Journal of Nonlinear and Convex Analysis
Volume 15, Number 6, 2014, 1221–1229

Yokohama Publishers
ISSN 1880-5221 ONLINE JOURNAL
© Copyright 2014

GENERALIZED n-METRIC SPACES AND FIXED POINT THEOREMS

KAMRAN ALAM KHAN

Abstract. Gähler ([3],[4]) introduced the concept of 2-metric as a possible generalization of usual notion of a metric space. In many cases the results obtained in the usual metric spaces and 2-metric spaces are found to be unrelated (see [5]). Mustafa and Sims [8] took a different approach and introduced the notion of G-metric. The author [6] generalized the notion of G-metric to more than three variables and introduced the concept of K-metric as a function $K\colon X^n \to \mathbb{R}^+$, $(n \geq 3)$. In this paper, We improve the definition of K-metric by making symmetry condition more general. This improved metric denoted by G_n is called the *Generalized n-metric*. We develop the theory for generalized n-metric spaces and obtain some fixed point theorems.

1. Introduction

Gähler ([3],[4]) introduced the concept of 2-metric as a possible generalization of usual notion of a metric space. K. S. Ha et al [5] have pointed out that the construction by Gähler is an independent approach and in many cases there is no connection between the results obtained in the usual metric spaces and 2-metric spaces. It was mentioned by Gähler [3] that the notion of a 2-metric is an extension of an idea of ordinary metric and geometrically $d(x, y, z)$ represents the area of a triangle formed by the points x,y and z in X as its vertices. But this is not always true. Sharma [9] showed that $d(x, y, z) = 0$ for any three distinct points x, y, z $\in \mathbb{R}^2$.

B. C. Dhage [2] introduced the concept of D-metric in order to translate results from usual metric space to D-metric space. Mustafa and Sims [7] showed that most of the results concerning D-metrics are incorrect. This led them to introduce a new class of generalized metrics called G-metric in which the tetrahedral inequality is replaced by an inequality involving repetition of indices (see [8]).Many authors(such as [1]) obtained fixed point results for G-metric spaces. Recently the author [6] generalized the notion of G-metric space to more than three variables and introduced the concept of K-metric. In the present work we improve the definition of K-metric by making symmetry condition more general. This improved metric denoted by G_n is called the *Generalized n-metric*. We develop the theory for generalized n-metric spaces and obtain some fixed point theorems.

Definition 1.1 ([6])**.** Let X be a non-empty set, and $\mathbb{R}^+$ denote the set of non-negative real numbers. Let $K\colon X^n \to \mathbb{R}^+$, $(n \geq 3)$ be a function satisfying the following properties:

[K 1] $K(x_1, x_2, \ldots, x_n) = 0$ if $x_1 = x_2 = \cdots = x_n$,

2010 *Mathematics Subject Classification.* Primary 54E35; Secondary 47H10.
Key words and phrases. 2-metric space, G-metric space, K-metric space, fixed point.

[K 2] $K(x_1, x_1, \dots, x_1, x_2) > 0$ for all x_1, $x_2 \in X$ with $x_1 \neq x_2$,

[K 3] $K(x_1, x_1, \dots, x_1, x_2) \leq K(x_1, x_2, \dots, x_n)$ for all x_1, $x_2, \dots, x_n \in X$ with the condition that any two of the points $x_2, \dots, x_n$ are distinct,

[K 4] $K(x_1, x_2, \dots, x_n) = K(x_{\pi^r(1)}, x_{\pi^r(2)}, \dots, x_{\pi^r(n)})$, for all x_1, $x_2, \dots, x_n \in X$ and a permutation π of $\{1, 2, \dots n\}$ such that $\pi(s) = s+1$ for all $1 \leq s < n$, $\pi(n) = 1$ and for all $r \in \mathbb{N}$,

[K 5] $K(x_1, x_2, \dots, x_n) \leq K(x_1, x_{n+1}, \dots, x_{n+1}) + K(x_{n+1}, x_2, \dots, x_n)$ for all $x_1, x_2, \dots, x_n, x_{n+1} \in X$.

