

(ii)$'$ *S is l.s.c. and F is u.s.c. with nonempty closed values.*

Then (GQVIP) is solvable. Moreover, the solution set of (GQVIP) is a compact subset of X.

Proof. We need only to show that the conditions (ii) and (iv) of Theorem 3.1 hold. For any $x \in X$, $y \in Y$, and for any $v \in S(x)$, let $M = \{u \in X : v \in F(x, y, u)\}$.

We shall show that M is closed in X. Let $\{u_\alpha\} \subseteq M$ be an arbitrary net such that $u_\alpha \to u \in X$. Then, we have

$$v \in F(x, y, u_\alpha), \text{ for all } \alpha.$$

Since F is *u.s.c.* with nonempty closed values, it follows from Lemma 2.2(i) that F is closed. Thus, we have $v \in F(x, y, u)$, i.e., $u \in M$, and so M is closed in X. Therefore, the condition (ii) of Theorem 3.1 holds.

Now we shall show that the condition (iv) of Theorem 3.1 also holds. Let $(x_\alpha, y_\alpha, u_\alpha) \in X \times Y \times X$, $(x_\alpha, y_\alpha, u_\alpha) \to (x, y, u) \in X \times Y \times X$, and $S(x_\alpha) \subseteq F(x_\alpha, y_\alpha, u_\alpha)$ for all α. We need to show that $S(x) \subseteq F(x, y, u)$. Indeed, for any $v \in S(x)$, since S is *l.s.c.*, it follows from Lemma 2.2(iv) that there exists a net $\{v_\alpha\}$ such that $v_\alpha \in S(x_\alpha)$ for all α and $v_\alpha \to v$. Noting that $S(x_\alpha) \subseteq F(x_\alpha, y_\alpha, u_\alpha)$ for all α, we have

$$v_\alpha \in F(x_\alpha, y_\alpha, u_\alpha), \text{ for all } \alpha.$$

Since F is closed, we get $v \in F(x, y, u)$. Since v is arbitrary, we have $S(x) \subseteq F(x, y, u)$. Thus the condition (iv) of Theorem 3.1 holds. This completes the proof. □

From Theorem 3.1, we can obtain the following result.

Theorem 3.4. *Let E, X and S be as in Theorem 3.1. Let $F : X \times X \to 2^E$ be a multi-valued mapping satisfying the following conditions:*

- (i) *for any $x \in X$, and for any finite subset $\{u_1, u_2, \ldots, u_n\} \subseteq X$ and any $u \in co\{u_1, u_2, \ldots, u_n\}$, there exists some i, such that $u_i \in F(x, u)$;*
- (ii) *for any $x \in X$, and for any $v \in S(x)$, the set $\{u \in X : v \in F(x, u)\}$ is closed in X;*
- (iii) *for any $x \in X$, the set $\{u \in X : S(x) \subseteq F(x, u)\}$ is empty or convex;*
- (iv) *if $(x_\alpha, u_\alpha) \in X \times X$, (x_α, u_α) converges to $(x, u) \in X \times X$, and $S(x_\alpha) \subseteq F(x_\alpha, u_\alpha)$ for all α, then $S(x) \subseteq F(x, u)$.*

Then, there exists $\bar{x} \in X$ such that $\bar{x} \in S(\bar{x})$ and $S(\bar{x}) \subseteq F(\bar{x}, \bar{x})$. Moreover, the solution set is a compact subset of X.

Proof. In Theorem 3.1, let $Z = E, Y = X$, $T = I$(identity mapping) and

$$F(x, y, u) = F(x, u), \text{ for all } (x, y, u) \in X \times Y \times X.$$

Then, it is easy to see that all the conditions of Theorem 3.1 are satisfied and so Theorem 3.1 yields the conclusion. This completes the proof. □

Remark 3.5. If S is *l.s.c.* and F is *u.s.c.* with nonempty closed values, then the conditions (ii) and (iv) of Theorem 3.4 hold.

In Theorem 3.4, let $S(x) \equiv X$ for all $x \in X$, then we have the following existence result for (EIP).

