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FIXED POINT RESULTS FOR CYCLIC CONTRACTION SATISFYING GENERALIZED ALTERING DISTANCES AND APPLICATION

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Abstract. A sufficient condition for the existence and uniqueness of fixed point for a new variant of cyclic contractive mapping, named as weakly cyclic contractive mappings, involving a generalized altering distance function in complete metric spaces are provided. Examples are given to support the usability of our results. At the end of the results, an application to the study of existence and uniqueness of solutions for a class of two-point boundary value problem of second order differential equation is discussed by using the fixed point results.

1. Introduction and preliminaries

The celebrated Banach Contraction Principle is a fundamental piece both in several branches of functional analysis and in many applications. This important fixed point theorem can be stated as follows

Theorem 1.1 ([2]). *Let $(\mathcal{X}, d)$ be a complete metric space and $\mathcal{T}$ be a self-map of $\mathcal{X}$ satisfying:*

$$d(\mathcal{T}x, \mathcal{T}y) \leq kd(x, y), \ \forall\, x, y \in \mathcal{X}, \tag{1.1}$$

where k is a constant in $(0, 1)$. Then, $\mathcal{T}$ has a unique fixed point $\xi \in \mathcal{X}$.

Due to its relevance, generalizations of Banach's fixed point theorem have been studied by many authors (see e.g. [15] and references cited therein).

The fact that condition (1.1) implies continuity of $\mathcal{T}$, suggests in a natural way the question of obtaining fixed point results for metric spaces where the involved self-map is not necessarily continuous. This question is answered by Kirk et al. [14] and turned the area of investigation of fixed point by introducing cyclic representations and cyclic contractions, which can be stated as follows:

A mapping $\mathcal{T} : \mathcal{A} \cup \mathcal{B} \to \mathcal{A} \cup \mathcal{B}$ is called cyclic if $\mathcal{T}(\mathcal{A}) \subseteq \mathcal{B}$ and $\mathcal{T}(\mathcal{B}) \subseteq \mathcal{A}$, where $\mathcal{A}, \mathcal{B}$ are nonempty subsets of a metric space $(\mathcal{X}, d)$. Moreover, $\mathcal{T}$ is called cyclic contraction if there exists $k \in (0, 1)$ such that $d(\mathcal{T}x, \mathcal{T}y) \leq kd(x, y)$ for all $x \in \mathcal{A}$ and $y \in \mathcal{B}$. Notice that although a contraction is continuous, cyclic contraction need not to be. This is one of the important gains of this theorem which motivates, in a natural way, the following notion:

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Definition 1.2 (See [14, 17]). Let $(\mathcal{X}, d)$ be a complete metric space. Let p be a positive integer, $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_p$ be nonempty subsets of $\mathcal{X}$, $\mathcal{Y} = \bigcup_{i=1}^{p} \mathcal{A}_i$ and $\mathcal{T} : \mathcal{Y} \to \mathcal{Y}$. Then $\mathcal{Y}$ is said to be a cyclic representation of $\mathcal{Y}$ with respect to $\mathcal{T}$ if

(i) $\mathcal{A}_i$, $i = 1, 2, \ldots, p$, are nonempty closed sets, and
(ii) $\mathcal{T}(\mathcal{A}_1) \subseteq \mathcal{A}_2, \ldots, \mathcal{T}(\mathcal{A}_{p-1}) \subseteq \mathcal{A}_p, \mathcal{T}(\mathcal{A}_p) \subseteq \mathcal{A}_1$.

Following [14], a number of fixed point theorems on cyclic representation of $\mathcal{Y}$ with respect to a self-mapping $\mathcal{T}$ have appeared (see e.g. [1, 6, 11, 12, 16–19]).

To continue the investigation specified in [14], a new variant of cyclic contractive mappings satisfying generalized altering distance function, which is followed by the existence and uniqueness of fixed points for such mappings is discussed here. The obtained result generalizes and improves many existing theorems in the literature. Some examples are given in the support of our results. Finally, an application to the study of existence and uniqueness of solutions for a class of two-point boundary value problem of second order differential equation is presented.

2. Main results

In the sequel, we designate the set of all real nonnegative numbers by $\mathbb{R}^+$ and the set of all natural numbers by $\mathbb{N}$.

To introduce a new variant of cyclic contraction we need the notion of an altering distance function.

Definition 2.1 ([13]). A function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is called an altering distance function if the following properties are satisfied:

(a) φ is continuous and non-decreasing, and
(b) $\varphi(t) = 0 \Leftrightarrow t = 0$.

Definition 2.2 ([20]). A function $\psi : {\mathbb{R}^+}^3 \to \mathbb{R}^+$ is said to be a generalized altering distance function if

(i) ψ is continuous,
(ii) ψ is non-decreasing in each variable,
(iii) $\psi(t_1, t_2, t_3) = 0 \Leftrightarrow t_1 = t_2 = t_3 = 0$.

$\mathcal{F}_3$ will denote the set of all functions ψ satisfying conditions (i)–(iii).

The following are examples of generalized altering distance functions with three variables.

Example 2.3. (a) $\psi(t_1, t_2, t_3) = k \max\{t_1, t_2, t_3\}, \;\; k > 0$;

(b) $\psi(t_1, t_2, t_3) = \frac{\max\{t_1, t_2, t_3\}}{1 + \max\{t_1, t_2, t_3\}}$;

(c) $\psi(t_1, t_2, t_3) = t_1^p + t_2^q + t_3^r, \;\; p, q, r \geq 1$.

Now we can state the notion of cyclic contraction under generalized altering distance function as :

Definition 2.4. Let $(\mathcal{X}, d)$ be a metric space. Let p be a positive integer, $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_p$ be nonempty subsets of $\mathcal{X}$ and $\mathcal{Y} = \cup_{i=1}^{p} \mathcal{A}_i$. An operator $\mathcal{T} : \mathcal{Y} \to \mathcal{Y}$ is called cyclic contractive under generalized altering distance function, if

(I) $\mathcal{Y} = \cup_{i=1}^{p} \mathcal{A}_i$ is a cyclic representation of $\mathcal{Y}$ with respect to $\mathcal{T}$;

(II) for any $(x, y) \in \mathcal{A}_i \times \mathcal{A}_{i+1}$, $i = 1, 2, \ldots, p$ (with $\mathcal{A}_{p+1} = \mathcal{A}_1$),

$$\Psi_1(d(\mathcal{T}x, \mathcal{T}y)) \leq \psi_1(d(x, y), d(x, \mathcal{T}x), d(y, \mathcal{T}y)) - \psi_2(d(x, y), d(x, \mathcal{T}x), d(y, \mathcal{T}y)),$$

where ψ_1 and ψ_2 are generalized altering distance functions (in $\mathcal{F}_3$) and $\Psi_1(t) = \psi_1(t, t, t)$.

It is easy to acquire the following examples of cyclic contractive mapping from Example 2.3(a):

Example 2.5. Let $\mathcal{X} = [0, 1]$ with the usual metric. Suppose $\mathcal{A}_1 = [0, \frac{1}{4}]$ and $\mathcal{A}_2 = [\frac{1}{4}, 1]$ and $\mathcal{X} = \bigcup_{i=1}^{2} \mathcal{A}_i$. Define $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ such that

$$\mathcal{T}x = \begin{cases} \frac{1}{4}, & x \in [0, 1), \\ 0, & x = 1. \end{cases} \tag{2.1}$$

Clearly, $\mathcal{A}_1$ and $\mathcal{A}_2$ are closed subsets of $\mathcal{X}$. Moreover $\mathcal{T}(\mathcal{A}_i) \subset \mathcal{A}_{i+1}$ for $i = 1, 2$, so that $\bigcup_{i=1}^{2} \mathcal{A}_i$ is a cyclic representation of $\mathcal{X}$ with respect to $\mathcal{T}$. Furthermore, if $\psi_1, \psi_2 : {\mathbb{R}^+}^3 \to \mathbb{R}^+$ denote

$$\psi_1(t_1, t_2, t_3) = \max\{t_1, t_2, t_3\} \text{ and } \psi_2(t_1, t_2, t_3) = \frac{1}{4} \max\{t_1, t_2, t_3\}.$$

Then $\psi_1, \psi_2 \in \mathcal{F}_3$.