Then the function K is called a *K-metric* on X, and the pair (X, K) a *K-metric space*.

Example 1.2. Let $\mathbb{R}$ denote the set of all real numbers. Define a function $\rho \colon \mathbb{R}^n \to \mathbb{R}^+, (n \geq 3)$ by

$$\rho(x_1, x_2, \dots, x_n) = \max\{|x_1 - x_2|, \dots, |x_{n-1} - x_n|, |x_n - x_1|\}$$

for all x_1, $x_2, \dots, x_n \in \mathbb{R}$. Then $(\mathbb{R}, \rho)$ is a K-metric space.

Example 1.3. For any metric space (X, d), the following functions define K-metrics on X:

(1) $K_1^d(x_1, x_2, \dots, x_n) = \frac{1}{n}\left[\sum_{r=1}^{n-1} d(x_r, x_{r+1}) + d(x_n, x_1)\right]$,

(2) $K_2^d(x_1, x_2, \dots, x_n) = \max\{d(x_1, x_2), d(x_2, x_3), \dots, d(x_{n-1}, x_n), d(x_n, x_1)\}$.

Geometrically the K-metric represents the notion of the perimeter of an oriented polygon with vertices $x_1, x_2, \dots, x_n$. Here we observe that the condition of symmetry is not satisfied in general as in G-metric. Thus the notion of a K-metric is not a straight forward translation of the concept of G-metric. Now we introduce an n point analogue of G-metric as follows.

2. Main results

Definition 2.1. Let X be a non-empty set, and $\mathbb{R}^+$ denote the set of non-negative real numbers. Let $G_n \colon X^n \to \mathbb{R}^+$, $(n \geq 3)$ be a function satisfying the following properties:

[G 1] $G_n(x_1, x_2, \dots, x_n) = 0$ if $x_1 = x_2 = \dots = x_n$,

[G 2] $G_n(x_1, x_1, \dots, x_1, x_2) > 0$ for all x_1, $x_2 \in X$ with $x_1 \neq x_2$,

[G 3] $G_n(x_1, x_1, \dots, x_1, x_2) \leq G_n(x_1, x_2, \dots, x_n)$ for all x_1, $x_2, \dots, x_n \in X$ with the condition that any two of the points $x_2, \dots, x_n$ are distinct,

[G 4] $G_n(x_1, x_2, \dots, x_n) = G_n(x_{\pi(1)}, x_{\pi(2)}, \dots, x_{\pi(n)})$, for all x_1, $x_2, \dots, x_n \in X$ and every permutation π of $\{1, 2, \dots n\}$,

[G 5] $G_n(x_1, x_2, \dots, x_n) \leq G_n(x_1, x_{n+1}, \dots, x_{n+1}) + G_n(x_{n+1}, x_2, \dots, x_n)$ for all $x_1, x_2, \dots, x_n, x_{n+1} \in X$.

Then the function G_n is called a *Generalized n-metric* on X, and the pair (X, G_n) a *Generalized n-metric space.*

From now on we always have $n \geq 3$ for (X, G_n) to be a generalized n-metric space.

Example 2.2. Define a function $\rho \colon \mathbb{R}^n \to \mathbb{R}^+, (n \geq 3)$ by

$$\rho(x_1, x_2, \dots, x_n) = \max\{|x_r - x_s| : r, s \in \{1, 2, \dots n\}, r \neq s\}$$

for all x_1, $x_2, \dots, x_n \in X$. Then $(\mathbb{R}, \rho)$ is a generalized n-metric space.

Example 2.3. For any metric space (X, d), the following functions define generalized n-metrics on X:

(1) $K_1^d(x_1, x_2, \dots, x_n) = \sum_r \sum_s d(x_r, x_s)$,
(2) $K_2^d(x_1, x_2, \dots, x_n) = \max\{d(x_r, x_s) \colon r, s \in \{1, 2, \dots, n\}, r \neq s\}$.