Corollary 3.6. *Let E and X be as in Theorem 3.1. Let $F : X \times X \to 2^E$ be a multi-valued mapping satisfying the following conditions:*

- (i) *for any $x \in X$, and for any finite subset $\{u_1, u_2, \ldots, u_n\} \subseteq X$ and any $u \in co\{u_1, u_2, \ldots, u_n\}$, there exists some i, such that $u_i \in F(x, u)$;*
- (ii) *for any $x, v \in X$, the set $\{u \in X : v \in F(x, u)\}$ is closed in X;*

(iii) *for any* $x \in X$*, the set* $\{u \in X : X \subseteq F(x,u)\}$ *is empty or convex;*
(iv) *if* $(x_\alpha, u_\alpha) \in X \times X$*,* (x_α, u_α) *converges to* $(x,u) \in X \times X$*, and* $X \subseteq F(x_\alpha, u_\alpha)$ *for all* α*, then* $X \subseteq F(x,u)$*.*

Then (EIP) is solvable, i.e., there exists $\bar{x} \in X$ *such that* $X \subseteq F(\bar{x}, \bar{x})$*. Moreover, the solution set of (EIP) is a compact subset of* X*.*

Remark 3.7. The extended inclusion problem (EIP) was studied by Fang and Huang [8]. We would like to point out that Corollary 3.6 is quite different from Theorem 2.3 of Fang and Huang [8]. In fact, in Corollary 3.6, the existence of solution for (EIP) is obtained in a locally convex Hausdorff topological vector space, while in Theorem 2.3 of [8], it was obtained in a real reflexive Banach space. Furthermore, Corollary 3.6 shows both the existence of solution and the compactness of the solution set, while Theorem 2.3 of [8] only showed the existence of solution.

Example 3.8. Let $E = R, X = [0,1]$ and $F : X \times X \to 2^E$ be defined as follows:

$$F(x,u) = [u-1, u+x], \ \forall\, (x,u) \in X \times X.$$

We choose $x^* = u^* = 1 \in X$. Then

$$F^C(x^*, u^*) = F^C(1,1) = R \backslash F(1,1) = R \backslash [0,2] = (-\infty, 0) \cup (2, +\infty).$$

Obviously, $F^C(1,1)$ is not convex and so the condition (ii) of Theorem 2.3 of Fang and Huang [8] does not hold. Thus, we can not use Theorem 2.3 of Fang and Huang [8] to show the solvability of (EIP). However, it is easy to check that all the conditions of Corollary 3.6 are satisfied and it follows from Corollary 3.6 that (EIP) is solvable. Indeed, we can see that $\bar{x} = 1$ is a solution of (EIP). Furthermore, the solution set $V = [\frac{1}{2}, 1]$ is a compact subset of X.

From Theorem 3.4, we can obtain the following result.

Theorem 3.9. *Let* E, X *and* S *be as in Theorem 3.1. Let* $F : X \to 2^E$ *be a multi-valued mapping satisfying the following conditions:*

(i) *for any finite subset* $\{u_1, u_2, \ldots, u_n\} \subseteq X$*, and for any* $u \in co\{u_1, u_2, \ldots, u_n\}$*, there exists some* i*, such that* $u_i \in F(u)$*;*
(ii) *for any* $x \in X$*, and for any* $v \in S(x)$*, the set* $\{u \in X : v \in F(u)\}$ *is closed in X;*
(iii) *for any* $x \in X$*, the set* $\{u \in X : S(x) \subseteq F(u)\}$ *is empty or convex;*
(iv) *if* $(x_\alpha, u_\alpha) \in X \times X$*,* (x_α, u_α) *converges to* $(x,u) \in X \times X$*, and* $S(x_\alpha) \subseteq F(u_\alpha)$ *for all* α*, then* $S(x) \subseteq F(u)$*.*

Then, there exists $\bar{x} \in X$ *such that* $\bar{x} \in S(\bar{x})$ *and* $S(\bar{x}) \subseteq F(\bar{x})$*. Moreover, the solution set is a compact subset of* X*.*

Proof. In Theorem 3.4, let $F(x,u) = F(u)$ for all $x, u \in X$. Then it is easy to check that all the conditions of Theorem 3.4 are satisfied and so Theorem 3.4 yields the conclusion. This completes the proof. □

Remark 3.10. If S is *l.s.c.* and F is *u.s.c.* with nonempty closed values, then the conditions (ii) and (iv) of Theorem 3.9 hold.