Now we show that $\mathcal{T}$ satisfies cyclic contractive condition (II).

For $x \in \mathcal{A}_1$, $y \in \mathcal{A}_2$ (or $x \in \mathcal{A}_2$, $y \in \mathcal{A}_1$).

• When $x \in [0, \frac{1}{4}]$ and $y \in [\frac{1}{4}, 1)$, we deduce $d(\mathcal{T}x, \mathcal{T}y) = 0$ and equation (II) is trivially satisfied.

• When $x \in [0, \frac{1}{4}]$ and $y = 1$, we deduce $d(\mathcal{T}x, \mathcal{T}y) = \frac{1}{2}$ and

$$t_1 = |x - 1|, \ t_2 = |x - \frac{1}{4}|, \ t_3 = 1,$$

then equation (II) holds as it reduces to $\frac{1}{4} < \frac{3}{8}$.

Our main result is the following.

Theorem 2.6. *Let $(\mathcal{X}, d)$ be a complete metric space, $p \in \mathbb{N}$, $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_p$ nonempty closed subsets of $\mathcal{X}$ and $\mathcal{Y} = \cup_{i=1}^{p} \mathcal{A}_i$. Suppose the mapping $\mathcal{T} : \mathcal{Y} \to \mathcal{Y}$ is cyclic contractive under generalized altering distance function, for some $\psi_1, \psi_2 \in \mathcal{F}_3$. Then $\mathcal{T}$ has a unique fixed point. Moreover, the fixed point of $\mathcal{T}$ belongs to $\cap_{i=1}^{p} \mathcal{A}_i$.*

Proof. Let $x_0 \in \mathcal{A}_1$ (such a point exists since $\mathcal{A}_1 \neq \emptyset$). Define a sequence $\{x_n\}$ in $\mathcal{X}$ by:

$$x_{n+1} = \mathcal{T}x_n, \quad n = 0, 1, 2, \ldots.$$

If there is $k \in \mathbb{N} \cup \{0\}$ such that $x_k = x_{k+1}$, then $x_k = x_n$ for all $n \geq k$, so x_k is a fixed point of $\mathcal{T}$ and $x_k \in \cap_{i=1}^{p} \mathcal{A}_i$.

Then, we assume that

$$x_n \neq x_{n+1}, \ \forall n \in \mathbb{N} \cup \{0\}. \tag{2.2}$$

We shall prove that the sequence $\{d(x_n, x_{n+1})\}$ is non-increasing with

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{2.3}$$

Indeed, suppose that, for some $n \in \mathbb{N}$,

$$d(x_{n+1}, x_{n+2}) > d(x_n, x_{n+1}).$$

Using this together with the properties of generalized altering distance functions ψ_1, ψ_2, we get

$$\begin{aligned}\Psi_1(d(x_{n+1}, x_{n+2})) &\leq \psi_1(d(x_n, x_{n+1}), d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})) \\ &\quad -\psi_2(d(x_n, x_{n+1}), d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})) \\ &\leq \Psi_1(d(x_{n+1}, x_{n+2})) \\ &\quad -\psi_2(d(x_n, x_{n+1}), d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})).\end{aligned}$$

This implies that

$$\psi_2(d(x_n, x_{n+1}), d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})) = 0$$

which yields $d(x_n, x_{n+1}) = d(x_{n+1}, x_{n+2}) = 0$. Hence, we obtain a contradiction with (2.2). We deduce that

$$d(x_{n+1}, x_{n+2}) \leq d(x_n, x_{n+1}), \quad \forall\, n \in \mathbb{N}. \tag{2.4}$$

Then, $\{d(x_{n+1}, x_{n+2})\}$ is a non-increasing sequence of positive real numbers. This implies that there exists $r \geq 0$ such that

$$\lim_{n\to\infty} d(x_{n+1}, x_{n+2}) = r. \tag{2.5}$$

Since

$$\begin{aligned}\Psi_1(d(x_{n+1}, x_{n+2})) &\leq \psi_1\big(d(x_n, x_{n+1}), d(x_n, x_{n+1}), d(x_n, x_{n+1})) \\ &\quad - \psi_2(d(x_n, x_{n+1}), d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})) \\ &= \Psi_1(d(x_n, x_{n+1})) - \psi_2(d(x_n, x_{n+1}), d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})),\end{aligned} \tag{2.6}$$

we deduce, passing to the limit as $n \to \infty$ in (2.6) and using continuity of Ψ_1 and ψ_2, that

$$\Psi_1(r) \leq \Psi_1(r) - \psi_2(r, r, r),$$

which implies that $\psi_2(r, r, r) = 0$, and thus $r = 0$. Hence, (2.3) is proved.

Now, we shall prove that $\{x_n\}$ is a Cauchy sequence in $(\mathcal{X}, d)$. Suppose to the contrary, that $\{x_n\}$ is not a Cauchy sequence. Then there exists $\varepsilon > 0$ for which we can find two sequences of positive integers $\{m(k)\}$ and $\{n(k)\}$ such that for all positive integers k,

$$n(k) > m(k) > k, \quad d(x_{m(k)}, x_{n(k)}) \geq \varepsilon, \quad d(x_{m(k)}, x_{n(k)-1}) < \varepsilon. \tag{2.7}$$

Using (2.7) and the triangle inequality, we get

$$\begin{aligned}\varepsilon &\leq d(x_{n(k)}, x_{m(k)}) \\ &\leq d(x_{m(k)}, x_{n(k)-1}) + d(x_{n(k)-1}, x_{n(k)}) \\ &< \varepsilon + d(x_{n(k)}, x_{n(k)-1}).\end{aligned}$$

Passing to the limit as $k \to \infty$ in the above inequality and using (2.3), we obtain

$$\lim_{k\to\infty} d(x_{n(k)}, x_{m(k)}) = \varepsilon. \tag{2.8}$$

On the other hand, for all k, there exists $j(k) \in \{1, .., p\}$ such that $n(k) - m(k) + j(k) \equiv 1[p]$. Then $x_{m(k)-j(k)}$ (for k large enough, $m(k) > j(k)$) and $x_{n(k)}$ lie in different adjacently labelled sets $\mathcal{A}_i$ and $\mathcal{A}_{i+1}$ for certain $i \in \{1, .., p\}$.