Proposition 2.4. *Let $G_n \colon X^n \to \mathbb{R}^+$,$(n \geq 3)$ be a generalized n-metric defined on X, then for x,$y \in X$ we have*

$$G_n(x, y, y, \dots, y) \leq (n-1)G_n(y, x, x, \dots, x). \tag{2.1}$$

Proof. Using [G 5] it is trivial to prove the result. □

Definition 2.5. Let (X, G_n) be a generalized n-metric space, then for $x_0 \in X, r > 0$, the *G_n-ball* with centre x_0 and radius r is

$$B_G(x_0, r) = \{y \in X \colon G_n(x_0, y, y, \dots, y) < r\}$$

Proposition 2.6. *Let (X, G_n) be a generalized n-metric space, then the G_n-ball is open in X.*

Proof. The proof is straightforward. □

Hence the collection of all such balls in X is closed under arbitrary union and finite intersection and therefore induces a topology on X called the generalized n-metric topology $\Im(G_n)$ generated by the generalized n-metric on X.

From example 2.3 it is clear that for a given metric we can always define generalized n-metrics. The converse is also true for if G_n is a generalized n-metric then we can define a metric d_G as follows

$$d_G(x, y) = G_n(x, y, y, \dots, y) + G_n(x, x, \dots, x, y).$$

Proposition 2.7. *Let $B_{d_G}(x, r)$ denote the open ball in the metric space (X, d_G) and $B_G(x, r)$ the G_n-ball in the correponding generalized n-metric space (X, G_n). Then we have*

$$B_G\Big(x, \frac{r}{n}\Big) \subseteq B_{d_G}(x, r)$$

Proof. Let $y \in B_G(x, \frac{r}{n})$ then $G_n(x, y, y, \dots, y) < \frac{r}{n}$. From (2.1) and [G 4] we have

$$G_n(x, x, \dots, x, y) \leq (n-1)G_n(x, y, y, \dots, y) < (n-1)r/n.$$

Therefore

$$d_G(x, y) = G_n(x, y, y, \dots, y) + G_n(x, x, \dots, x, y) < \frac{r}{n} + (n-1)\frac{r}{n} = r.$$

Hence we have $y \in B_{d_G}(x, r)$ and therefore $B_G(x, \frac{r}{n}) \subseteq B_{d_G}(x, r)$ □

This indicates that the topology induced by the generalized n-metric on X coincides with the metric topology induced by the metric d_G. Thus every generalized n-metric space is topologically equivalent to a metric space.

Definition 2.8. Let (X, G_n) be a generalized n-metric space. A sequence $\langle x_m \rangle$ in X is said to be *G_n-convergent* if it converges to a point x in the generalized n-metric topology $\Im(G_n)$ generated by the G_n-metric on X.

Proposition 2.9. *Let $G_r\colon X^r \to \mathbb{R}^+$, $(r \geq 3)$ be a generalized r-metric defined on X. Then for a sequence $\langle x_n \rangle$ in X and $x \in X$ the following are equivalent:*

(1) *The sequence $\langle x_n \rangle$ is G_r-convergent to x.*
(2) $d_G(x_n, x) \to 0$ *as* $n \to \infty$.
(3) $G_r(x_n, x_n, \dots, x_n, x) \to 0$ *as* $n \to \infty$.
(4) $G_r(x_n, x, \dots, x) \to 0$ *as* $n \to \infty$.

Proof. Since the topology induced by the G_r-metric on X coincides with the metric topology induced by the metric d_G, hence (1)⇔(2). Now

$$d_G(x_n, x) = G_r(x_n, x, \dots, x) + G_r(x_n, x_n, \dots, x_n, x). \tag{2.2}$$

Hence $G_r(x_n, x, \dots, x) \to 0$ and $G_r(x_n, x_n, \dots, x_n, x) \to 0$ whenever $d_G(x_n, x) \to 0$. Thus (2)⇒(3) and (2)⇒(4). From (2.1) we have

$$G_r(x_n, x, \dots, x) \leq (r-1)G_r(x, x_n, \dots, x_n). \tag{2.3}$$

Thus (3)⇒(4). Similarly (4)⇒(3). Also from (2.2) and (2.3) we have

$$d_G(x_n, x) \leq rG_r(x_n, x_n, \dots, x_n, x).$$

Therefore (3)⇒(2). □

Definition 2.10. Let (X, G_n^X) and (Y, G_n^Y) be generalized n-metric spaces. A function $f\colon X \to Y$ is said to be *Generalized n-continuous* at a point $x \in X$ if $f^{-1}(B_{G_n^Y}(f(x), r)) \in \Im(G_n^X)$, for all $r > 0$. The function f is said to be generalized n-continuous if it is generalized n-continuous at all points of X.

