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PAINLEVÉ-KURATOWSKI CONVERGENCES OF THE SOLUTION SETS FOR PERTURBED GENERALIZED SYSTEMS*

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Abstract. In this paper, under new assumptions, which are weaker than the assumption of C-strictly monotone mapping, we obtain the Painlevé-Kuratowski convergence of the weak efficient solution sets and global efficient solution sets for the perturbed generalized system with a sequence of mappings converging in a real locally convex Hausdorff topological vector space. These results extend and improve the recent ones in the literature [10]. Several examples are given for the illustration of our results.

1. Introduction

The vector equilibrium problem is a very general mathematical model, which embraces the formats of several disciplines, as those for equilibrium problems of mathematical physics, those from game theory, those from vector variational inequality problem, (vector) complementarity problem and (vector) saddle point problem, and so on(see [9, 12]). In the literature, existence results for various types of (generalized) vector equilibrium problems have been investigated intensively (see [12, 13, 20] and the references therein).

The stability analysis of solution maps for vector equilibrium problems is an important topic in optimization theory and applications. There are some papers to discuss the upper and/or lower semicontinuity of solution maps. Cheng and Zhu [8] discussed both upper and lower semicontinuity of the solution set mapping for a weak vector variational inequality in finite dimensional spaces. Huang et al. [19] discussed the upper semicontinuity and lower semicontinuity of the solution map for parametric implicit vector equilibrium problems. By virtue of a density result and scalarization technique, Gong and Yao [14] first obtained the lower semicontinuity of the set of the efficient solutions to a parametric vector equilibrium problem with vector-valued mappings. By using the idea of Cheng and Zhu [8], Gong [17] studied the continuity of the solution mappings to parametric weak vector equilibrium problems in topological vector spaces. Chen et al. [7] studied the lower semicontinuity and continuity of the solution mapping to a parametric generalized vector equilibrium problem involving set-valued mappings. Hou et al. [20] obtained results on the existence and semicontinuity of solutions for generalized Ky Fan inequality problems with trifunctions. Chen and Li [6] discussed the continuity of various efficient

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solution sets for a parametric generalized vector equilibrium problem without the uniform compactness assumption and improved the results of [14, 17]. Under a key assumption, Peng and Yang [27] obtained the lower semicontinuity of the solution maps for two classes of weak parametric generalized vector equilibrium problems when the f-solution set is a general set by removing the assumption of C-strict monotonicity.

Recently, there are some stability results for the vector optimization and vector variational inequality with a sequence of sets converging in the sense of Painlevé-Kuratowski, e.g., [10, 11, 21, 24, 25]. Huang [21] discussed the convergence of the approximate efficient sets to the efficient set of vector-valued and set-valued optimization problems in the sense of Painlevé-Kuratowski and Mosco. Lucchetti and Miglierina [25] investigated the Painlevé-Kuratowski set convergence of the solution sets of the perturbed problems both in the given space and its image space for a convex vector optimization problem. Fang et al. [11] established Painlevé-Kuratowski upper and lower convergences of the solution sets of the perturbed set-valued weak variational inequality with a sequence of converging mappings in a Banach space. Very recently, Fang and Li [10] obtained the Painlevé-Kuratowski convergence of the efficient solution sets, the weak efficient solution sets and various proper efficient solution sets for the perturbed generalized system with a sequence of mapping converging.

In this paper, under new assumptions, which are weaker than the assumption of C-strict monotonicity mapping, we obtain the Painlevé-Kuratowski convergence of the weak efficient solution sets and global efficient solution sets for the perturbed generalized system with a sequence of mappings converging in a real locally convex Hausdorff topological vector space. These results improve the recent ones in the literature [10]. Several examples are given for the illustration of our results.

2. Preliminaries

Throughout this paper, let X be a real Hausdorff topological vector space, Y be a real locally convex Hausdorff topological vector space and Y^* be the topological dual space of Y, Z be a topological space. Let C be a closed convex pointed cone in Y with nonempty topological interior int$C$.