Now, we consider the problem of (IP). In order to make the conditions brief, we can invoke the FKKM theorem rather than Theorem 3.9 to obtain the existence result of solution for (IP).

Theorem 3.11. *Let E be a Hausdorff topological vector space, $X \subseteq E$ a nonempty compact convex subset. Let $F : X \to 2^E$ be a multi-valued mapping satisfying the following conditions:*

(i) *for any $v \in X$, the set $\{u \in X : v \in F(u)\}$ is closed in X;*
(ii) *for any finite subset $\{u_1, u_2, \dots, u_n\} \subseteq X$, and for any $u \in co\{u_1, u_2, \dots, u_n\}$, there exists some i, such that $u_i \in F(u)$.*

Then (IP) is solvable, i.e., there exists $\bar{x} \in X$ such that $X \subseteq F(\bar{x})$. Moreover, the solution set of (IP) is a compact subset of X.

Proof. Define a multi-valued mapping $G : X \to 2^X$ as follows:

$$G(v) = \{u \in X : v \in F(u)\}, \ \forall\, v \in X.$$

Then, it follows from assumption (i) that for any $v \in X$, $G(v)$ is closed in X. Since X is compact, $G(v)$ is compact for all $v \in X$.

Now we shall show that G is a KKM mapping. Suppose that it is not the case, then there exist $u_1, u_2, \dots, u_n \in X$ and $\lambda_i \geq 0,\ i = 1, 2, \dots, n$ with $\sum_{i=1}^n \lambda_i = 1$ such that $u = \sum_{i=1}^n \lambda_i u_i \notin \cup_{i=1}^n G(u_i)$. Since X is convex, we know that $u \in X$ and so

$$u_i \notin F(u),\ i = 1, 2, \dots, n,$$

which contradicts assumption (ii). Thus G is a KKM mapping. It follows from Lemma 2.7 that $\cap_{v \in X} G(v) \neq \emptyset$. Let $\bar{x} \in \cap_{v \in X} G(v)$. Then, for any $v \in X$, we have $v \in F(\bar{x})$. Thus $X \subseteq F(\bar{x})$ and so $\bar{x}$ is a solution of (IP). Denote by V the solution set of (IP). Then, we have

$$\begin{aligned} V &= \{u \in X : X \subseteq F(u)\} \\ &= \{u \in X : v \in F(u),\ \forall\, v \in X\} \\ &= \cap_{v \in X} \{u \in X : v \in F(u)\}. \end{aligned}$$

By assumption (i), for any $v \in X$, the set $\{u \in X : v \in F(u)\}$ is closed in X, and so V is closed in X. Note that X is compact. Thus V is compact. This completes the proof. □

Remark 3.12. If F is *u.s.c.* with nonempty closed values, then the condition (i) of Theorem 3.11 holds.

Remark 3.13. The inclusion problem (IP) was studied by Di Bella [6]. However, Theorem 3.11 is quite different from Theorem 1 of Di Bella [6] in the following aspects:

(a) The assumptions of Theorem 3.11 are much more simple than that of Theorem 1 in Di Bella [6];
(b) Theorem 3.11 shows both the existence of solution and the compactness of the solution set, while Theorem 1 of Di Bella [6] only showed the existence of solution;
(c) The method of proof is different. In fact, Theorem 3.11 is proved by using the famous FKKM Theorem, while Theorem 1 of Di Bella [6] was proved by using the Michael continuous selection theorem and the well-known Brouwer fixed point theorem.

At the end of this section, we give the following example to illustrate Theorem 3.11.

Example 3.14. Let $E = R$, $X = [0, 1]$ and $F : X \to 2^E$ be defined by $F(x) = [0, x]$ for all $x \in X$. Then, by simple computation, it is easy to see that all the conditions of Theorem 3.11 are satisfied and so Theorem 3.11 shows that there exists $\bar{x} \in X$ such that $X \subseteq F(\bar{x})$. Indeed, we can see that $\bar{x} = 1$ is a solution. Moreover, the solution set $V = \{1\}$ is a compact subset of X.