Using the triangle inequality, we get

$$\begin{aligned}
&|d(x_{m(k)-j(k)}, x_{n(k)}) - d(x_{n(k)}, x_{m(k)})| \leq d(x_{m(k)-j(k)}, x_{m(k)}) \\
&\leq \sum_{l=0}^{j(k)-1} d(x_{m(k)-j(k)+l}, x_{m(k)-j(k)+l+1}) \\
&\leq \sum_{l=0}^{p-1} d(x_{m(k)-j(k)+l}, x_{m(k)-j(k)+l+1}) \longrightarrow 0 \text{ as } k \longrightarrow \infty \text{ (from (2.3))},
\end{aligned}$$

which, by (2.8), implies that

$$\lim_{k\to\infty} d(x_{m(k)-j(k)}, x_{n(k)}) = \varepsilon. \tag{2.9}$$

Again, using the triangle inequality, we get

$$|d(x_{m(k)-j(k)}, x_{n(k)+1}) - d(x_{m(k)-j(k)}, x_{n(k)})| \leq d(x_{n(k)}, x_{n(k)+1}).$$

Passing to the limit as $k \to \infty$ in the above inequality, and using (2.9), we get

$$\lim_{k\to\infty} d(x_{m(k)-j(k)}, x_{n(k)+1}) = \varepsilon. \tag{2.10}$$

Therefore, from the inequality

$$|d(x_{n(k)+1}, x_{m(k)-j(k)+1}) - d(x_{m(k)-j(k)}, x_{n(k)+1})| \leq d(x_{m(k)-j(k)}, x_{m(k)-j(k)+1}),$$

we deduce, passing to the limit as $k \to \infty$, and using (2.3) and (2.10), that

$$\lim_{k\to\infty} d(x_{m(k)-j(k)+1}, x_{n(k)+1}) = \varepsilon. \tag{2.11}$$

Hence, by the continuity of Ψ_1 and (2.11), we get

$$\Psi_1(\varepsilon) = \lim_{k\to\infty} \Psi_1(d(\mathcal{T}x_{m(k)-j(k)}, \mathcal{T}x_{n(k)})). \tag{2.12}$$

Using (II), we obtain

$$\begin{aligned}
&\Psi_1(d(x_{m(k)-j(k)+1}, x_{n(k)+1})) \\
&\quad \leq \psi_1(d(x_{m(k)-j(k)}, x_{n(k)}), d(x_{m(k)-j(k)+1}, x_{m(k)-j(k)}), d(x_{n(k)+1}, x_{n(k)})) \\
&\quad -\psi_2(d(x_{m(k)-j(k)}, x_{n(k)}), d(x_{m(k)-j(k)+1}, x_{m(k)-j(k)}), d(x_{n(k)+1}, x_{n(k)})),
\end{aligned} \tag{2.13}$$

holds only when k is sufficiently large that $m(k) - j(k) > 0$. Now, it follows from (2.9) that

(2.14)
$$\lim_{k\to\infty} \Psi_1(d(\mathcal{T}x_{m(k)-j(k)}, \mathcal{T}x_{n(k)})) \leq \psi_1(\varepsilon, 0, 0) - \psi_2(\varepsilon, 0, 0) \leq \Psi_1(\varepsilon) - \psi_2(\varepsilon, 0, 0).$$

Now, combining (2.12) with the above inequality, we get

$$\Psi_1(\varepsilon) \leq \Psi_1(\varepsilon) - \psi_2(\varepsilon, 0, 0) \tag{2.15}$$

which implies that $\psi_2(\varepsilon,0,0)=0$, a contradiction since $\varepsilon>0$. Thus we proved that $\{x_n\}$ is a Cauchy sequence in $(\mathcal{X},d)$.

Since $(\mathcal{X},d)$ is complete, there exists $\xi\in\mathcal{X}$ such that

$$\lim_{n\to\infty} x_n=\xi. \tag{2.16}$$

We shall prove that

$$\xi\in\bigcap_{i=1}^{p}\mathcal{A}_i. \tag{2.17}$$

From condition (I), and since $x_0\in\mathcal{A}_1$, we have $\{x_{np}\}_{n\geq 0}\subseteq\mathcal{A}_1$. Since $\mathcal{A}_1$ is closed, from (2.16), we get that $\xi\in\mathcal{A}_1$. Again, from the condition (I), we have $\{x_{np+1}\}_{n\geq 0}\subseteq\mathcal{A}_2$. Since $\mathcal{A}_2$ is closed, from (2.16), we get that $\xi\in\mathcal{A}_2$. Continuing this process, we obtain (2.17).

Now, we shall prove that ξ is a fixed point of $\mathcal{T}$. Indeed, from (2.17), since for all n, there exists $i(n)\in\{1,2,\dots,p\}$ such that $x_n\in\mathcal{A}_{i(n)}$, applying (II) with $x=\xi$ and $y=x_n$, we obtain

$$\begin{aligned}\Psi_1(d(\mathcal{T}\xi,x_{n+1})) &= \Psi_1(d(\mathcal{T}\xi,\mathcal{T}x_n))\\ &\leq \psi_1(d(\xi,x_n),d(\xi,\mathcal{T}\xi),d(x_n,x_{n+1}))\\ &\quad -\psi_2(d(\xi,x_n),d(\xi,\mathcal{T}\xi),d(x_n,x_{n+1})),\end{aligned} \tag{2.18}$$

for all n. Passing to the limit as $n\to\infty$ in (2.18), and using (2.16), we get

$$\begin{aligned}\Psi_1(d(\xi,\mathcal{T}\xi)) &\leq \psi_1(0,d(\xi,\mathcal{T}\xi),0)-\psi_2(0,d(\xi,\mathcal{T}\xi),0)\\ &\leq \Psi_1(d(\xi,\mathcal{T}\xi))-\psi_2(0,d(\xi,\mathcal{T}\xi),0)\end{aligned}$$

which holds unless $\psi_2(0,d(\xi,\mathcal{T}\xi),0)=0$, so

$$\xi=\mathcal{T}\xi \tag{2.19}$$

that is, ξ is a fixed point of $\mathcal{T}$.

Finally, we prove that ξ is the unique fixed point of $\mathcal{T}$. Assume that ζ is another fixed point of $\mathcal{T}$, that is, $\mathcal{T}\zeta=\zeta$. By the condition (I), this implies that $\zeta\in\bigcap_{i=1}^{p}\mathcal{A}_i$. Then we can apply (II) for $x=\xi$ and $y=\zeta$. We obtain

$$\begin{aligned}\Psi_1(d(\xi,\zeta)) &= \Psi_1(d(\mathcal{T}\xi,\mathcal{T}\zeta))\\ &\leq \psi_1(d(\xi,\zeta),d(\xi,\mathcal{T}\xi),d(\zeta,\mathcal{T}\zeta))\\ &\quad -\psi_2(d(\xi,\zeta),d(\xi,\mathcal{T}\xi),d(\zeta,\mathcal{T}\zeta)).\end{aligned}$$

Since ξ and ζ are fixed points of $\mathcal{T}$, we can show easily that

$$\Psi_1(d(\xi,\zeta))\leq\Psi_1(d(\xi,\zeta))-\psi_2(d(\xi,\zeta),0,0)$$

which implies $\psi_2(d(\xi,\zeta),0,0)=0$, and thus $d(\xi,\zeta)=0$, that is, $\xi=\zeta$. Thus we proved the uniqueness of the fixed point. □

In the following, we deduce some fixed point theorems from our main result given by Theorem 2.6.

If we take $p=1$ and $\mathcal{A}_1=\mathcal{X}$ in Theorem 2.6, then we get immediately the following fixed point theorem.

Corollary 2.7. *Let $(\mathcal{X}, d)$ be a complete metric space and let $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ satisfying the following condition: there exist $\psi_1, \psi_2 \in \mathcal{F}_3$ with $\Psi_1(t) = \psi_1(t, t, t)$ in Theorem 2.6 such that*

$$\Psi_1(d(\mathcal{T}x, \mathcal{T}y)) \leq \psi_1(d(x, y), d(x, \mathcal{T}x), d(y, \mathcal{T}y)) - \psi_2(d(x, y), d(x, \mathcal{T}x), d(y, \mathcal{T}y)),$$

for all $x, y \in \mathcal{X}$. Then $\mathcal{T}$ has a unique fixed point.