Let

$$C^* := \{f \in Y^* : f(y) \geq 0,\ \forall y \in C\}$$

be the dual cone of C. Denote the quasi-interior of C^* by $C^\sharp$, i.e.,

$$C^\sharp := \{f \in Y^* : f(y) > 0,\ \forall y \in C \setminus \{0\}\}.$$

It is easy to see that $C^\sharp \neq \emptyset$ if and only if C has a base.

Let A be a nonempty subset of X and $F : A \times A \to Y$ be a bifunction. We consider the following generalized system

$$(GS) \text{ Find } x \in A \text{ such that } F(x, y) \notin -K,\ \forall y \in A,$$

where $K \cup \{0\}$ is a convex cone in Y.

For a sequence of bifunctions $F_n : A \times A \to Y, n = 1, 2, \ldots$, we define a sequence of generalized system

$$(GS)_n \text{ Find } x_n \in A \text{ such that } F_n(x_n, y) \notin -K,\ \forall y \in A,$$

where $K \cup \{0\}$ is a convex cone in Y.

For each $f \in C^* \setminus \{0\}$, let V_f and V_f^n denote the set of f-efficient solutions to the (GS) and $(GS)_n$, i.e.,

$$V_f = \{x \in A : \ f(F(x,y)) \geq 0, \ \forall y \in A\},$$

$$V_f^n = \{x_n \in A : \ f(F_n(x_n,y)) \geq 0, \ \forall y \in A\}.$$

A vector $x \in A$ is called a global efficient solution to $(GS)_n$ if there exists a point convex cone $H \subset Y$, with $C \setminus \{0\} \subset \text{int}H$, such that $F_n(x,A) \cap ((-H) \setminus \{0\}) = \emptyset$. The set of global efficient solutions to $(GS)_n$ is denoted by I_n^G.

A vector $x \in A$ is called a positive proper efficient solution to the (GS) if there exists $f \in C^\sharp$ such that

$$f(F(x,y)) \geq 0, \text{ for all } y \in A.$$

Throughout this paper, we always assume $V_f \neq \emptyset$ and $V_f^n \neq \emptyset$.

In this paper, under new assumptions, we discuss the Painlevé-Kuratowski convergence of the weak efficient solution sets and the global efficient solution sets of $(GS)_n$.

Definition 2.1 ([21])**.** Let X be a normed space. A sequence of sets $\{A_n \subset X : n \in N\}$ is said to converge in the sense of Painlevé-Kuratowski(P.K.) to A(denoted as $A_n \xrightarrow{P.K.} A$) if

$$\limsup_{n\to\infty} A_n \subset A \subset \liminf_{n\to\infty} A_n$$

with

$$\liminf_{n\to\infty} A_n := \{x \in X | \exists (x_n), x_n \in A_n, \forall n \in N, x_n \to x\},$$

$$\limsup_{n\to\infty} A_n := \{x \in X | \exists (n_k), \exists (x_{n_k}), x_{n_k} \in A_{n_k}, \forall k \in N, x_{n_k} \to x\}.$$

Definition 2.2 ([10, 26])**.** Let $F_n, F : X \to Y (n \in N)$ be vector-valued mappings and let $U(x)$ be the family of neighborhoods of x. We say that $(F_n)_{n\in N}\Gamma_C$-converges to F(denoted as $F_n \xrightarrow{\Gamma_C} F$) if for every $x \in X$:

(i) $\forall\ U \in U(x), \forall \epsilon \in \text{int}C, \exists n_{\epsilon,U} \in N$ such that $\forall n \geq n_{\epsilon,U}, \exists x_n \in U$ such that

$$F_n(x_n) \in F(x) + \epsilon - C;$$

(ii) $\forall \epsilon \in \text{int}C, \exists U_\epsilon \in U(x), k_\epsilon \in N$ such that $\forall x' \in U_\epsilon, \forall n \geq k_\epsilon$,

$$F_n(x') \in F(x) - \epsilon + C.$$

Definition 2.3 ([29])**.** Let $F_n, F : X \to Y (n \in N)$ be vector-valued mappings. We say that F_n continuously converges to F if the fact that $x_n \to x$ implies that $F_n(x_n) \to F(x)$.