4. Applications

In this section, we shall apply Theorem 3.1 to present some existence results for generalized upper quasivariational inclusion problem and quasioptimization problem.

Theorem 4.1. *Let E_1 and E_2 be locally convex Hausdorff topological vector spaces, $X \subseteq E_1$ and $Y \subseteq E_2$ be nonempty compact convex subsets. Let Z be a Hausdorff topological vector space. Suppose that*

(i) $S : X \to 2^X$ *is a continuous multi-valued mapping such that, for any $x \in X$, $S(x)$ is a nonempty closed convex subset of X;*

(ii) $T : X \to 2^Y$ *is an upper semicontinuous multi-valued mapping such that, for any $x \in X$, $T(x)$ is a nonempty closed convex subset of Y;*

(iii) $C : X \times Y \to 2^Z$ *is an upper semicontinuous multi-valued mapping such that, for any $(x, y) \in X \times Y$, $C(x, y)$ is a nonempty closed convex cone with apex at the origin of Z;*

(iv) $F : X \times Y \times X \to 2^Z$ *is upper $C(x, y)$-continuous and lower $-C(x, y)$-continuous for each $(x, y) \in X \times Y$ such that, for any $(x, y, u) \in X \times Y \times X$, $F(x, y, u) + C(x, y)$ is a closed set, and for any $(x, y) \in X \times Y$, $F(x, y, u)$ is upper properly $C(x, y)$-quasiconvex in $u \in X$.*

Then, the following generalized upper quasivariational inclusion problem (for short, GUQVIP) is solvable, i.e., there exist $\bar{x} \in X$ and $\bar{y} \in T(\bar{x})$ such that $\bar{x} \in S(\bar{x})$ and

$$\text{(GUQVIP)} \qquad F(\bar{x}, \bar{y}, x) \subseteq F(\bar{x}, \bar{y}, \bar{x}) + C(\bar{x}, \bar{y}), \ \forall\, x \in S(\bar{x}).$$

Moreover, the solution set of (GUQVIP) is a compact subset of X.

Proof. Define a multi-valued mapping $G : X \times Y \times X \to 2^X$ as follows:

$$G(x, y, u) = \{v \in X : F(x, y, v) \subseteq F(x, y, u) + C(x, y)\}, \quad \forall\, (x, y, u) \in X \times Y \times X.$$

Then, it is sufficient to show that S, T and G satisfy all the conditions of Theorem 3.1.

(I) For any $(x, y) \in X \times Y$, and for any finite subset $\{u_1, u_2, \dots, u_n\} \subseteq X$ and any $u \in co\{u_1, u_2, \dots, u_n\}$, since $F(x, y, u)$ is upper properly $C(x, y)$-quasiconvex in $u \in X$, there exists some i, such that

$$F(x, y, u_i) \subseteq F(x, y, u) + C(x, y).$$

Thus $u_i \in G(x, y, u)$.

(II) For any $(x, y) \in X \times Y$, let $M = \{u \in X : S(x) \subseteq G(x, y, u)\}$. We shall show that M is empty or convex. Let $u_1, u_2 \in M$, $\lambda \in [0, 1]$, $u_\lambda = \lambda u_1 + (1 - \lambda) u_2$. Since

$u_1, u_2 \in X$ and X is convex, we have $u_\lambda \in X$. Noting that $u_1, u_2 \in M$, for any $v \in S(x)$, we have $v \in G(x,y,u_i)$ for $i = 1,2$, i.e.,

$$F(x,y,v) \subseteq F(x,y,u_i) + C(x,y),\ i = 1,2.$$

Since $F(x,y,u)$ is upper properly $C(x,y)$-quasiconvex in $u \in X$, there exists some $i_0 \in \{1,2\}$ such that

$$F(x,y,u_{i_0}) \subseteq F(x,y,u_\lambda) + C(x,y).$$

Note that $C(x,y)$ is a convex cone with apex at the origin of Z. Thus, we have

$$\begin{aligned} F(x,y,v) &\subseteq F(x,y,u_{i_0}) + C(x,y) \\ &\subseteq F(x,y,u_\lambda) + C(x,y) + C(x,y) \\ &\subseteq F(x,y,u_\lambda) + C(x,y). \end{aligned}$$

This implies that $v \in G(x,y,u_\lambda)$. Then, by the arbitrary of v, we have $S(x) \subseteq G(x,y,u_\lambda)$. It follows that $u_\lambda \in M$ and so M is convex.