Remark 2.8. Corollary 2.7 extends and generalizes many existing fixed point theorems in the literature [2], [3–5, 7, 8, 10–13, 21–23].

Now, it is easy to state a corollary of Theorem 2.6 involving a contraction of integral type.

Corollary 2.9. *Let $\mathcal{T}$ satisfy the conditions of Theorem 2.6, except that condition (II) is replaced by the following: there exists a positive Lebesgue integrable function u on $\mathbb{R}_+$ such that $\int_0^\varepsilon u(t)dt > 0$ for each $\varepsilon > 0$ and that*

$$\begin{aligned}\int_0^{\Psi_1(d(\mathcal{T}x,\mathcal{T}y))} u(t)dt &\leq \int_0^{\psi_1(d(x,y),d(x,\mathcal{T}x),d(y,\mathcal{T}y))} u(t)\,dt \\ &\quad - \int_0^{\psi_2(d(x,y),d(x,\mathcal{T}x),d(y,\mathcal{T}y))} u(t)\,dt. \qquad (2.20)\end{aligned}$$

Then $\mathcal{T}$ has a unique fixed point. Moreover, the fixed point of $\mathcal{T}$ belongs to $\cap_{i=1}^p \mathcal{A}_i$.

Remark 2.10. If we take

$$\psi_1(t_1, t_2, t_3) = \max\{t_1, t_2, t_3\} \text{ and } \psi_2(t_1, t_2, t_3) = (1 - k)\max\{t_1, t_2, t_3\},$$

for $k \in (0, 1)$ then $\Psi_1(t) = t$ for all $t \geq 0$, and the contractive condition (II) becomes

$$d(\mathcal{T}x, \mathcal{T}y) \leq k \max\left\{d(x, y), d(x, \mathcal{T}x), d(y, \mathcal{T}y)\right\}.$$

A number of fixed point results may be obtained by assuming different forms for the functions ψ_1 and ψ_2. In particular, fixed point results under various contractive conditions may be derived from the above theorems. For example, if we consider

$$\begin{aligned}\psi_1(x, y, z) &= k_1x^q + k_2y^q + k_3z^q, \\ \psi_2(x, y, z) &= (1 - k)[k_1x^q + k_2y^q + k_3z^q],\end{aligned}$$

where $q > 0$ and $0 < k = k_1 + k_2 + k_3 < 1$, we obtain the following results.

The next result is an immediate consequence of Theorem 2.6.

Corollary 2.11. *Let $(\mathcal{X}, d)$ be a complete metric space, $p \in \mathbb{N}$, $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_p$ nonempty closed subsets of $\mathcal{X}$, $\mathcal{Y} = \cup_{i=1}^p \mathcal{A}_i$ and $\mathcal{T} : \mathcal{Y} \to \mathcal{Y}$ such that*

(I)' *$\mathcal{Y} = \cup_{i=1}^p \mathcal{A}_i$ is a cyclic representation of $\mathcal{Y}$ with respect to $\mathcal{T}$;*
(II)' *for any $(x, y) \in \mathcal{A}_i \times \mathcal{A}_{i+1}$, $i = 1, 2, \ldots, p$ (with $\mathcal{A}_{p+1} = \mathcal{A}_1$),*

$$(d(\mathcal{T}x, \mathcal{T}y))^q \leq k_1(d(x, y))^q + k_2(d(x, \mathcal{T}x))^q + k_3(d(y, \mathcal{T}y))^q,$$

where $q > 0$ and $0 < k_1 + k_2 + k_3 < 1$. Then $\mathcal{T}$ has a unique fixed point. Moreover, the fixed point of $\mathcal{T}$ belongs to $\cap_{i=1}^p \mathcal{A}_i$.

Remark 2.12. Other fixed point results may also be obtained under specific choices of ψ_1 and ψ_2.

Next we present some examples showing how our Theorem 2.6 can be used.

Example 2.13. Let $\mathcal{X} = \mathbb{R}$ endowed with the usual metric. Assume $\mathcal{A}_1 = \mathcal{A}_3 = [-1, 0]$ and $\mathcal{A}_2 = \mathcal{A}_4 = [0, 1]$ so that $\mathcal{Y} = \bigcup_{i=1}^4 \mathcal{A}_i = [-1, 1]$. Define $\mathcal{T} : \mathcal{Y} \to \mathcal{Y}$ by $\mathcal{T}x = -\frac{x}{32}$, for all $x \in \mathcal{Y}$. It is clear that $(\mathcal{X}, d)$ is a complete metric space and $\mathcal{Y} = \bigcup_{i=1}^4 \mathcal{A}_i$ is a cyclic representation of $\mathcal{Y}$ with respect to $\mathcal{T}$. Then, for any $x \in \mathcal{A}_i$, $y \in \mathcal{A}_{i+1}$, $i = 1, 2, 3, 4$ we have

$$\begin{aligned}
\frac{|x-y|}{32} &\leq \frac{1}{16}\frac{31}{32}(|x| + |y|) \\
&\leq \frac{1}{8}\left(\frac{1}{2}\left(|x-y| + \left|x - \frac{x}{32}\right| + \left|y - \frac{y}{32}\right|\right)\right) \\
&\leq \frac{1}{8}\max\left\{d(x,y), d(x,\mathcal{T}x), d(y,\mathcal{T}y)\right\}.
\end{aligned}$$

Take $\psi_1(t_1, t_2, t_3) = \max\{t_1, t_2, t_3\}$ and $\psi_2(t_1, t_2, t_3) = \frac{7}{8}\max\{t_1, t_2, t_3\}$. Then $\mathcal{T}$ is a cyclic contractive map. Therefore, all the hypotheses of Theorem 2.6 are satisfied and 0 is a unique fixed point of $\mathcal{T}$.

Example 2.14. Let $\mathcal{X} = [\frac{-\pi}{2}, \frac{\pi}{2}]$ endowed with the standard metric $d(x, y) = |x-y|$ for all $x, y \in \mathcal{X}$. Consider the closed subsets $\mathcal{A}_1$ and $\mathcal{A}_2$ defined by $\mathcal{A}_1 = [\frac{-\pi}{2}, 0]$ and $\mathcal{A}_2 = [0, \frac{\pi}{2}]$. Define the mapping $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ by

$$\mathcal{T}x = \begin{cases} -\frac{1}{5}x|\cos(1/x)| & \text{if} \quad x \in [\frac{-\pi}{2}, 0) \cup (0, \frac{\pi}{2}], \\ 0 & \text{if} \quad x = 0. \end{cases}$$

Clearly, we have $\mathcal{T}(\mathcal{A}_1) \subset \mathcal{A}_2$ and $\mathcal{T}(\mathcal{A}_2) \subset \mathcal{A}_1$.