Definition 2.4. Let A be a nonempty convex subset in X. The mapping $F(\cdot, y)$ is called to be C-concave on A if, for each fixed $y \in A$, for every $x_1, x_2 \in A, t \in [0,1], tF(x_1,y) + (1-t)F(x_2,y) \in F(tx_1 + (1-t)x_2, y) - C$.

Definition 2.5. Let A be a nonempty convex subset in X. The mapping $F(\cdot, y)$ is called to be C-strictly concave on A if, for each fixed $y \in A$, for every $x_1, x_2 \in A(x_1 \neq x_2), t \in (0,1), tF(x_1,y) + (1-t)F(x_2,y) \in F(tx_1 + (1-t)x_2, y) - \text{int}C$.

3. Painlevé-Kuratowski convergence for f-efficient solution sets

The next part in this paper, set $F_0 = F, V_f^0 = V_f$.

Lemma 3.1. *Let $f \in C^* \setminus \{0\}$, A be a convex set. For $n = 0, 1, 2, \dots,$ assume that the following conditions are satisfied:*
(i) *For each $x \in A, F_n(x, x) = \{0\}$;*
(ii) *For each $y \in A, F_n(\cdot, y)$ is C-strictly concave mapping on A.*
Then, V_f^n is a singleton.

Proof. Suppose that there exist $x_1, x_2 \in V_f^n$ and $x_1 \neq x_2$, then we have

$$f(F_n(x_1, y)) \geq 0, \ \forall \ y \in A. \tag{3.1}$$

$$f(F_n(x_2, y)) \geq 0, \ \forall \ y \in A. \tag{3.2}$$

As $x_1, x_2 \in A$, and A is a convex set, then for all $t \in [0, 1]$,

$$tx_1 + (1 - t)x_2 \in A.$$

By (3.1) and (3.2), we get

$$f(F_n(x_1, tx_1 + (1 - t)x_2)) \geq 0. \tag{3.3}$$

$$f(F_n(x_2, tx_1 + (1 - t)x_2)) \geq 0. \tag{3.4}$$

By assumptions (i) and (ii), we have

$$\begin{aligned} & tF_n(x_1, tx_1 + (1 - t)x_2) + (1 - t)F_n(x_2, tx_1 + (1 - t)x_2) \\ \subset \ & F_n(tx_1 + (1 - t)x_2, tx_1 + (1 - t)x_2) - \text{int}C = -\text{int}C. \end{aligned}$$

Since $f \in C^* \setminus \{0\}$, we have

$$f(tF_n(x_1, tx_1 + (1 - t)x_2) + (1 - t)F_n(x_2, tx_1 + (1 - t)x_2)) < 0. \tag{3.5}$$

By (3.3) and (3.4), we get

$$f(tF_n(x_1, tx_1 + (1 - t)x_2) + (1 - t)F_n(x_2, tx_1 + (1 - t)x_2)) \geq 0,$$

which contradicts to (3.5). Thus, V_f^n is a singleton. □

Remark 3.2. In Lemma 3.1, we obtain V_f^n is a singleton under new assumptions, where the assumptions are weaker than the assumption of C-strict monotonicity. Therefore, Lemma 3.1 extends the corresponding results of [6, 7, 10, 14, 17, 18].

The following example is given to illustrate this case.

Example 3.3. Let $X = R, Y = R^2, C = R^2_+ = \{x = (x_1, x_2) : x_1 \geq 0, x_2 \geq 0\}$. Let $A = [0, 1]$, it is clear that A is a convex set.