Since every generalized n-metric space is topologically equivalent to a metric space, hence we have the following result:

Proposition 2.11. *Let (X, G_n^X) and (Y, G_n^Y) be generalized n-metric spaces. A function $f\colon X \to Y$ is said to be generalized n-continuous at a point $x \in X$ if and only if it is generalized n-sequentially continuous at x; that is, whenever the sequence $\langle x_m \rangle$ is G_n^X-convergent to x, the sequence $\langle f(x_m) \rangle$ is G_n^Y-convergent to $f(x)$.*

Proposition 2.12. *Let (X, G_n) be a generalized n-metric space, then the function $G_n(x_1, x_2, \dots, x_n)$ is jointly continuous in the variables $x_1, x_2, \dots, x_n$.*

Proof. Let $\langle x_{m_1} \rangle, \langle x_{m_2} \rangle, \dots, \langle x_{m_n} \rangle$ be the sequences in the generalized n-metric space (X, G_n) such that $x_{m_1} \to x_1$, $x_{m_2} \to x_2, \dots, x_{m_n} \to x_n$. Then by [G 4] and [G 5] we can show that

$$\begin{aligned} G_n(x_{m_1}, x_{m_2}, \dots, x_{m_n}) - G_n(x_1, x_2, \dots, x_n) \leq\ & G_n(x_{m_1}, x_1, \dots, x_1) \\ & + G_n(x_{m_2}, x_2, \dots, x_2) \\ & + \dots + G_n(x_{m_n}, x_n, \dots, x_n). \end{aligned}$$

Similarly

$$\begin{aligned} G_n(x_1, x_2, \dots, x_n) - G_n(x_{m_1}, x_{m_2}, \dots, x_{m_n}) \leq\ & G_n(x_1, x_{m_1}, \dots, x_{m_1}) \\ & + G_n(x_2, x_{m_2}, \dots, x_{m_2}) \\ & + \dots + G_n(x_n, x_{m_n}, \dots, x_{m_n}). \end{aligned}$$

Therefore on using (2.1) we have

$$\begin{aligned}|G_n(x_{m_1}, x_{m_2}, \ldots, x_{m_n}) - G_n(x_1, x_2, \ldots, x_n)| \leq & (n-1)\{G_n(x_1, x_{m_1}, \ldots, x_{m_1}) \\ & + G_n(x_2, x_{m_2}, \ldots, x_{m_2}) \\ & + \cdots + G_n(x_n, x_{m_n}, \ldots, x_{m_n})\}.\end{aligned}$$

Making $m_1 \to \infty$, $m_2 \to \infty$,$\ldots$,$m_n \to \infty$ we have

$$G_n(x_{m_1}, x_{m_2}, \ldots, x_{m_n}) \to G_n(x_1, x_2, \ldots, x_n).$$

Hence the result follows. □

Definition 2.13. Let (X, G_m) be a generalized m-metric space. A sequence $\langle x_n \rangle$ in X is said to be G_m-Cauchy if for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$G_m(x_{n_1}, x_{n_2}, \ldots, x_{n_m}) < \epsilon \text{ for all } n_1, n_2, \ldots, n_m \geq N.$$

Proposition 2.14. *Let (X, G_m) be a generalized m-metric space. A sequence $\langle x_n \rangle$ in X is G_m-Cauchy if and only if for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that*

$$G_m(x_{n_1}, x_{n_2}, \ldots, x_{n_2}) < \epsilon \textit{ for all } n_1, n_2 \geq N. \tag{2.4}$$