(III) For any $(x,y) \in X \times Y$, and for any $v \in S(x)$, let $N = \{u \in X : v \in G(x,y,u)\}$. We shall show that N is closed in X. Let $\{u_\alpha\} \subseteq N$ be an arbitrary net such that $u_\alpha \to u$. Since $\{u_\alpha\} \subseteq X$ and X is closed, we have $u \in X$. In addition, for each α, we have $v \in G(x,y,u_\alpha)$, i.e.,

$$F(x,y,v) \subseteq F(x,y,u_\alpha) + C(x,y),\ \forall\, \alpha.$$

For any neighborhood V of the origin in Z, since F is upper $C(x,y)$-continuous, there exists a neighborhood U of u such that

$$F(x,y,\tilde{u}) \subseteq F(x,y,u) + V + C(x,y),\ \forall\, \tilde{u} \in U \cap X.$$

Since $u_\alpha \to u$, there exists α_0 such that, for every $\alpha \geq \alpha_0$, $u_\alpha \in U \cap X$. Then, by noting that $C(x,y)$ is a convex cone with apex at the origin of Z, for any $\alpha \geq \alpha_0$, we have

$$\begin{aligned} F(x,y,v) &\subseteq F(x,y,u_\alpha) + C(x,y) \\ &\subseteq F(x,y,u) + V + C(x,y) + C(x,y) \\ &\subseteq F(x,y,u) + V + C(x,y). \end{aligned} \tag{4.1}$$

By the arbitrary of V, we can prove that $F(x,y,v) \subseteq F(x,y,u) + C(x,y)$. Indeed, suppose that it is not the case, then there exists some $z \in F(x,y,v)$ such that $z \notin F(x,y,u) + C(x,y)$. Since $F(x,y,u) + C(x,y)$ is a closed set, there exists a neighborhood V_0 of the origin in Z such that

$$(z + V_0) \cap (F(x,y,u) + C(x,y)) = \emptyset. \tag{4.2}$$

Note that Z is a topological vector space. There exists a balanced neighborhood V_1 of the origin in Z such that $V_1 \subseteq V_0$. Then, by (4.2), we have

$$(z + V_1) \cap (F(x,y,u) + C(x,y)) = \emptyset.$$

Since V_1 is balanced, we obtain that

$$(z - V_1) \cap (F(x,y,u) + C(x,y)) = \emptyset.$$

It follows that

$$0 \notin (F(x,y,u) + C(x,y)) - (z - V_1) = -z + F(x,y,u) + V_1 + C(x,y),$$

i.e.,

$$z \notin F(x,y,u) + V_1 + C(x,y),$$

which contradicts (4.1). So $F(x,y,v) \subseteq F(x,y,u) + C(x,y)$, i.e., $v \in G(x,y,u)$. It follows that $u \in N$ and so N is closed.

(IV) Let $(x_\alpha, y_\alpha, u_\alpha) \in X \times Y \times X$ with $(x_\alpha, y_\alpha, u_\alpha)$ converges to $(x,y,u) \in X \times Y \times X$, and $S(x_\alpha) \subseteq G(x_\alpha, y_\alpha, u_\alpha)$ for all α. We shall show that $S(x) \subseteq G(x,y,u)$. In fact, for any $v \in S(x)$, since S is *l.s.c.*, it follows from Lemma 2.2 (iv) that there exists a net $\{v_\alpha\}$ such that $v_\alpha \in S(x_\alpha)$ for all α and $v_\alpha \to v$. Then, we have

$$F(x_\alpha, y_\alpha, v_\alpha) \subseteq F(x_\alpha, y_\alpha, u_\alpha) + C(x_\alpha, y_\alpha),\ \forall\, \alpha. \tag{4.3}$$