Define the mappings $F_n : A \times A \longrightarrow R^2$ by

$$F_n(x, y) = \left(-\frac{1}{2}\left(\left(x - \frac{1}{n}\right)^2 - \left(y - \frac{1}{n}\right)^2\right), -2\left(\left(x - \frac{1}{n}\right)^2 - \left(y - \frac{1}{n}\right)^2\right)\right), \ \forall x, y \in A.$$

It is clear that condition (i) of Lemma 3.1 is satisfied. For each $y \in A$, $F_n(\cdot, y)$ is C-strictly concave mappings on A. In fact, for every $x_1, x_2 \in A(x_1 \neq x_2)$ and $t \in (0,1)$, we have

$$\begin{aligned} F_n(tx_1+(1-t)x_2, y) = &\Big(-\frac{1}{2}\Big(\Big(tx_1+(1-t)x_2-\frac{1}{n}\Big)^2-\Big(y-\frac{1}{n}\Big)^2\Big),\\ &-2\Big(\Big(tx_1+(1-t)x_2-\frac{1}{n}\Big)^2-\Big(y-\frac{1}{n}\Big)^2\Big)\Big)\\ \in&\Big(-\frac{1}{2}\Big(t\Big(x_1-\frac{1}{n}\Big)^2+(1-t)\Big(x_2-\frac{1}{n}\Big)^2-\Big(y-\frac{1}{n}\Big)^2\Big),\\ &-2\Big(t\Big(x_1-\frac{1}{n}\Big)^2+(1-t)\Big(x_2-\frac{1}{n}\Big)^2-\Big(y-\frac{1}{n}\Big)^2\Big)\Big)+\text{int C}\\ =&tF_n(x_1,y)+(1-t)F_n(x_2,y)+\text{int C}. \end{aligned}$$

Thus, condition (ii) of Lemma 3.1 is satisfied. Hence, it is easy to obtain that $V_f^n = \{\frac{1}{n}\}$ is a singleton.

But $F_n(\cdot,\cdot)$ is not C-strictly monotone on $A \times A$. In fact, for any $x, y \in A$ and $x \neq y$, and

$$F_n(x,y)+F_n(y,x) = (0,0) \notin -\text{int}C.$$

Hence, the corresponding results in [6,7,10,14,17,18] are not applicable.

Lemma 3.4. *Let A be a nonempty compact convex set. For $n = 0,1,2,\dots,$ assume that the following conditions are satisfied:*

(i) *For each $x \in A, F_n(x,x) = \{0\}$;*

(ii) *For each $y \in A, F_n(\cdot,y)$ is C-strictly concave mapping on A;*

(iii) *For each $y \in A$, $-F_n(\cdot,y) \xrightarrow{\Gamma_C} -F(\cdot,y)$.*

Then, for each $f \in C^* \setminus \{0\}$, $\lim_{n\to\infty} V_f^n = V_f$.

Proof. In view of Lemma 3.1, for $n = 0,1,2,\dots, V_f^n$ is a singleton. Let $\{x_n\} = V_f^n$ and $\{x\} = V_f$. Then, we have $x_n \in A$ and $f(F_n(x_n,y)) \geq 0,\ \forall\ y \in A$. Take an arbitrary subnet $\{x_{n_k}\} \subset \{x_n\} \subset A$, we may assume that $x_{n_k} \longrightarrow z(z \neq x)$. Thus, we have

$$f(F_{n_k}(x_{n_k},y)) \geq 0,\ \forall\ y \in A.$$

Since $-F_n(\cdot,y) \xrightarrow{\Gamma_C} -F(\cdot,y)$, it follows from definition 2.2 that for any $\epsilon \in \text{int}C$, $\exists N_\epsilon \in N$ such that

$$-F_{n_k}(x_{n_k},y) \in -F(z,y)-\epsilon+C,\ \forall\ n_k \geq N_\epsilon.$$

Furthermore, we have $f(F(z,y)) \geq f(F_{n_k}(x_{n_k},y)) - f(\epsilon)$.