Now, let $(x, y) \in (\mathcal{A}_1 \times \mathcal{A}_2)$ with $x \neq 0$ and $y \neq 0$, we have

$$\begin{aligned}
d(\mathcal{T}x, \mathcal{T}y) &= |\mathcal{T}x - \mathcal{T}y| \\
&= \left|\frac{1}{5}x|\cos(1/x)| + \frac{1}{5}y|\cos(1/y)|\right| \\
&= \frac{1}{5}\Big||x||\cos(1/x)| + |y||\cos(1/y)|\Big| \\
&\leq \frac{1}{5}(|x| + |y|).
\end{aligned}$$

On the other hand, we have

$$|x| = -x \leq -x + \frac{1}{5}|x\cos(1/x)| = -x - \frac{1}{5}x|\cos(1/x)| \leq \left|x + \frac{1}{5}x|\cos(1/x)|\right| = d(x, \mathcal{T}x)$$

and

$$|y| = y \leq y + \frac{1}{5}|y\cos(1/y)| = \left|y + \frac{1}{5}y|\cos(1/y)|\right| = d(y, \mathcal{T}y).$$

Then we have

$$d(\mathcal{T}x, \mathcal{T}y) \leq \frac{2}{5}\max\{d(x, \mathcal{T}x), d(y, \mathcal{T}y)\}$$

$$\leq \frac{2}{5}\max\{d(x,y),d(x,\mathcal{T}x),d(y,\mathcal{T}y)\}.$$

Take $\psi_1(t_1,t_2,t_3)=\max\{t_1,t_2,t_3\}$ and $\psi_2(t_1,t_2,t_3)=\frac{3}{5}\max\{t_1,t_2,t_3\}$. Then $\mathcal{T}$ is a cyclic contractive map. Moreover, we can show that condition (II) holds if $x=0$ or $y=0$.

Now, all conditions of Theorem 2.6 are satisfied (with $p=2$), we deduce that $\mathcal{T}$ has a unique fixed point $x^*\in\mathcal{A}_1\cap\mathcal{A}_2=\{0\}$.

We conclude this section by applying Corollary 2.11 to the study of solutions for the functional equation $x(t)=f(t,x(t))$, under the conditions (A), (B) and (C) described below.

Given a metric space (X,d) we denote, as usual, by $C(X,\mathbb{R})$ the set of all real-valued continuous functions on (X,d).

Example 2.15. Consider the functional equation $x(t)=f(t,x(t))$, where $f\in C([0,1]\times\mathbb{R},\mathbb{R})$ satisfies the following three conditions:

(A) f is non-increasing in the second variable, i.e., for each $t\in[0,1]$,

$$x,y\in\mathbb{R},\ x\leq y \implies f(t,x)\geq f(t,y).$$

(B) There exist two functions $\alpha,\beta\in C([0,1],\mathbb{R})$ and two numbers $\alpha_0,\beta_0\in\mathbb{R}$ such that

$$\alpha_0\leq\alpha(t)\leq\beta(t)\leq\beta_0,$$

and

$$\alpha(t)\leq f(t,\beta(t)) \quad\text{and}\quad f(t,\alpha(t))\leq\beta(t),$$

for all $t\in[0,1]$.

(C) There exist $q>0$ and $k_i>0$, $i=1,2,3$, such that $k_1+k_2+k_3<1$ and

$$|f(t,x)-f(t,y)|^q\leq k_1\,|x-y|^q+k_2\,|x-f(t,x)|^q+k_3\,|y-f(t,y)|^q\,,$$

for all $t\in[0,1]$, $x\leq\beta_0$ and $y\geq\alpha_0$.

If we define

$$\mathcal{A}_1=\{u\in C([0,1],\mathbb{R}):u(t)\leq f(t,\alpha(t)),\ \text{for all } t\in[0,1]\},$$

and

$$\mathcal{A}_2=\{u\in C([0,1],\mathbb{R}):f(t,\beta(t))\leq u(t),\ \text{for all } t\in[0,1]\},$$

then, we shall prove that there is a unique $u^*\in\mathcal{A}_1\cup\mathcal{A}_2$ such that

$$u^*(t)=f(t,u^*(t)), \tag{2.21}$$

for all $t\in[0,1]$. Furthermore $u^*\in\mathcal{A}_1\cap\mathcal{A}_2$, i.e., $f(t,\beta(t))\leq u^*(t)\leq f(t,\alpha(t))$ for all $t\in[0,1]$.

Indeed, let d_∞ be the metric on $(C([0,1],\mathbb{R})$ given by

$$d_\infty(x,y)=\max_{t\in[0,l]}|x(t)-y(t)|,$$

for all $x,y\in C([0,1],\mathbb{R})$. It is well known that $(C([0,1],\mathbb{R}),d_\infty)$ is a complete metric space.

Now define the self-map $\mathcal{T} : (C[0,1],\mathbb{R}) \to (C[0,1],\mathbb{R})$ by

$$\mathcal{T}x(t) = f(t, x(t)), \text{ for } x \in (C[0,1],\mathbb{R}).$$

We shall show the existence of a unique fixed point of $\mathcal{T}$ in $\mathcal{A}_1 \cup \mathcal{A}_2$, which is the unique solution of (2.21) in $\mathcal{A}_1 \cup \mathcal{A}_2$.

To this end, first note that $\mathcal{A}_1$ and $\mathcal{A}_2$ are non-empty closed subsets of $C([0,1],\mathbb{R})$, so $\mathcal{X} = \mathcal{A}_1 \cup \mathcal{A}_2$ is closed in $C([0,1],\mathbb{R})$, and thus the metric space $(\mathcal{X}, d_\infty)$ is complete.

Moreover $\mathcal{X} = \mathcal{A}_1 \cup \mathcal{A}_2$ is a cyclic representation of the restriction of $\mathcal{T}$ to $\mathcal{X}$, which will be also denoted by $\mathcal{T}$. In fact, for each $u \in \mathcal{A}_1$ we have, by conditions (A) and (B),

$$f(t, \beta(t)) \leq f(t, u(t)), \text{ for } t \in [0,1],$$

and consequently $\mathcal{T}(\mathcal{A}_1) \subseteq \mathcal{A}_2$.

Similarly, for any $u \in \mathcal{A}_2$ we have

$$f(t, u(t)) \leq f(t, \alpha(t)), \text{ for } t \in [0,1],$$

and consequently $\mathcal{T}(\mathcal{A}_2) \subseteq \mathcal{A}_1$.

Finally, let $u \in \mathcal{A}_1$, $v \in \mathcal{A}_2$ and $t \in [0,1]$. Since $u(t) \leq \beta_0$ and $v(t) \geq \alpha_0$, we deduce by condition (C) that

$$\begin{aligned} |\mathcal{T}u(t) - \mathcal{T}v(t)|^q &= |f(t,u(t)) - f(t,v(t))|^q \\ &\leq k_1\,|u(t) - v(t)|^q + k_2\,|u(t) - f(t,u(t))|^q \\ &\quad + k_3\,|v(t) - f(t,v(t))|^q . \end{aligned}$$

Therefore

$$\begin{aligned} (d_\infty(\mathcal{T}u,\mathcal{T}v))^q &= \left(\max_{t\in[0,1]} |\mathcal{T}u(t) - \mathcal{T}v(t)|\right)^q = \max_{t\in[0,1]} |\mathcal{T}u(t) - \mathcal{T}v(t)|^q \\ &\leq k_1(d_\infty(u,v))^q + k_2(d_\infty(u,\mathcal{T}u))^q + k_3(d_\infty(v,\mathcal{T}v))^q. \end{aligned}$$

It follows form Corollary 2.11 that $\mathcal{T}$ has a unique fixed point u^* in $\mathcal{A}_1 \cup \mathcal{A}_2$. In fact $u^* \in \mathcal{A}_1 \cap \mathcal{A}_2$, i.e., $f(t,\beta(t)) \leq u^*(t) \leq f(t,\alpha(t))$ for all $t \in [0,1]$.