Proof. If $\langle x_n \rangle$ is G_m-Cauchy then the result follows from definition 2.13. Conversely suppose that the condition (2.4) holds for a sequence $\langle x_n \rangle$ in X. Then for n_1,n_2,$n_3 \geq N$ we have from [G 5]

$$\begin{aligned}G_m(x_{n_1}, x_{n_2}, x_{n_3}, \ldots, x_{n_3}) &\leq G_m(x_{n_1}, x_{n_3}, \ldots, x_{n_3}) + G_m(x_{n_3}, x_{n_2}, x_{n_3}, \ldots, x_{n_3}) \\ &< \epsilon + \epsilon = 2\epsilon.\end{aligned}$$

Continuing the above argument, for n_1,n_2,$\ldots$,$n_m \geq N$ we have

$$G_m(x_{n_1}, x_{n_2}, \ldots, x_{n_m}) < (m-1)\epsilon.$$

i.e. $\langle x_n \rangle$ is G_m-Cauchy. □

Proposition 2.15. *Every G_n-convergent sequence in a generalized n-metric space is G_n-Cauchy.*

Proof. The result follows from proposition 2.9 and (2.4). □

Definition 2.16. A generalized n-metric space (X, G_n) is said to be G_n-*complete* if every G_n-Cauchy sequence in (X, G_n) is G_n-convergent in (X, G_n).

Theorem 2.17. *Let $G_r \colon X^r \to \mathbb{R}^+, (r \geq 3)$ be a generalized r-metric and (X, G_r) be a G_r-complete generalized r-metric space. Let f and g be self mappings on X satisfying the following conditions:*

(1) *$f(X) \subseteq g(X)$,*
(2) *g is continuous,*
(3) *$G_r(f\xi_1, f\xi_2, \ldots, f\xi_r) \leq q\, G_r(g\xi_1, g\xi_2, \ldots, g\xi_r)$ for every $\xi_1, \xi_2, \ldots, \xi_r \in X$ and $0 < q < 1$*

Then f and g have a unique common fixed point in X provided f and g commute.

Proof. Let x_0 be an arbitrary point in X. Since $f(X) \subseteq g(X)$ hence there exists a point x_1 such that $fx_0 = gx_1$. In general we can choose x_{n+1} such that $y_n = fx_n = gx_{n+1}$.Using (3) we have

$$\begin{aligned} G_r(fx_n, fx_{n+1}, \dots, fx_{n+1}) &\leq q\, G_r(gx_n, gx_{n+1}, \dots, gx_{n+1}) \\ &= q\, G_r(fx_{n-1}, fx_n, \dots, fx_n) \end{aligned}$$

Proceeding in above manner we have

$$\begin{aligned} G_r(fx_n, fx_{n+1}, \dots, fx_{n+1}) &\leq q^n\, G_r(fx_0, fx_1, \dots, fx_1) \\ \Rightarrow G_r(y_n, y_{n+1}, \dots, y_{n+1}) &\leq q^n\, G_r(y_0, y_1, \dots, y_1). \end{aligned}$$

We claim that the sequence $\langle y_n \rangle$ in X is G_r-Cauchy in X. For all natural numbers n and $m(> n)$ we have from [G 5]

$$\begin{aligned} G_r(y_n, y_m, \dots, y_m) &\leq G_r(y_n, y_{n+1}, \dots, y_{n+1}) + G_r(y_{n+1}, y_{n+2}, \dots, y_{n+2}) \\ &\quad + \dots + G_r(y_{m-1}, y_m, \dots, y_m) \\ &\leq (q^n + q^{n+1} + \dots + q^{m-1})\, G_r(y_0, y_1, \dots, y_1) \\ &\leq (q^n + q^{n+1} + \dots)\, G_r(y_0, y_1, \dots, y_1) \\ &= \frac{q^n}{1-q}\, G_r(y_0, y_1, \dots, y_1) \to 0 \,\mathrm{as}\, n, m \to \infty. \end{aligned}$$