By the arbitrariness of ϵ, we get

$$f(F(z,y)) \geq f(F_{n_k}(x_{n_k},y)) \geq 0. \tag{3.6}$$

Noting that $z \notin V_f$, there exists $y_0 \in A$ such that

$$f(F(z,y_0)) < 0, \tag{3.7}$$

which contradicts to (3.6). Therefore $z = x$, i.e., $x_{n_k} \longrightarrow x$ as $n_k \longrightarrow \infty$. Since the subnet $\{x_{n_k}\} \subset \{x_n\}$ is arbitrary, we have $x_n \longrightarrow x$ as $n \longrightarrow \infty$. That is, $\lim_{n\longrightarrow\infty} V_f^n = V_f$. □

Now we give an example to show this case.

Example 3.5. Let $X = R, Y = R^2, C = R^2_+ = \{x = (x_1, x_2) : x_1 \geq 0, x_2 \geq 0\}$. Let $A = [0, 1]$, it is clear that A is a compact convex set.

Define the mappings $F_n : A \times A \longrightarrow R^2$ by

$$F(x, y) = \Big(-\frac{1}{3}(x^2 - y^2), -\frac{1}{2}(x^2 - y^2)\Big), \ n = 0, \ \forall x, y \in A.$$

$$F_n(x, y) = \bigg(-\frac{1}{3}\Big(\Big(x - \frac{1}{n}\Big)^2 - \Big(y - \frac{1}{n}\Big)^2\Big), -\frac{1}{2}\Big(\Big(x - \frac{1}{n}\Big)^2 - \Big(y - \frac{1}{n}\Big)^2\Big)\bigg), \quad n = 1, 2, \dots, \ \forall x, y \in A.$$

It is clear that conditions (i) (ii) of Lemma 3.4 are satisfied. It is easy to obtain that $V_f^n = \frac{1}{n}$ and $V_f = 0$. The condition (iii) of Lemma 3.4 can be checked as follows: In fact, (i) $\forall \epsilon \in \text{int}C, \exists U_\epsilon = (x - \frac{1}{n}, x + \frac{1}{n}) \in U(x), \exists N_\epsilon$ such that $\forall\ x' \in U_\epsilon = (x - \frac{1}{n}, x + \frac{1}{n}), \forall\ n \geq N_\epsilon$, we have

$$-F_n(x', y) = \bigg(\frac{1}{3}\Big(x'^2 - y^2 - \frac{2x'}{n} + \frac{2y}{n}\Big), \frac{1}{2}\Big(x'^2 - y^2 - \frac{2x'}{n} + \frac{2y}{n}\Big)\bigg),$$

$$-F_n(x', y) + F(x, y) = \bigg(\frac{1}{3}(x'^2 - x^2) + \frac{2}{3n}(y - x'), \frac{1}{2}(x'^2 - x^2) + \frac{1}{n}(y - x')\bigg)$$

From the above two inequalities, we obtain

$$-F_n(x', y) \in -F(x, y) - \epsilon + C.$$

(ii) $\forall\ U \in U(x), \forall \epsilon \in \text{int}C, \exists n_{\epsilon,U} \in N$ such that $\forall n \geq n_{\epsilon,U}, \exists x_n = x \in U$ such that

$$-F_n(x, y) + F(x, y) = \Big(\frac{2y}{3n} - \frac{2x}{3n}, \frac{y}{n} - \frac{x}{n}\Big),$$

Thus, we have

$$-F_n(x_n, y) \in -F(x, y) + \epsilon - C.$$

Therefore, it follows from Lemma 3.4, for each $f \in C^* \setminus \{0\}$, $\lim_{n\to\infty} V_f^n = V_f = 0$.

By virtue of Lemma 3.1 and Lemma 3.4, using the similar method of Lemma 2.3 in [10], we can easily obtain the following Lemma.