For any neighborhood V of the origin in Z, since Z is a topological vector space, there exists a balanced neighborhood V_0 of the origin in Z such that $V_0 + V_0 + V_0 \subseteq V$. Moreover, since C is *u.s.c.* and F is upper $C(x,y)$-continuous and lower $-C(x,y)$-continuous for each $(x,y) \in X \times Y$, there exists α_0 such that, for any $\alpha \geq \alpha_0$,

$$C(x_\alpha, y_\alpha) \subseteq C(x,y) + V_0; \tag{4.4}$$

$$F(x_\alpha, y_\alpha, u_\alpha) \subseteq F(x,y,u) + V_0 + C(x,y); \tag{4.5}$$

and

$$F(x,y,v) \subseteq F(x_\alpha, y_\alpha, v_\alpha) + V_0 + C(x,y). \tag{4.6}$$

Noting that $C(x,y)$ is a convex cone with apex at the origin of Z, for any $\alpha \geq \alpha_0$, by (4.6),(4.3),(4.5) and (4.4), we have

$$\begin{aligned}
F(x,y,v) &\subseteq F(x_\alpha, y_\alpha, v_\alpha) + V_0 + C(x,y) \\
&\subseteq F(x_\alpha, y_\alpha, u_\alpha) + C(x_\alpha, y_\alpha) + V_0 + C(x,y) \\
&\subseteq F(x,y,u) + V_0 + C(x,y) + C(x_\alpha, y_\alpha) + V_0 + C(x,y) \\
&\subseteq F(x,y,u) + V_0 + C(x,y) + V_0 + C(x,y) + V_0 + C(x,y) \\
&= F(x,y,u) + C(x,y) + C(x,y) + C(x,y) + V_0 + V_0 + V_0 \\
&\subseteq F(x,y,u) + C(x,y) + V.
\end{aligned}$$

i.e., $F(x,y,v) \subseteq F(x,y,u) + C(x,y) + V$. Since V is arbitrary, we obtain

$$F(x,y,v) \subseteq F(x,y,u) + C(x,y)$$

and so $v \in G(x,y,u)$. Since v is arbitrary, we know that $S(x) \subseteq G(x,y,u)$.

Thus, all the conditions of Theorem 3.1 are satisfied and it follows from Theorem 3.1 that there exist $\bar{x} \in X$ and $\bar{y} \in T(\bar{x})$ such that $\bar{x} \in S(\bar{x})$ and

$$S(\bar{x}) \subseteq G(\bar{x}, \bar{y}, \bar{x}),$$

i.e.,

$$F(\bar{x}, \bar{y}, x) \subseteq F(\bar{x}, \bar{y}, \bar{x}) + C(\bar{x}, \bar{y}),\ \forall\, x \in S(\bar{x}).$$

Moreover, the solution set of (GUQVIP) is a compact subset of X. This completes the proof. □

In Theorem 4.1, if for any $(x,y) \in X \times Y$, $C(x,y) \equiv C$ ($C \subseteq Z$ is a nonempty closed convex cone with apex at the origin of Z), then we have the following result.

Corollary 4.2. *Let E_1, E_2, X and Y be as in Theorem 4.1. Let Z be a Hausdorff topological vector space, $C \subseteq Z$ a nonempty closed convex cone with apex at the origin of Z. Suppose that*

(i) *$S : X \to 2^X$ is a continuous multi-valued mapping such that, for any $x \in X$, $S(x)$ is a nonempty closed convex subset of X;*

(ii) *$T : X \to 2^Y$ is an upper semicontinuous multi-valued mapping such that, for any $x \in X$, $T(x)$ is a nonempty closed convex subset of Y;*

(iii) *$F : X \times Y \times X \to 2^Z$ is upper C-continuous and lower $-C$-continuous such that, for any $(x, y, u) \in X \times Y \times X$, $F(x, y, u) + C$ is a closed set, and for any $(x, y) \in X \times Y$, $F(x, y, u)$ is upper properly C-quasiconvex in $u \in X$.*

Then, the following upper quasivariational inclusion problem (for short, UQVIP) is solvable, i.e., there exist $\bar{x} \in X$ and $\bar{y} \in T(\bar{x})$ such that $\bar{x} \in S(\bar{x})$ and

$$\text{(UQVIP)} \qquad F(\bar{x}, \bar{y}, x) \subseteq F(\bar{x}, \bar{y}, \bar{x}) + C, \ \forall\, x \in S(\bar{x}).$$

Moreover, the solution set of (UQVIP) is a compact subset of X.