Thus the sequence $\langle y_n \rangle$ is a G_r-Cauchy sequence in X. By completeness of (X, G_r), there exists a point $u \in X$ such that $\langle y_n \rangle$ is G_r-convergent to u. Since $y_n = fx_n = gx_{n+1}$ hence we have $\lim_{n\to\infty} y_n = \lim_{n\to\infty} gx_n = \lim_{n\to\infty} fx_n = u$. Now g is continuous hence

$$\lim_{n\to\infty} ggx_n = \lim_{n\to\infty} gfx_n = gu.$$

Also f and g commute, therefore

$$\lim_{n\to\infty} fgx_n = \lim_{n\to\infty} gfx_n = \lim_{n\to\infty} ggx_n = gu.$$

Taking $\xi_1 = gx_n$, $\xi_k = x_n \,(2 \leq k \leq r)$ in (3) we have

$$G_r(fgx_n, fx_n, \dots, fx_n) \leq q\, G_r(ggx_n, gx_n, \dots, gx_n).$$

Making $n \to \infty$ we have

$$G_r(gu, u, \dots, u) \leq q\, G_r(gu, u, \dots, u).$$

Which gives $gu = u$. For otherwise $q \geq 1$ contradicting the fact that $0 < q < 1$. Now by taking $\xi_1 = x_n$, $\xi_k = u \,(2 \leq k \leq r)$ in (3) we have

$$G_r(fx_n, fu, \dots, fu) \leq q\, G_r(gx_n, gu, \dots, gu).$$

Making $n \to \infty$ we have $fu = u$. Therefore we have $fu = gu = u$, i.e. u is a common fixed point of f and g.

For uniqueness of u, suppose that $v \neq u$ is such that $fv = gu = v$. Then we have $G_r(u, v, \dots, v) > 0$ and

$$\begin{aligned} G_r(u, v, \dots, v) &= G_r(fu, fv, \dots, fv) \leq q\, G_r(gu, gv, \dots, gv) = q\, G_r(u, v, \dots, v) \\ &< G_r(u, v, \dots, v). \end{aligned}$$

Thus we get a contradiction, hence we have $u = v$. □

Theorem 2.18. *Let $G_r\colon X^r \to \mathbb{R}^+$,$(r \geq 3)$ be a generalized r-metric and (X, G_r) be a G_r-complete generalized r-metric space. Let $f\colon X \to X$ be a mapping which satisfies the following condition for all $x_1, x_2, \dots, x_r \in X$*

$$\begin{aligned} G_r(fx_1, fx_2, \dots, fx_r) \leq k\, max\{&G_r(x_1, x_2, \dots, x_r), G_r(x_1, fx_1, \dots, fx_1), \dots,\\ &G_r(x_r, fx_r, \dots, fx_r), G_r(x_1, fx_2, \dots, fx_2),\\ &G_r(x_2, fx_3, \dots, fx_3), \dots, G_r(x_r, fx_1, \dots, fx_1)\}. \end{aligned}$$

Where $0 \leq k < 1/2$. Then f has a unique fixed point (say u) and f is generalized r-continuous at u.

Proof. Let $f\colon X \to X$ be a mapping satisfying the given condition. Let $y_0 \in X$ be an arbitrary point. Define a sequence $\langle y_n \rangle$ by the relation $y_n = f^n y_0$, then by the given condition we have

$$\begin{aligned} G_r(fy_{n-1}, fy_n, \dots, fy_n) \leq k\max\{&G_r(y_{n-1}, y_n, \dots, y_n),\\ &G_r(y_{n-1}, fy_{n-1}, \dots, fy_{n-1}), \dots,\\ &G_r(y_n, fy_n, \dots, fy_n),\\ &G_r(y_{n-1}, fy_n, \dots, fy_n),\\ &G_r(y_n, fy_n, \dots, fy_n), \dots,\\ &G_r(y_n, fy_{n-1}, \dots, fy_{n-1})\} \end{aligned}$$

which gives
(2.5)
$$G_r(y_n, y_{n+1}, \dots, y_{n+1}) \leq k \max\{G_r(y_{n-1}, y_n, \dots, y_n), G_r(y_{n-1}, y_{n+1}, \dots, y_{n+1})\}.$$