Lemma 3.6. *Let A be a nonempty compact convex set. For $n = 0, 1, 2, \dots,$ assume that the following conditions are satisfied:*

(i) *For each $x \in A, F_n(x, x) = \{0\}$;*

(ii) *For each $y \in A, F_n(\cdot, y)$ is C-strictly concave mapping on A;*

(iii) *For each $y \in A$, $-F_n(\cdot, y) \xrightarrow{\Gamma_C} -F(\cdot, y)$.*

Then, we have $\cup_{f\in S} V_f^n \longrightarrow \cup_{f\in S} V_f$ in the sense of Painlevé-Kuratowski, that is,

$$\limsup_{n\to\infty} \cup_{f\in S} V_f^n \subset \cup_{f\in S} V_f \subset \liminf_{n\to\infty} \cup_{f\in S} V_f^n,$$

where $S \in \{C^* \setminus \{0\}, C^\sharp\}$.

Remark 3.7. In [10], Fang et al. obtained the results under the assumption of C-strict monotonicity, but in Lemma 3.6, we obtain the same results only using the C-strict concavity of F with respect to the first variable.

4. Painlevé-Kuratowski convergence of the weak efficient solution sets

Let I^W and I_n^W denote the set of the weak efficient solution sets to (GS) and $(GS)_n$, i.e.,

$$I^W = \{x \in A | F(x, y) \in Y \setminus -\text{int C}, \forall\ y \in A\},$$

$$I_n^W = \{x_n \in A | F_n(x_n, y) \in Y \setminus -\text{int C}, \forall\ y \in A\}.$$

Similarly as in the proof of Lemma 2.1 in Ref. [15], we can get the following Lemma.

Lemma 4.1. *Suppose that int$C \neq \emptyset$ and for $n = 0, 1, 2, \ldots, \forall\ x \in A, F_n(x, \cdot)$ is C-convexlike on A. Then,*

$$I_n^W = \cup_{f \in C^* \setminus \{0\}} V_f^n.$$

Theorem 4.2. *Let A be a nonempty compact convex set. For $n = 0, 1, 2, \ldots,$ assume that the following conditions are satisfied:*

(i) *For each $x \in A, F_n(x, x) = \{0\}$;*

(ii) *For each $y \in A, F_n(\cdot, y)$ is C-strictly concave mapping on A;*

(iii) *For each $y \in A$, $-F_n(\cdot, y) \xrightarrow{\Gamma_C} -F(\cdot, y)$;*

(iv) *For each $x \in A, F_n(x, \cdot)$ is C-convexlike on A;*

(v) *int$C \neq \emptyset$.*

Then, we have $I_n^W \longrightarrow I^W$ in the sense of Painlevé-Kuratowski.

Proof. (i) We need to prove $\limsup_{n\to\infty} I_n^W \subset I^W$. Suppose to the contrary that there exists $x \in \limsup_{n\to\infty} I_n^W$ with $x \notin I^W$.

From $x \in \limsup_{n\to\infty} I_n^W$, we have $x = \lim_{k\to\infty} x_{n_k}$, where $x_{n_k} \in I_{n_k}^W$ and $\{n_k\}$ is a subnet of N. Thus, we have

$$F_{n_k}(x_{n_k}, y) \in Y \setminus -\text{int}C,\ \forall\ y \in A. \tag{4.1}$$

Noting that $x \notin I^W$, there exists $y_0 \in A$ such that

$$F(x, y_0) \in -\text{int}C. \tag{4.2}$$

For (4.1), in particular, we have

$$F_{n_k}(x_{n_k}, y_0) \in Y \setminus -\text{int}C.$$

Since $-F_n(\cdot, y_0) \xrightarrow{\Gamma_C} -F(\cdot, y_0)$, we have $\forall \epsilon \in \text{int}C, \exists U_\epsilon \in U(x), k_\epsilon \in N$ such that $\forall x_{n_k} \in U_\epsilon, \forall n_k \geq k_\epsilon$,

$$F(x, y_0) \in F_{n_k}(x_{n_k}, y_0) - \epsilon + C. \tag{4.3}$$

By the arbitrariness of ϵ, we get

$$F(x, y_0) \in Y \setminus -\text{int}C,$$

which contradicts to (4.2). Hence, the proof is complete.