As a particular case of the preceding example let

$$x(t) = f(t, x(t)),$$

where $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is defined by

$$f(t,x) = g(t) \quad \text{if} \quad x < 0,$$

$$f(t,x) = \frac{g(t)}{1 + \sin x} \quad \text{if} \quad 0 \leq x \leq \pi/2,$$

and

$$f(t,x) = \frac{1}{2} g(t) \quad \text{if} \quad x > \pi/2,$$

with $g : [0,1] \to \mathbb{R}$ continuous and satisfying $0 \leq g(t) < 1/2$ for all $t \in [0,1]$.

It is clear that f satisfies condition (A) above.

Let $\alpha, \beta : [0,1] \to \mathbb{R}$ defined by $\alpha(t) = g(t)/2$ and $\beta(t) = g(t)$ for all $t \in [0,1]$, and let $\alpha_0 = 0$, $\beta_0 = 1$. Obviously

$$\alpha_0 \le \alpha(t) \le \beta(t) \le \beta_0,$$

for all $t \in [0,1]$. Moreover,

$$\alpha(t) \le f(t, \beta(t)) \quad \text{and} \quad f(t, \alpha(t)) \le \beta(t),$$

for all $t \in [0, l]$, because

$$\begin{aligned} f(t,\beta(t)) &= \frac{g(t)}{1+\sin(\beta(t))} \ge \frac{g(t)}{2} = \alpha(t) \qquad \text{and} \\ f(t,\alpha(t)) &= \frac{g(t)}{1+\sin(\alpha(t))} \le g(t) = \beta(t). \end{aligned}$$

So condition (B) is also satisfied.

Finally, since g is continuous on $[0,1]$ and $0 \le g(t) < 1/2$ for all $t \in [0,1]$, there is $k \in (0, 1/2)$ such that $g(t) \le k$ for all $t \in [0,1]$. We shall show that for every $x \le 1$, $y \ge 0$ and $t \in [0,1]$, condition (C) follows with $q = 1$, $k_1 = k_3 = k$ and $k_2 = 0$.

- Case 1. $x < 0$, $0 \le y \le \pi/2$. Then we have

$$\begin{aligned} |f(t,x) - f(t,y)| &= \left| g(t) - \frac{g(t)}{1+\sin y} \right| \\ &\le k \frac{\sin y}{1+\sin y} \le k \sin y \le ky < k\,|y - x|\,. \end{aligned}$$

- Case 2. $0 \le x \le 1$, $0 \le y \le \pi/2$. Then we have

$$\begin{aligned} |f(t,x) - f(t,y)| &= g(t) \left| \frac{\sin x - \sin y}{(1+\sin x)(1+\sin y)} \right| \\ &\le k\,|\sin x - \sin y| \le k\,|x-y|\,, \end{aligned}$$

  where, as is well-known, the last inequality can be immediately deduced from The Mean Value Theorem applied to the function $x \to \sin x$.
- Case 3. $x < 0$, $y > \pi/2$. Then we have

$$\begin{aligned} |f(t,x) - f(t,y)| &= \frac{g(t)}{2} < k\left(\frac{\pi}{2} - \frac{g(t)}{2}\right) \\ &< k\left(y - \frac{g(t)}{2}\right) = k\,|y - f(t,y)|\,. \end{aligned}$$

- Case 4. $0 \le x \le 1$, $y > \pi/2$. Then we have

$$\begin{aligned} |f(t,x) - f(t,y)| &= \left| \frac{g(t)}{1+\sin x} - \frac{g(t)}{2} \right| \le \frac{k}{2} \frac{1-\sin x}{1+\sin x} \\ &\le \frac{k}{2} < k\,|y - x|\,. \end{aligned}$$

Thus condition (C) is also satisfied and hence there is a unique solution $u^* \in C([0,1])$ of (2.21) such that

$$\frac{g(t)}{1+\sin(g(t))} \le u^*(t) \le \frac{g(t)}{1+\sin(g(t)/2)},$$

for all $t \in [0,1]$.

Remark 2.16. In case that f is non-decreasing in the second variable, a suitable modification of conditions (A), (B) and (C) above also allows us to deduce the existence and uniqueness of solution for the functional equation (2.21).

Indeed, let $f \in C([0,1]\times\mathbb{R},\mathbb{R})$ satisfying the following conditions:

(A') f is non-decreasing in the second variable.

(B') There exist two functions $\alpha, \beta \in C([0,1],\mathbb{R})$ and two numbers $\alpha_0, \beta_0 \in \mathbb{R}$ such that

$$\alpha_0 \le \alpha(t) \le f(t,\alpha(t)) \le f(t,\beta(t)) \le \beta(t) \le \beta_0,$$

for all $t \in [0,1]$.

(C') There exist $q > 0$ and $k_i > 0$, $i = 1,2,3$, such that $k_1 + k_2 + k_3 < 1$ and

$$|f(t,x) - f(t,y)|^q \le k_1 |x-y|^q + k_2 |x - f(t,x)|^q + k_3 |y - f(t,y)|^q,$$

for all $t \in [0,1]$, and $x, y \in [\alpha_0, \beta_0]$.

If we define

$$\mathcal{A}_1 = \{u \in C([0,1],\mathbb{R}) : f(t,\beta_0) \le u(t) \le \beta_0, \text{ for all } t \in [0,1]\},$$

and

$$\mathcal{A}_2 = \{u \in C([0,1],\mathbb{R}) : \alpha_0 \le u(t) \le f(t,\alpha_0), \text{ for all } t \in [0,1]\},$$

then $\mathcal{A}_1$ and $\mathcal{A}_2$ are non-empty closed subsets of $C([0,1],\mathbb{R})$, so $(\mathcal{X}, d_\infty)$ is a complete metric space where $\mathcal{X} = \mathcal{A}_1 \cup \mathcal{A}_2$.

Now define the self-map $\mathcal{T} : (C[0,1],\mathbb{R}) \to (C[0,1],\mathbb{R})$ as in Example 2.15. Then $\mathcal{T}(\mathcal{A}_1) \subseteq \mathcal{T}(\mathcal{A}_2)$ and $\mathcal{T}(\mathcal{A}_2) \subseteq \mathcal{T}(\mathcal{A}_1)$ by conditions (A') and (B'), so $\mathcal{X} = \mathcal{A}_1 \cup \mathcal{A}_2$ is a cyclic representation of the restriction of $\mathcal{T}$ to $\mathcal{X}$, which is also denoted by $\mathcal{T}$.

Finally, from condition (C') we deduce, as above, that

$$\begin{aligned}(d_\infty(\mathcal{T}u,\mathcal{T}v))^q &= \left(\max_{t\in[0,1]} |\mathcal{T}u(t) - \mathcal{T}v(t)|\right)^q = \max_{t\in[0,1]} |\mathcal{T}u(t) - \mathcal{T}v(t)|^q \\ &\le k_1(d_\infty(u,v))^q + k_2(d_\infty(u,\mathcal{T}u))^q + k_3(d_\infty(v,\mathcal{T}v))^q.\end{aligned}$$

for all $u \in \mathcal{A}_1$ and $v \in \mathcal{A}_2$.

By Corollary 2.11 we conclude that $\mathcal{T}$ has a unique fixed point u^* in $\mathcal{A}_1 \cup \mathcal{A}_2$. Furthermore $u^* \in \mathcal{A}_1 \cap \mathcal{A}_2$, i.e., $f(t,\alpha(t))) \le u^*(t) \le f(t,\beta(t))$ for all $t \in [0,1]$.

3. An application to second order differential equations

In this section we shall apply Corollary 2.11 to the study of existence and uniqueness of solution for a type of second order differential equations. Our approach is inspired by Section 3 of [9].