Remark 4.3. The upper quasivariational inclusion problem (UQVIP) was studied by Lin and Shei [30], Lin and Tan [31], and Tan [38]. We note that Theorem 3.1 of Tan [38] required the following assumption (A):

$$(A) \qquad \text{the dual } C^* \text{ of the cone } C \text{ has a weak}^* \text{ compact base.}$$

However, Corollary 4.2 does not require assumption (A). It is well known that the assumption (A) is equivalent to $int\, C \neq \emptyset$, which can not be satisfied in many cases (see, for example, [16, 25]). Furthermore, Corollary 4.2 shows both the existence of solution and the compactness of the solution set, while Theorem 3.1 of Tan [38] only showed the existence of solution. Therefore, Corollary 4.2 is quite different from Theorem 3.1 of Tan [38].

Corollary 4.4. *Let E_1, E_2, X, Y, Z and C be as in Corollary 4.2. Assume that the conditions (i)-(iii) of Corollary 4.2 and the following condition hold:*

(iv) *for any $(x, y) \in X \times Y$, $F(x, y, x) \subseteq C$.*

Then, the following generalized strong vector quasiequilibrium problem (for short, GSVQEP) is solvable, i.e., there exist $\bar{x} \in X$ and $\bar{y} \in T(\bar{x})$ such that $\bar{x} \in S(\bar{x})$ and

$$\text{(GSVQEP)} \qquad F(\bar{x}, \bar{y}, x) \subseteq C, \ \forall\, x \in S(\bar{x}).$$

Moreover, the solution set of (GSVQEP) is a compact subset of X.

Proof. By Corollary 4.2, there exist $\bar{x} \in X$ and $\bar{y} \in T(\bar{x})$ such that $\bar{x} \in S(\bar{x})$ and

$$F(\bar{x}, \bar{y}, x) \subseteq F(\bar{x}, \bar{y}, \bar{x}) + C, \ \forall\, x \in S(\bar{x}).$$

Noting that $F(\bar{x}, \bar{y}, \bar{x}) \subseteq C$ and C is a convex cone with apex at the origin of Z, we have

$$F(\bar{x}, \bar{y}, x) \subseteq C + C \subseteq C, \ \forall\, x \in S(\bar{x}).$$

Thus, $\bar{x}$ is a solution of (GSVQEP).

Now we shall show that the solution set V of (GSVQEP) is a compact subset of X. In fact, by noting that X is compact and $V \subseteq X$, we need only to show that V is closed in X. Let $\{x_\alpha\} \subseteq V$ be an arbitrary net such that $x_\alpha \to x \in X$. Then, for every α, we have $x_\alpha \in S(x_\alpha)$ and there exists $y_\alpha \in T(x_\alpha)$ such that

$$F(x_\alpha, y_\alpha, u) \subseteq C, \ \forall\, u \in S(x_\alpha). \tag{4.7}$$

Since S is *u.s.c.* and close-valued, it follows from Lemma 2.2(i) that S is closed. Thus, we have $x \in S(x)$. Since Y is compact and T is *u.s.c.* with closed values, it follows from Lemma 2.2(iii) that there exist $y \in T(x)$ and a subnet $\{y_\beta\}$ of $\{y_\alpha\}$ such that $y_\beta \to y$. So $(x_\beta, y_\beta) \to (x, y)$. For any $u \in S(x)$, since S is *l.s.c.*, it follows from Lemma2.2(iv) that there exists a net $\{u_\beta\}$ such that $u_\beta \in S(x_\beta)$ for all β and $u_\beta \to u$. Thus $(x_\beta, y_\beta, u_\beta) \to (x, y, u)$. Moreover, by (4.7), we have

$$F(x_\beta, y_\beta, u_\beta) \subseteq C, \ \forall\ \beta.$$

For any neighborhood U of the origin in Z, since F is lower $-C$-continuous, there exists β_0 such that, for any $\beta \geq \beta_0$,

$$\begin{aligned} F(x, y, u) &\subseteq F(x_\beta, y_\beta, u_\beta) + U + C \\ &\subseteq C + U + C \\ &\subseteq C + U. \end{aligned}$$

i.e., $F(x, y, u) \subseteq C + U$. Since U is arbitrary, we obtain that $F(x, y, u) \subseteq C$. Therefore,

$$F(x, y, u) \subseteq C, \ \forall\, u \in S(x).$$

It follows that $x \in V$ and so V is closed. This completes the proof. □

Now we consider the following quasioptimization problem (for short, QOP)(see, for example, [17, 38]):