(ii) We need to prove that $I^W \subset \liminf_{n\to\infty} I_n^W$. In view of Lemma 4.1, we have

$$I_n^W = \cup_{f\in C^*\setminus\{0\}} V_f^n, \ n = 0, 1, 2, \ldots.$$

From Lemma 3.6, it is easy to obtain the conclusion. Hence, the proof is complete. □

Example 4.3. Let $X = R, Y = R^2, C = R_+^2 = \{x = (x_1, x_2) : x_1 \geq 0, x_2 \geq 0\}$. Let $A = [0, 1]$, it is clear that A is a compact convex set.

Define the mappings $F_n : A \times A \longrightarrow R^2$ by

$$F(x, y) = (-(e^y + 10)(\sin(x-1) - \sin(y-1)), -(\sin(x-1) - \sin(y-1))), \quad n = 0, \ \forall\ x, y \in A.$$

$$F_n(x,y) = \Big(-(e^y+10)\Big(\sin\Big(x-1-\frac{1}{(n+10)^2}\Big) - \sin\Big(y-1-\frac{1}{(n+10)^2}\Big)\Big), -\Big(\sin\Big(x-1-\frac{1}{(n+10)^2}\Big) - \sin\Big(y-1-\frac{1}{(n+10)^2}\Big)\Big)\Big), \quad n = 1, 2, \ldots, \forall\ x, y \in A.$$

Clearly, it is easy to show conditions (i) (iii) (v) in Theorem 4.2 are satisfied.

The condition (ii) of Theorem 4.2 can be checked as follows: Let $n = 0, f(x) = -(e^y + 10)(\sin(x-1) - \sin(y-1)), g(x) = -(\sin(x-1) - \sin(y-1))$, we have $f'(x) = -(e^y+10)\cos(x-1), f''(x) = (e^y+10)\sin(x-1) < 0, g'(x) = -\cos(x-1), g''(x) = \sin(x-1) < 0$. Thus, for each $y \in A, F(\cdot, y)$ is a C-strictly concave mapping on A.

The condition (iv) of Theorem 4.2 can be checked as follows: Let $h(y) = -(e^y + 10)(\sin(x-1) - \sin(y-1)), p(y) = -(\sin(x-1) - \sin(y-1))$, we have $h'(y) = -e^y\sin(x-1) + e^y\sin(y-1) + (e^y+10)\cos(y-1), h''(y) = -e^y\sin(x-1) + 2e^y\cos(y-1) - 10\sin(y-1) > 0, p'(y) = \cos(y-1), p''(y) = -\sin(y-1) > 0$. Thus, for each $x \in A, F(x, \cdot)$ is C-convexlike on A.

Using the similar method, it is easy to check F_n satisfies conditions (ii) (iv) of Theorem 4.2.

Therefore, by Theorem 4.2, $I_n^W \longrightarrow I^W$ in the sense of Painlevé-Kuratowski.

5. Painlevé-Kuratowski convergence of the global efficient solution sets

In this section, we discuss the Painlevé-Kuratowski convergence of the global efficient solution sets of $(GS)_n$.

By the Theorem 2.1 of [16], we know the following Lemma establishes.

Lemma 5.1. *Suppose that for each $x \in A, F(x, \cdot)$ is C-convexlike on A. If C has a base, then*

$$I^G = \cup_{f\in C^\sharp} V_f.$$

By virtue of Lemma 3.6 and Lemma 5.1, we can also obtain the Painlevé-Kuratowski convergence of the global efficient solution sets.

Theorem 5.2. *Let A be a nonempty compact convex set. For $n = 0, 1, 2, \dots$, assume that the following conditions are satisfied:*
(i) *For each $x \in A, F_n(x, x) = \{0\}$;*
(ii) *For each $y \in A, F_n(\cdot, y)$ is C-strictly concave mapping on A;*
(iii) *For each $y \in A$, $-F_n(\cdot, y) \xrightarrow{\Gamma_C} -F(\cdot, y)$;*
(iv) *For each $x \in A, F_n(x, \cdot)$ is C-convexlike on A;*
(v) *$intC \neq \emptyset$ and $C^\sharp \neq \emptyset$.*
Then, we have $I_n^G \longrightarrow I^G$ in the sense of Painlevé-Kuratowski.