Consider the two-point boundary value problem for second order differential equation

$$\begin{cases} x''(t) = -f(t,x(t)), \\ x(0) = x(1) = 0. \end{cases} \tag{3.1}$$

where $f \in C([0,1] \times \mathbb{R}, \mathbb{R})$.

It is known, and easy to check, that problem (3.1) is equivalent to the integral equation

$$x(t) = \int_0^1 G(t,s) f(s, x(s))\, ds, \text{ for } t \in [0,1], \tag{3.2}$$

where G is the Green function defined by

$$G(t,s) = (1-t)s \quad \text{if} \quad 0 \le s \le t \le 1,$$

and

$$G(t,s) = (1-s)t \quad \text{if} \quad 0 \le t \le s \le 1.$$

That is, if $x \in C^2([0,1], \mathbb{R})$, then x is a solution of problem (3.1) if and only if it is a solution of the integral equation (3.2).

Now assume that the following three conditions are satisfied:

(i) f is non-increasing in the second variable.

(ii) There exist two functions $\alpha, \beta \in C([0,1], \mathbb{R})$ and two numbers $\alpha_0, \beta_0 \in \mathbb{R}$ such that

$$\alpha_0 \le \alpha(t) \le \beta(t) \le \beta_0,$$

and

$$\alpha(t) \le \int_0^1 G(t,s) f(s, \beta(s)) \quad \text{and} \quad \int_0^1 G(t,s) f(s, \alpha(s)) \le \beta(t),$$

for all $t \in [0,1]$.

(iii) For

$$\mathcal{A}_1 = \{u \in C([0,1], \mathbb{R}) : u(t) \le \beta(t), \text{ for all } t \in [0,1]\},$$

and

$$\mathcal{A}_2 = \{u \in C([0,1], \mathbb{R}) : \alpha(t) \le u(t), \text{ for all } t \in [0,1]\},$$

there exist $q \ge 1$ and $a_i > 0$, $i = 1,2,3$, such that $a_1 + a_2 + a_3 < 8$ and

$$\begin{aligned} |f(t,u(t)) - f(t,v(t))|^q \quad &\le \quad a_1 |u(t) - v(t)|^q + a_2 \left| u(t) - \int_0^1 G(t,s) f(s,u(s)) ds \right|^q \\ &\quad + a_3 \left| v(t) - \int_0^1 G(t,s) f(s,v(s)) ds \right|^q, \end{aligned}$$

for all $u \in \mathcal{A}_1$, $v \in \mathcal{A}_2$ and $t \in [0,1]$.

Then, we can prove the following.

Theorem 3.1. *Under the conditions (i), (ii) and (iii) above, the problem (3.1) has one and only one solution u^* in $\mathcal{A}_1 \cup \mathcal{A}_2$. In fact, $u^* \in \mathcal{A}_1 \cap \mathcal{A}_2$.*

Proof. Define the self-map $\mathcal{T} : (C[0,1], \mathbb{R}) \to (C[0,1], \mathbb{R})$ by

$$\mathcal{T}x(t) = \int_0^1 G(t,s) f(s, x(s))\, ds, \text{ for } x \in (C[0,1], \mathbb{R}).$$

We shall prove the existence of a unique fixed point of $\mathcal{T}$ in $\mathcal{A}_1 \cup \mathcal{A}_2$.

To this end, put $\mathcal{X} = \mathcal{A}_1 \cup \mathcal{A}_2$. Since $\mathcal{A}_1$ and $\mathcal{A}_2$ are non-empty closed subsets of $C([0,1],\mathbb{R})$, it follows that $\mathcal{X}$ is closed in $C([0,1],\mathbb{R})$, and thus the metric space $(\mathcal{X}, d_\infty)$ is complete.

Moreover $\mathcal{X} = \mathcal{A}_1 \cup \mathcal{A}_2$ is a cyclic representation of the restriction of $\mathcal{T}$ to $\mathcal{X}$, which will be also denoted by $\mathcal{T}$. In fact, for each $u \in \mathcal{A}_1$ we have by conditions (i) and (ii),

$$\alpha(t) \le \mathcal{T}\beta(t) \le \mathcal{T}u(t), \text{ for } t \in [0,1],$$

and consequently $\mathcal{T}(\mathcal{A}_1) \subseteq \mathcal{A}_2$, and for any $u \in \mathcal{A}_2$ we have similarly,

$$\mathcal{T}u(t) \le \mathcal{T}\alpha(t) \le \beta(t), \text{ for } t \in [0,1],$$

and consequently $\mathcal{T}(\mathcal{A}_2) \subseteq \mathcal{A}_1$.

Next, we recall that for each $t \in [0,1]$ one has

$$\int_0^1 G(t,s)ds = \frac{t(1-t)}{2},$$

and then

$$\max_{t\in[0,1]} \int_0^1 G(t,s)ds = \frac{1}{8}.$$

Finally, for each $u \in \mathcal{A}_1$, $v \in \mathcal{A}_2$ and $t \in [0,1]$, we have $u(s) \le \beta_0$ and $v(s) \ge \alpha_0$ for all $s \in [0,1]$, so by the inequality of Cauchy-Schwarz, condition (iii) and the fact that $G(t,s) \le 1$, we deduce

$$|\mathcal{T}u(t) - \mathcal{T}v(t)|^q \le \left(\int_0^1 (G(t,s))^q ds\right)\left(\int_0^1 |f(s,u(s)) - f(s,v(s))|^q\, ds\right)$$

$$\le \left(\int_0^1 (G(t,s))ds\right)\left((a_1(d_\infty(u,v))^q + a_2(d_\infty(u,\mathcal{T}u))^q + a_3(d_\infty(v,\mathcal{T}v))^q\right).$$

Hence

$$(d_\infty(\mathcal{T}u,\mathcal{T}v))^q \le \frac{1}{8}(a_1(d_\infty(u,v))^q + a_2(d_\infty(u,\mathcal{T}u))^q + a_3(d_\infty(v,\mathcal{T}v))^q).$$

By applying Corollary 2.11, with $k_i = a_i/8$, $i = 1,2,3$, we deduce that the problem (3.1) has a unique solution u^* such that $\alpha(t) \le u^*(t) \le \beta(t)$ for all $t \in [0,1]$. □

In case that f is non-decreasing in the second variable, a suitable modification of conditions (i), (ii) and (iii) above also allows us to deduce the existence and uniqueness of solution of the problem (3.1).

Indeed, let $f \in (C[0,1] \times \mathbb{R}, \mathbb{R})$ satisfying the following three conditions:

(i') f is non-decreasing in the second variable.

(ii') There exist two functions $\alpha, \beta \in C([0,1],\mathbb{R})$ and two numbers $\alpha_0, \beta_0 \in \mathbb{R}$ such that

$$\alpha_0 \le \alpha(t) \le \int_0^1 G(t,s)f(s,\alpha(s))\, ds \le \int_0^1 G(t,s)f(s,\beta(s))\, ds \le \beta(t) \le \beta_0,$$

for all $t \in [0,1]$.