By [G 5] we have

$$G_r(y_{n-1}, y_{n+1}, \dots, y_{n+1}) \leq G_r(y_{n-1}, y_n, \dots, y_n) + G_r(y_n, y_{n+1}, \dots, y_{n+1}).$$

Hence from (2.5) we have

$$\begin{aligned} G_r(y_n, y_{n+1}, \dots, y_{n+1}) \leq k \max\{&G_r(y_{n-1}, y_n, \dots, y_n), G_r(y_{n-1}, y_n, \dots, y_n)+\\ &G_r(y_n, y_{n+1}, \dots, y_{n+1})\}. \end{aligned}$$

Thus

$$G_r(y_n, y_{n+1}, \dots, y_{n+1}) \leq k\, \{G_r(y_{n-1}, y_n, \dots, y_n) + G_r(y_n, y_{n+1}, \dots, y_{n+1})\}.$$

Which gives

$$G_r(y_n, y_{n+1}, \dots, y_{n+1}) \leq \frac{k}{1-k}\, G_r(y_{n-1}, y_n, \dots, y_n). \tag{2.6}$$

Let $q = \frac{k}{1-k}$, then $q < 1$ since $0 \leq k < 1/2$ and by repeated application of (2.6) we have

$$G_r(y_n, y_{n+1}, \dots, y_{n+1}) \leq q^n\, G_r(y_0, y_1, \dots, y_1). \tag{2.7}$$

For all natural numbers n and $m(> n)$ we have by repeated use of [G 5] and (2.7) that

$$G_r(y_n, y_m, \dots, y_m) \leq \frac{q^n}{1-q}\, G_r(y_0, y_1, \dots, y_1) \to 0 \;\text{ as }\; n, m \to \infty.$$

Thus the sequence $\langle y_n \rangle$ is a G_r-Cauchy sequence in X. By completeness of (X, G_r), there exists a point $u \in X$ such that $\langle y_n \rangle$ is G_r-convergent to u. Suppose that $fu \neq u$, then

$$\begin{aligned} G_r(y_n, fu, \ldots, fu) \leq k \max\{&G_r(y_{n-1}, u, \ldots, u), G_r(y_{n-1}, y_n, \ldots, y_n), \ldots, \\ &G_r(u, fu, \ldots, fu), G_r(y_{n-1}, fu, \ldots, fu), \\ &G_r(u, fu, \ldots, fu), \ldots, G_r(u, y_n, \ldots, y_n)\} \end{aligned}$$

or

$$\begin{aligned} G_r(y_n, fu, \ldots, fu) \leq k \max\{&G_r(y_{n-1}, u, \ldots, u), G_r(y_{n-1}, y_n, \ldots, y_n), \\ &G_r(u, fu, \ldots, fu), G_r(y_{n-1}, fu, \ldots, fu), \\ &G_r(u, y_n, \ldots, y_n)\}. \end{aligned}$$

Taking the limit as $n \to \infty$, and using the fact that the function G_r is continuous on its variables, we have $G_r(u, fu, \ldots, fu) \leq k\, G_r(u, fu, \ldots, fu)$, which is a contradiction, since $0 \leq k < 1/2$. So we have $u = fu$.

For uniqueness of u, suppose that $v \neq u$ is such that $fv = v$, then we have

$$G_r(u, v, \ldots, v) = G_r(fu, fv, \ldots, fv) \leq k \max\{G_r(u, v, \ldots, v), G_r(v, fu, \ldots, fu)\}$$

or

$$G_r(u, v, \ldots, v) \leq k \max\{G_r(u, v, \ldots, v), G_r(v, u, \ldots, u)\}.$$

So, it must be the case that $G_r(u, v, \ldots, v) \leq k\, G_r(v, u, \ldots, u)$.

Again by the same argument we find that $G_r(v, u, \ldots, u) \leq k\, G_r(u, v, \ldots, v)$. Thus we have $G_r(u, v, \ldots, v) \leq k^2\, G_r(u, v, \ldots, v)$. Which implies that $u = v$, since $0 \leq k < 1/2$.