Proof. In a similar way to the proof of Theorem 4.2, with suitable modifications, we can obtain the conclusion. □

Remark 5.3. In [10], under the assumption of C-strict monotonicity, the Painlevé-Kuratowski convergence of the global efficient solution sets is obtained. In our paper, we obtain the Painlevé-Kuratowski convergence of the global efficient solution sets by using assumption (ii) in Theorem 5.2, which is weaker than C-strict monotonicity. Therefore, the results extend and improve the corresponding ones obtained in [10].

The following example is given to illustrate Theorem 5.2.

Example 5.4. Let $X = R, Y = R^2, C = R_+^2 = \{x = (x_1, x_2) : x_1 \geq 0, x_2 \geq 0\}$. Let $A = [-1, 0]$, it is clear that A is a compact convex set.

Define the mappings $F_n : A \times A \longrightarrow R^2$ by

$$F(x,y) = \left(-\frac{3}{5}\Big(\sin x - \sin y\Big), -\Big(\sin x - \sin y\Big)\right), \ n = 0, \ \forall\ x, y \in A.$$

$$F_n(x,y) = \left(-\frac{3}{5}\Big(\sin\Big(x - \frac{1}{n+15}\Big) - \sin\Big(y - \frac{1}{n+15}\Big)\Big), -\Big(\sin\Big(x - \frac{1}{n+15}\Big) - \sin\Big(y - \frac{1}{n+15}\Big)\Big)\right),$$
$$n = 1, 2, \dots, \forall x, y \in A.$$

Clearly, it is easy to show all conditions of Theorem 5.2 are satisfied. Therefore, by Theorem 5.2, we know $I_n^G \longrightarrow I^G$ in the sense of Painlevé-Kuratowski.

The following example is given to show the assumption (iii) in Theorem 5.2 is essential.

Example 5.5. Let $X = R, Y = R^2, C = R_+^2 = \{x = (x_1, x_2) : x_1 \geq 0, x_2 \geq 0\}$. Let $A = [0, 1]$, it is clear that A is a compact convex set.

Define the mappings $F_n : A \times A \longrightarrow R^2$ by

$$F(x,y) = (-y(x^2 - y^2), -y(x^2 - y^2)), \ n = 0, \ \text{for all } x, y \in A.$$

$$F_n(x,y) = \left(-\Big(32 + \frac{1}{n}\Big)((x-1)^2 - (y-1)^2), -\Big(32 + \frac{1}{n}\Big)((x-1)^2 - (y-1)^2)\right),$$
$$n = 1, 2, \dots, \ \forall x, y \in A.$$

Thus, conditions (i) (ii) (iv) (v) in Theorem 5.2 are satisfied.

It follows from a direct computation that

$$I_n^G = \{1\}, \ I^G = \{0\}.$$

Obviously, I_n^G is not convergent to I^G, the reason is that it does not satisfy the assumption (iii) in Theorem 5.2. Indeed, let $x = \frac{1}{4}, y = 0, \exists \epsilon = (\frac{1}{2}, \frac{1}{2}) \in \text{int}C, \forall\, U_\epsilon \in U(x), \exists x = \frac{1}{4} \in U_\epsilon, \forall\, n \geq 1$ such that

$$\begin{aligned} -F_n(x,y) + F(x,y) = & \Bigg(\Big(32 + \frac{1}{n}\Big)(x^2 - y^2 - 2x + 2y) - yx^2 + y^3, \\ & \Big(32 + \frac{1}{n}\Big)(x^2 - y^2 - 2x + 2y) - yx^2 + y^3 \Bigg) \\ = & \Big(-14 - \frac{7}{16n}, -14 - \frac{7}{16n} \Big) \notin -\epsilon + C. \end{aligned}$$

Therefore, the assumption (iii) in Theorem 5.2 is essential.

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