(iii') For

$$\mathcal{A}_1 = \{u \in C([0,1],\mathbb{R}) : \int_0^1 G(t,s)f(s,\alpha(s))\,ds \le u(t) \le \beta(t),\ \text{for all } t \in [0,1]\},$$

and

$$\mathcal{A}_2 = \{u \in C([0,1],\mathbb{R}) : \alpha(t) \le u(t) \le \int_0^1 G(t,s)f(s,\beta(s))\,ds,\ \text{for all } t \in [0,1]\},$$

there exist $q \ge 1$ and $a_i > 0$, $i = 1,2,3$, such that $a_1 + a_2 + a_3 < 8$ and

$$\begin{aligned} |f(t,u(t)) - f(t,v(t))|^q &\le a_1 |u(t) - v(t)|^q + a_2 \left| u(t) - \int_0^1 G(t,s)f(s,u(s))ds \right|^q \\ &\quad + a_3 \left| v(t) - \int_0^1 G(t,s)f(s,v(s))ds \right|^q, \end{aligned}$$

for all $u \in \mathcal{A}_1$, $v \in \mathcal{A}_2$ and $t \in [0,1]$.

Theorem 3.2. *Under the conditions (i'), (ii') and (iii') above, the problem (3.1) has one and only one solution u^* in $\mathcal{A}_1 \cup \mathcal{A}_2$. In fact, $u^* \in \mathcal{A}_1 \cap \mathcal{A}_2$.*

Proof. We omit some details because the proof follows similarly to the one given in Theorem 3.1.

Define the self-map $\mathcal{T} : (C[0,1],\mathbb{R}) \to (C[0,1],\mathbb{R})$ by

$$\mathcal{T}x(t) = \int_0^1 G(t,s)f(s,x(s))\,ds,\ \text{for } x \in (C[0,1],\mathbb{R}).$$

We shall prove the existence of a unique fixed point of $\mathcal{T}$ in $\mathcal{A}_1 \cup \mathcal{A}_2$.

To this end, put $\mathcal{X} = \mathcal{A}_1 \cup \mathcal{A}_2$. Since $\mathcal{A}_1$ and $\mathcal{A}_2$ are non-empty closed subsets of $C([0,1],\mathbb{R})$, it follows that $\mathcal{X}$ is closed in $C([0,1],\mathbb{R})$, and thus the metric space $(\mathcal{X}, d_\infty)$ is complete.

Moreover $\mathcal{X} = \mathcal{A}_1 \cup \mathcal{A}_2$ is a cyclic representation of the restriction of $\mathcal{T}$ to $\mathcal{X}$, which will be also denoted by $\mathcal{T}$. In fact, for each $u \in \mathcal{A}_1$ we have by conditions (i') and (ii'),

$$\alpha(t) \le \mathcal{T}\alpha(t) \le \mathcal{T}u(t) \le \mathcal{T}\beta(t),\ \text{for } t \in [0,1],$$

and consequently $\mathcal{T}(\mathcal{A}_1) \subseteq \mathcal{A}_2$, and for any $u \in \mathcal{A}_2$ we have similarly,

$$\mathcal{T}\alpha(t) \le \mathcal{T}u(t) \le \mathcal{T}\beta(t) \le \beta(t),\ \text{for } t \in [0,1],$$

and consequently $\mathcal{T}(\mathcal{A}_2) \subseteq \mathcal{A}_1$.

Finally, for each $u \in \mathcal{A}_1$, $v \in \mathcal{A}_2$ and $t \in [0,1]$, we have $u(s), v(s) \in [\alpha_0, \beta_0]$ for all $s \in [0,1]$, so, by using (iii'), we deduce, as in the proof of Theorem 3.1, that

$$(d_\infty(\mathcal{T}u, \mathcal{T}v))^q \le \frac{1}{8}(a_1(d_\infty(u,v))^q + a_2(d_\infty(u,\mathcal{T}u))^q + a_3(d_\infty(v,\mathcal{T}v))^q).$$

By Corollary 2.11, $\mathcal{T}$ has a unique fixed point u^* such that $f(t,\alpha(t)) \le u^*(t) \le f(t,\beta(t))$ for all $t \in [0,1]$. □

References

[1] R. P. Agarwal, M.A. Alghamdi and N. Shahzad, *Fixed point theory for cyclic generalized contractions in partial metric spaces*, Fixed Point Theory Appl. **2012** (2012), doi:10.1186/1687-1812-2012-40.

[2] S. Banach, *Sur les opérations dans les ensembles abstraits et leur application aux equations integrales*, Fund. Math. **3** (1922), 133–181.

[3] R. M. T. Bianchini, *Su un problema di S. Reich riguardante la teoria dei punti fissi*, Bollettino U. M. I. **5** (1972), 103–108.

[4] D. W. Boyd and J. S. Wong, *On nonlinear contractions*, Proc. Amer. Math. Soc. **20** (1969), 458–469.

[5] S. K. Chatterjea, *Fixed point theorems*, C. R. Acad. Bulgare Sci. **25** (1972), 727–730.

[6] C. M. Chen, *Fixed point theory for the cyclic weaker Meir-Keeler function in complete metric spaces,* Fixed Point Theory and Appl. **2012**, (2012), 17, doi:10.1186/1687-1812-2012-17.

[7] P. N. Dutta and B. S. Choudhury, *A generalisation of contraction principle in metric spaces*, Fixed Point Theory Appl. (2008), Article ID 406368.

[8] G. E. Hardy and T. D. Rogers, *A generalization of a fixed point theorem of Reich*, Canad. Math. Bull. **16** (1973), 201–206.

[9] J. Harjani and K. Sadarangani, *Generalized contractions in partially ordered metric spaces and applications to ordinary differential equations*, Nonlinear Anal. **72** (2010), 1188–1197.

[10] R. Kannan, *Some results on fixed points*, Bull. Clacutta. Math. Soc. **10** (1968), 71–76.

[11] E. Karapınar, *Fixed point theory for cyclic weak ϕ-contraction*, Appl. Math. Lett. **24** (2011), 822–825.

[12] E. Karapınar and K. Sadaranagni, *Fixed point theory for cyclic (ϕ-ψ)-contractions*, Fixed Point Theory Appl. **2011** (2011), doi:10.1186/1687-1812-2011-69.

[13] M. S. Khan, M. Swaleh and S. Sessa, *Fixed point theorems by altering distances between the points*, Bull. Aust. Math. Soc. **30** (1984), 1–9.

[14] W. A. Kirk, P. S. Srinivasan and P. Veeramani, *Fixed points for mappings satisfying cyclical contractive conditions*, Fixed Point Theory **4** (2003), 79–89.

[15] H. K. Nashine, *New fixed point theorems for mappings satisfying generalized weakly contractive condition with weaker control functions*, Ann. Polonici Math. **104** (2012), 109–119.

[16] H. K. Nashine, *Cyclic generalized ψ-weakly contractive mappings and fixed point results with applications to integral equations*, Nonlinear Anal. **75** (2012), 6160–6169.

[17] M. Pacurar and I. A. Rus, *Fixed point theory for cyclic φ-contractions*, Nonlinear Anal. **72** (2010), 1181–1187.

[18] M. A. Petric, *Some results concerning cyclical contractive mappings*, General Math. **18** (2010), 213–226.

[19] I. A. Rus, *Cyclic representations and fixed points*, Ann. T. Popoviciu, Seminar Funct. Eq. Approx. Convexity **3** (2005), 171–178.

[20] K. P. R. Rao, G. R. Babu and D. V. Babu, *Common fixed point theorems through generalized altering distance functions*, Math. Communications **13** (2008), 67–73.

[21] S. Reich, *Some remarks concerning contraction mappings*, Canad. Math. Bull. **14** (1971), 121–124.

[22] S. Reich, *Kannan's fixed point theorem*, Boll. Un. Mat. Ital. **4** (1971), 1–11.

[23] B. E. Rhoades, *Some theorems on weakly contractive maps*, Nonlinear Anal. **47** (2001), 2683–2693.

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