Now to prove that f is generalized r-continuous at u, let $\langle y_n \rangle$ be any sequence in X such that it is G_r-convergent to u, then

$$\begin{aligned} G_r(fy_n, fu, \ldots, fu) \leq k \max\{&G_r(y_n, u, \ldots, u), G_r(y_n, fy_n, \ldots, fy_n), \ldots, \\ &G_r(u, fu, \ldots, fu), G_r(y_n, fu, \ldots, fu), \ldots, \\ &G_r(u, fu, \ldots, fu), G_r(u, fy_n, \ldots, fy_n)\} \end{aligned}$$

or

$$\begin{aligned} G_r(fy_n, u, \ldots, u) \leq k \max\{&G_r(y_n, u, \ldots, u), G_r(y_n, fy_n, \ldots, fy_n), \\ &G_r(u, fy_n, \ldots, fy_n)\}. \end{aligned}$$

By [G 5] we have

$$G_r(y_n, fy_n, \ldots, fy_n) \leq G_r(y_n, u, \ldots, u) + G_r(u, fy_n, \ldots, fy_n).$$

Thus we deduce that

$$G_r(fy_n, u, \ldots, u) \leq k\, \{G_r(y_n, u, \ldots, u) + G_r(u, fy_n, \ldots, fy_n)\}.$$

Using proposition 2.4 we have

$$G_r(fy_n, u, \ldots, u) \leq k\, \{G_r(y_n, u, \ldots, u) + (r-1)\, G_r(fy_n, u, \ldots, u)\}$$

or

$$G_r(fy_n, u, \ldots, u) \leq \frac{k}{1-(r-1)k}\, G_r(y_n, u, \ldots, u).$$

Taking the limit as $n \to \infty$, we see that $G_r(fy_n, u, \ldots, u) \to 0$ and so by proposition 2.9 the sequence $\langle fy_n \rangle$ is G_r-convergent to $u = fu$. Therefore proposition 2.11 implies that f is generalized r-continuous at u.

□

Acknowledgement

The author is thankful to the referee for his valuable comments and suggestions on this manuscript.

References

[1] M. Abbas, W. Sintunavarat and P. Kumam, *Coupled fixed point of generalized contractive mappings on partially ordered G-metric spaces*, Fixed Point Theory Appl.2012, (2012) Article ID:31.

[2] B. C. Dhage, *A study of some fixed point theorem*, Ph.D. Thesis, Marathwada Univ. Aurangabad, 1984.

[3] S. Gähler, *2-metrische räume und ihre topologische struktur*, Math. Nachr. **26** (1963), 115–148.

[4] S. Gähler, *Zur geometric 2-metrische räume*, Rev. Roum. Math. Pures et Appl. **11** (1966), 664–669.

[5] K. S. Ha, Y. J. Cho and A. White, *Strictly convex and 2-convex 2-normed spaces*, Math. Japonica **33** (1988), 375–384.

[6] K. A. Khan, *On the possibility of N-topological spaces*, Int. J. Math. Arc. **3** (2012), 2520–2523.

[7] Z. Mustafa and B. Sims, *Some remarks concerning D-metric spaces*, in Proceedings of the International Conferences on Fixed Point Theory and Applications J. G. Falset, E. L. Fuster, B. Sims (eds), Valencia (Spain), July 2003, Yokohama Publishers, Yokohama, 2004, pp. 189–198.

[8] Z. Mustafa and B. Sims, *A new approach to generalized metric spaces*, J. Nonlinear Convex Anal. **7** (2006), 289–297.

[9] A. K. Sharma, *A note on fixed points in 2-metric spaces*, Indian J. Pure Appl. Math. **11** (1980), 1580–1583.

Manuscript received May 3, 2013
revised May 8, 2013

K. A. Khan
Department of Mathematics, V. R. A. L. Govt. Girls P. G. College, Bareilly (U.P.)-INDIA
E-mail address: kamran12341@yahoo.com