Find $\bar{x} \in X$ and $\bar{y} \in T(\bar{x})$ such that $\bar{x} \in S(\bar{x})$ and

$$\text{(QOP)} \qquad F(\bar{x}, \bar{y}, \bar{x}) \cap \text{Min}\{F(\bar{x}, \bar{y}, S(\bar{x}))/C\} \neq \emptyset,$$

where E_1, E_2 and Z are Hausdorff topological vector spaces, $X \subseteq E_1$ and $Y \subseteq E_2$ are nonempty subsets, $C \subseteq Z$ is a cone, $S : X \to 2^X$, $T : X \to 2^Y$ and $F : X \times Y \times X \to 2^Z$ are multi-valued mappings, and $\text{Min}\{A/C\}$ denotes the set of Pareto efficient points of the set $A \subseteq Z$ with respect to the cone C.

As a consequence of Corollary 4.2, we can obtain the following sufficient conditions for the solution existence of (QOP).

Corollary 4.5. *Let E_1, E_2, X, Y, Z and C be as in Corollary 4.2. Assume that the conditions (i)-(iii) of Corollary 4.2 and the following condition hold:*

(iv) *for any $(x, y) \in X \times Y$, $F(x, y, x)$ is a compact set of Z.*

Then (QOP) is solvable.

Proof. By Corollary 4.2, there exist $\bar{x} \in X$ and $\bar{y} \in T(\bar{x})$ such that $\bar{x} \in S(\bar{x})$ and

$$F(\bar{x}, \bar{y}, x) \subseteq F(\bar{x}, \bar{y}, \bar{x}) + C, \ \forall\, x \in S(\bar{x}). \tag{4.8}$$

We claim that

$$F(\bar{x}, \bar{y}, \bar{x}) \cap \text{Min}\{F(\bar{x}, \bar{y}, S(\bar{x}))/C\} \neq \emptyset. \tag{4.9}$$

In fact, since $F(\bar{x}, \bar{y}, \bar{x})$ is a compact set, $\text{Min}\{F(\bar{x}, \bar{y}, \bar{x})/C\} \neq \emptyset$. Assume that

$$\bar{v} \in \text{Min}\{F(\bar{x}, \bar{y}, \bar{x})/C\} \quad \text{but} \quad \bar{v} \notin \text{Min}\{F(\bar{x}, \bar{y}, S(\bar{x}))/C\}.$$

It follows that there exist some $x \in S(\bar{x})$ and $v \in F(\bar{x}, \bar{y}, x)$ such that

$$\bar{v} - v \in C \backslash (C \cap (-C)).$$

By virtue of (4.8), we have $v \in F(\bar{x}, \bar{y}, \bar{x}) + C$, i.e., $v = v^* + c$ for some $v^* \in F(\bar{x}, \bar{y}, \bar{x})$ and $c \in C$. It follows that

$$\bar{v} - v^* = \bar{v} - v + v - v^* \in C \backslash (C \cap (-C)) + C \subseteq C \backslash (C \cap (-C)),$$

which contradicts the fact that $\bar{v} \in \mathrm{Min}\{F(\bar{x}, \bar{y}, \bar{x})/C\}$. Therefore, (4.9) holds and so (QOP) is solvable. This completes the proof. □

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San-hua Wang
Department of Mathematics, Nanchang University, Nanchang, Jiangxi 330031, P. R. China
E-mail address: wsh_315@163.com

Nan-jing Huang
Department of Mathematics, Sichuan University, Chengdu, Sichuan 610064, P. R. China
E-mail address: nanjinghuang@hotmail.com

Donal O'Regan
Department of Mathematics, National University of Ireland, Galway, Ireland
E-mail address: donal.oregan@nuigalway.ie